

A Symmetry-Enabled Direct Quantum Protocol for Many-Body Green's Functions

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We present a symmetry-enabled direct quantum protocol for computing many-body Green's functions, a central tool for studying strongly correlated quantum systems. Our protocol relies only on native time evolution and straightforward measurements available on current hardware platforms. By exploiting parity symmetry, a property satisfied by a broad class of Hamiltonians in condensed matter physics and quantum chemistry, including the Fermi–Hubbard and Heisenberg models, we introduce a tailored quench spectroscopy scheme that recovers both the real and imaginary parts of two-point time correlators, from which Green's functions can be reconstructed via efficient classical signal processing. We further develop a tailored parity-preserving quantum sampler that prepares parity-resolved (symmetric and antisymmetric) thermal states, enabling finite-temperature extensions within the same framework. Finally, we show that the same symmetry-based measurement primitive extends naturally to out-of-time-ordered correlators. Our results provide a practical route to estimating symmetry-resolved dynamical correlators on near-term and early fault-tolerant quantum hardware.

1 Introduction

Many-body Green's functions are essential tools for probing strongly interacting many-body systems [1]. They compactly describe the propagation of particles and excitations without explicit wave-function representations, and they encode key information about the density of states and band structure, which underpin phenomena such as superconductivity, magnetism, and phase transitions. Experimentally, spectroscopic techniques can be used to extract Green's-function information: for example, angle-resolved photoemission spectroscopy (ARPES) [65] measures the momentum-resolved spectral function associated with single-particle Green's functions, while resonant inelastic X-ray scattering (RIXS) [2] probes collective excitations encoded in higher-order correlation functions. On the computational side, classical approaches face severe limitations when calculating these dynamical quantities. For example, exact diagonalization [57] is restricted to small clusters due to the exponential growth of the Hilbert space. While quantum Monte Carlo methods [21] scale better for static properties, they struggle with the dynamical sign problem and the

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ill-conditioned analytic continuation required to extract real-time Green’s functions from imaginary time. Similarly, tensor network states [54], though powerful in 1D, are limited by the rapid growth of entanglement entropy during non-equilibrium time evolution, which requires an exponentially growing bond dimension to maintain accuracy.

At a fundamental level, Green’s functions are time-ordered correlation functions. Quantum computing offers a promising route to access them by enabling coherent real-time evolution and the measurement of dynamical correlators, potentially bypassing several bottlenecks faced by classical simulations. Many prominent quantum algorithms have been proposed, such as methods based on phase estimation [3, 4, 11, 12, 36, 50] and variational approaches [8, 15, 28]. However, such methods often rely on ancilla-assisted interference circuits and controlled time evolution to extract complex-valued correlators, which can be prohibitively resource-intensive on near-term hardware. In particular, controlled implementations of large unitaries substantially increase circuit depth and error rates, while also placing stringent demands on gate fidelity and coherence. These challenges are especially acute for current noisy intermediate-scale quantum (NISQ) devices [45], and remain nontrivial even in the early fault-tolerant regime [30].

To circumvent these overheads, a complementary class of quantum algorithms has recently gained traction by avoiding controlled unitaries and, in many cases, ancilla qubits. For instance, *quench spectroscopy* (QS) [14, 31, 37, 50, 56, 61], a widely used protocol inspired by Ramsey interferometry, can be used to probe dynamical response functions. However, early studies such as [56] focused on fermionic correlation functions under translation symmetry, which restricts the accessible information to the imaginary part of time correlators. Closely related, the *linear-response framework* (LRF) [5, 32, 33, 44] exploits the connection between Green’s functions and response theory by introducing a weak time-dependent perturbation, $H(t) = H + \varepsilon A(t)$, and extracting correlators from the leading-order contribution to measured expectation values such as $\langle A(t) \rangle$. While conceptually simple, this approach typically requires simulating time-dependent Hamiltonians, and the recovered correlators are generally approximate due to higher-order corrections (except in [5]). Another route is *quantum imaginary time evolution* (QITE) [39, 58, 60], which implements normalized imaginary-time evolution (e.g., via Trotter–Suzuki decompositions) and introduces an imaginary-time shift $t \rightarrow t + i\tau$. Multi-time correlators can then be obtained from derivatives with respect to τ , but the accuracy is controlled by the asymptotic limit $\tau \rightarrow 0$. The *direct Krylov space* (DKS) method [25, 29] instead computes Green’s functions recursively using analytically derived coefficients estimated from states, observables, and the Hamiltonian, thereby avoiding explicit quantum dynamics. However, this advantage comes at the cost of a high sample complexity that grows with the problem size. Moreover, subspace methods such as DKS can be sensitive to noise and estimation errors, particularly as the recursion depth increases [20, 29]. Finally, the *direct measurement* scheme [38] leverages mid-circuit measurements (MCMs) to access the real part of time correlators without assumptions on the initial state. Despite its generality and simplicity, MCM-based protocols remain challenging on current hardware due to slow and noisy MCMs and crosstalk that can disturb nearby qubits.

In this work, we introduce an ancilla-free and noise-robust direct quantum protocol for estimating both the real and imaginary parts of two-point time correlators by exploiting *parity symmetry* [1], from which Green’s functions can be reconstructed via efficient classical signal analysis. We refer to this approach as *tailored quench spectroscopy* (TQS). Specifically, we consider Hamiltonians that commute with a parity operator P , while the observables of interest A and B anti-commute with P . This condition is satisfied by a broad class of models central to condensed matter physics and quantum chemistry, includ-

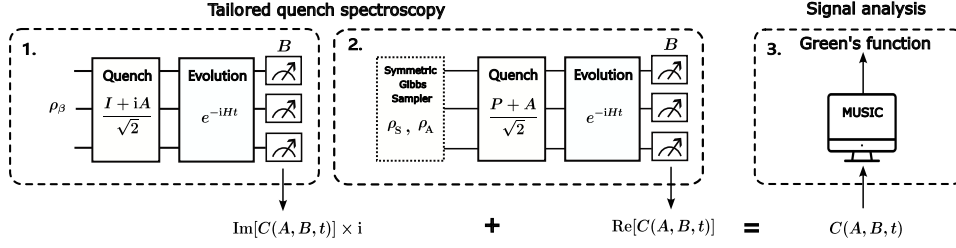


Figure 1: Direct quantum protocol for computing many-body Green's functions via tailored quench spectroscopy. For an initial thermal state ρ_β (with $\beta \rightarrow \infty$ corresponding to the zero-temperature limit, so that the thermal state becomes the ground state), the imaginary and real parts of the two-point time correlator $C(A, B, t)$ are obtained by applying the tailored quench operators $(I + iA)/\sqrt{2}$ and $(P + A)/\sqrt{2}$, respectively, followed by time evolution under e^{-iHt} and measurements in the eigenbasis of the observable B . For a thermal initial state, the symmetric and antisymmetric thermal states ρ_S and ρ_A are prepared using a tailored parity-preserving quantum sampler (step 2). For a ground-state initial condition, this thermal-state preparation step is not needed. The full Green's function is then reconstructed via classical signal-processing routines (e.g., MULTiple Signal Classification (MUSIC)).

ing the Fermi–Hubbard and Heisenberg models. An overview of the protocol is provided in Fig. 1.

By designing tailored quench operators, we extend the standard QS method [56] to access bosonic correlation functions, enabling direct evaluation of the real part of time correlators. Compared to other ancilla-free protocols, such as LRF and QITE, our TQS approach offers several advantages: i) it requires only time-independent Hamiltonian evolution, thereby avoiding the simulation of time-dependent dynamics and reducing experimental complexity; ii) it enables exact, non-perturbative recovery of correlators, substantially mitigating systematic errors; and iii) unlike imaginary-time methods, which perturb the state via $e^{-\tau O}$ and typically require additional overhead (e.g., Trotterization or dissipative evolution), it relies on considerably simpler operations.

In developing TQS protocols for finite-temperature Green's functions, we also introduce a tailored *parity-preserving quantum sampler* for preparing symmetric and antisymmetric thermal states, which may be of independent interest.

The remainder of the paper is organized as follows. In Sec. 2, we review the relationship between two-point time correlators and many-body Green's functions, parity symmetry, and the quench spectroscopy approach. In Sec. 3, we present the symmetry-enabled TQS protocol in detail. We generalize the method to out-of-time-ordered correlators (OTOCs) in Sec. 3.3. Complexity and error analyses are provided in Sec. 4, and numerical simulations are reported in Sec. 5. We conclude in Sec. 6 with a discussion of future directions.

2 Background

2.1 Many-body Green's functions and two-point time correlators

We begin by clarifying the relationship between two-point time correlators and the Lehmann representation of many-body Green's functions. Consider a Hamiltonian H with spectral decomposition $H = \sum_{n=0}^{d-1} E_n |n\rangle\langle n|$, and let O_a and O_b be two operators. In many-body physics, the two-point correlator is

$$C(O_a, O_b, t_1, t_2) := \text{tr}[\rho O_a(t_1) O_b(t_2)], \quad (1)$$

where $O(t) := e^{iHt} O e^{-iHt}$. If the initial state is stationary with respect to H , e.g., a Gibbs state $\rho = \rho_\beta := e^{-\beta H} / \text{tr}(e^{-\beta H})$ that commutes with H , then the correlator depends only on the time difference and can be written as

$$C(O_a, O_b, t) = \text{tr}[\rho O_a O_b(t)]. \quad (2)$$

Using the spectral decomposition of H , we obtain the expansion

$$C(O_a, O_b, t) = \sum_{m,n} \langle n | O_b | m \rangle \langle m | \rho O_a | n \rangle e^{i(E_n - E_m)t}. \quad (3)$$

We define the Green's function by a one-sided Fourier (Laplace) transform,

$$G(z) = -i \int_0^\infty dt e^{izt} C(O_a, O_b, -t), \quad \text{Im } z > 0, \quad (4)$$

which yields the Lehmann representation [40]

$$G(z) = \sum_{m,n} \frac{\langle n | O_b | m \rangle \langle m | \rho O_a | n \rangle}{z - (E_n - E_m)}, \quad \text{Im } z > 0. \quad (5)$$

Green's functions are commonly expressed either in the time domain, through the correlator C , or in the frequency domain, through Eq. (5); in this work we focus on the Lehmann representation. See Appendix A.1 for additional background.

In fermionic systems, O_a and O_b are typically linear combinations of fermionic annihilation and creation operators $\{c_i, c_i^\dagger\}_i$. For example, if $O_a = \sum_i a_i c_i$ and $O_b = \sum_{i'} b_{i'} c_{i'}^\dagger$, then by linearity

$$C(O_a, O_b, t) = \sum_{i,i'} a_i b_{i'} C(c_i, c_{i'}^\dagger, t). \quad (6)$$

Thus, it suffices to estimate correlators of the form $C(c_i, c_{i'}^\dagger, t)$. Under the Jordan–Wigner transformation (see Appendix A.2), each fermionic operator can be written as a linear combination of two Pauli strings (equivalently, Majorana operators), i.e., $c_i = (A + iB)/2$ with Pauli operators A and B that anticommute. Writing $c_i = (A_0 + iB_0)/2$ and $c_{i'} = (A_1 + iB_1)/2$, we obtain

$$\begin{aligned} \frac{1}{2} C(A_0 + iB_0, A_1 - iB_1, t) &= \frac{1}{2} C(A_0, A_1, t) + \frac{1}{2} C(B_0, B_1, t) \\ &\quad - \frac{i}{2} C(A_0, B_1, t) + \frac{i}{2} C(B_0, A_1, t). \end{aligned} \quad (7)$$

Consequently, the problem reduces to estimating general two-point correlators of the form $C(A, B, t)$, which will be the primary focus of the following sections.

2.2 Parity symmetry

If a unitary Hermitian operator P satisfies $[H, P] = 0$, then we say that H has a parity symmetry and refer to P as a parity operator. This condition is common in many lattice models. For example, the Heisenberg (XXZ) chain

$$H_{\text{XXZ}} = J_X \left(\sum_{n=0}^{N-2} X_n X_{n+1} + Y_n Y_{n+1} \right) + J_Z \sum_{n=0}^{N-2} Z_n Z_{n+1} + h \sum_{n=0}^{N-1} Z_n. \quad (8)$$

commutes with the global $\mathbb{Z}_2$ parity operator $P := \prod_{j=0}^{N-1} Z_j$.

As another example, consider the spinful Fermi–Hubbard model

$$H_{\text{FH}} = - \sum_{j=0}^{N-2} \sum_{\sigma=\uparrow,\downarrow} \left(c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma} \right) + h_U \sum_{j=0}^{N-1} n_{j,\uparrow} n_{j,\downarrow}. \quad (9)$$

A natural parity symmetry here is fermion-number parity,

$$P := (-1)^{\hat{N}} = \prod_{j,\sigma} e^{i\pi n_{j,\sigma}}, \quad \hat{N} := \sum_{j,\sigma} n_{j,\sigma}. \quad (10)$$

Under the Jordan–Wigner transformation, this maps to the Pauli- Z string over all spin-orbitals (qubits),

$$P = \prod_{q=0}^{2N-1} Z_q. \quad (11)$$

Throughout the remainder of the paper, we choose $\{|n\rangle\}_{n=0}^{d-1}$ to be a common eigenbasis of H and P , so that

$$H|n\rangle = E_n|n\rangle, \quad P|n\rangle = p_n|n\rangle, \quad p_n \in \{\pm 1\}. \quad (12)$$

We say that $|n\rangle$ has even parity if $p_n = +1$ and odd parity if $p_n = -1$. Let $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$ denote the corresponding parity sectors (subspaces). We also define the parity projectors

$$\Pi_S := \frac{I + P}{2}, \quad \Pi_A := \frac{I - P}{2}, \quad (13)$$

which form a two-outcome parity POVM. Here I denotes the identity operator.

We will use the following selection rule (see Appendix B.1 for a proof).

Lemma 1. *Suppose $[H, P] = 0$ and $\{A, P\} = 0$, and let $|m\rangle, |n\rangle$ be eigenstates of H (and hence of P). If $\langle n|A|m\rangle \neq 0$, then $|m\rangle$ and $|n\rangle$ have opposite parity, i.e., $p_m p_n = -1$.*

2.3 Quench spectroscopy

Quench spectroscopy (QS) is a simple yet powerful technique for probing dynamical properties of quantum systems [14, 37]. The central idea is to apply a sudden perturbation to a prepared state and then monitor the ensuing unitary dynamics under a time-independent Hamiltonian. By measuring suitable observables as a function of time, one obtains a signal whose oscillation frequencies correspond to energy differences in the spectrum. Classical post-processing of this time-domain signal then yields spectral information and dynamical correlation functions, without requiring controlled time evolution or other time-dependent operations.

Building on the QS method in Refs. [55, 56], one can directly access the *imaginary part* of a two-point time correlator under minimal assumptions: the initial state may be either equilibrium or nonequilibrium. To formalize the protocol, we define the *quench function* as the expectation value of a measurement outcome,

$$Q_{\rho,M}(U) := \text{tr}(U \rho U^\dagger M), \quad (14)$$

where U denotes the unitary implemented in the QS experiment and M is the measured observable. Assuming A is a Hermitian operator satisfying $A^2 = I$, and applying the quench operator

$$U_{\text{Im}} := e^{i\pi A/4} = \frac{I + iA}{\sqrt{2}}, \quad (15)$$

one obtains

$$\begin{aligned}\text{Im}[C(A, B, t)] &= -\frac{i}{2}\text{tr}(\rho[A, B(t)]) \\ &= Q_{\rho, B} \left(e^{-iHt} U_{\text{Im}} \right) - \frac{1}{2} Q_{\rho, B} \left(e^{-iHt} A \right) - \frac{1}{2} Q_{\rho, B} \left(e^{-iHt} \right). \quad (16)\end{aligned}$$

All terms on the right-hand side can be measured directly on a quantum device, making QS a practical route to extracting imaginary time-correlator information, equivalently bosonic response functions of the form $iC(A, B, t) - iC(B, A, t)$. The nontrivial step is to access the *real part* of time correlators, or equivalently fermionic response functions of the form $C(A, B, t) + C(B, A, t)$. In the following sections, we show how to achieve this by introducing a *tailored quench operator* and exploiting symmetries inherent to the system.

3 Symmetry-enabled protocols for time correlators

From now on, we focus on systems with a $\mathbb{Z}_2$ (parity) symmetry. Concretely, we assume there exists a parity operator P such that

$$[P, H] = 0, \quad P^\dagger = P, \quad P^2 = I. \quad (17)$$

We consider Hermitian observables A and B satisfying

$$A^2 = I, \quad \{A, P\} = \{B, P\} = 0, \quad (18)$$

and we assume the initial state is parity-symmetric,

$$[\rho, P] = 0. \quad (19)$$

Prominent examples include equilibrium (Gibbs) states $\rho_\beta = e^{-\beta H} / \text{tr}(e^{-\beta H})$, and the ground state recovered in the limit $\beta \rightarrow \infty$.

3.1 Tailored quench spectroscopy

To access the real part of a two-point time correlator, we introduce the quench operator

$$U_{\text{Re}} := \frac{P + A}{\sqrt{2}}. \quad (20)$$

Since $\{P, A\} = 0$ and $P^2 = A^2 = I$, the operator PA satisfies $(PA)^\dagger = -PA$ and $(PA)^2 = -I$, so $G := -iPA$ is Hermitian and obeys $G^2 = I$. Consequently, U_{Re} admits the decomposition

$$\frac{P + A}{\sqrt{2}} = P \exp\left(i\frac{\pi}{4} G\right) = P \exp\left[i\frac{\pi}{4} (-iPA)\right]. \quad (21)$$

In particular, when P and A are Pauli strings, G is also a Pauli string and the implementation reduces to a single Pauli rotation (with Bloch-sphere rotation angle $\pi/2$).

We refer to the protocol based on the two quench operators U_{Im} and U_{Re} as *tailored quench spectroscopy* (TQS). For an N -qubit system, both operators can be implemented with circuit depth $\mathcal{O}(\log N)$ using only CNOTs and single-qubit gates [26].

With the quench operator U_{Re} , one obtains

$$\begin{aligned}\text{Re}[C(PA, B, t)] &= \frac{1}{2}\text{tr}(\rho P\{A, B(t)\}) \\ &= Q_{\rho, B} \left(e^{-iHt} U_{\text{Re}} \right) - \frac{1}{2} Q_{\rho, B} \left(e^{-iHt} P \right) - \frac{1}{2} Q_{\rho, B} \left(e^{-iHt} A \right). \quad (22)\end{aligned}$$

Importantly, if the initial state ρ has a well-defined parity, i.e., $P\rho = p\rho$ with $p \in \{+1, -1\}$, then $C(PA, B, t) = pC(A, B, t)$. Consequently, the left-hand side of Eq. (22) directly yields $\text{Re}[C(A, B, t)]$ up to the known sign p , which can be fixed from the parity sector of the initial state. Together with the QS identity for $\text{Im}[C(A, B, t)]$, the full complex correlator can therefore be reconstructed from a linear combination of six experimentally accessible quench functions.

Moreover, when $\{B, P\} = 0$, several terms in Eq. (16) and Eq. (22) vanish identically by symmetry. For instance,

$$\{P, B\} = 0 \implies PBP = -B \implies PB(t)P = -B(t) \implies \text{tr}(\rho B(t)) = 0. \quad (23)$$

As a result, the full complex correlator can be reconstructed using only two quench functions. This simplification is summarized in the following theorem; see Appendix B.2 for the proof.

Theorem 1. *Under the assumptions Eqs. (17)–(19), for all $t \in \mathbb{R}$,*

$$\text{Im}[C(A, B, t)] = Q_{\rho, B} \left(e^{-iHt} U_{\text{Im}} \right). \quad (24)$$

Moreover, if the initial state has definite parity, i.e., $P\rho = p\rho$ with $p \in \{\pm 1\}$, then

$$\text{Re}[C(A, B, t)] = p Q_{\rho, B} \left(e^{-iHt} U_{\text{Re}} \right). \quad (25)$$

The full protocol is demonstrated in Algorithm 1 in the form of pseudocode.

Algorithm 1 Tailored quench spectroscopy

Input: Initial state ρ with definite parity, times t , Hamiltonian H , observables A, B , parity operator P , error bound ε .
Output: Estimate of $C(A, B, t)$ with error bound ε .
1: Measure ρ in the eigenbasis of P to obtain its parity $\hat{p}$.
2: **for** $n = 1, 2, \dots, N = \mathcal{O}(\varepsilon^{-2})$ **do**
3: Prepare ρ ;
4: Apply quench operator U_{Im} on the state ρ ;
5: Let the state evolve under $\exp(-iHt)$;
6: Measure the evolved state in the eigenbasis of B to obtain outcome $c_n^{(i)}$;
7: **end for**
8: Compute $\overline{c^{(i)}} = \sum_{n=1}^N c_n^{(i)} / N$.
9: **for** $n = 1, 2, \dots, N = \mathcal{O}(\varepsilon^{-2})$ **do**
10: Prepare ρ ;
11: Apply quench operator U_{Re} on ρ ;
12: Let the state evolve under $\exp(-iHt)$;
13: Measure the evolved state in the eigenbasis of B to obtain outcome $c_n^{(r)}$;
14: **end for**
15: Compute $\overline{c^{(r)}} = \sum_{n=1}^N c_n^{(r)} / N$.
16: Return $\hat{p} \overline{c^{(r)}} + i \overline{c^{(i)}}$.

Theorem 1 applies in particular when the initial state ρ is a non-degenerate eigenstate of H , in which case ρ has a definite parity $p \in \{\pm 1\}$. In the degenerate case, since $[H, P] = 0$, one can choose an energy eigenbasis that simultaneously diagonalizes P within each degenerate eigenspace, and hence prepare a *pure* eigenstate with definite parity. In practice, parity can be assessed by estimating $\langle P \rangle_\rho = \text{tr}(\rho P)$ using standard end-of-circuit measurements. If parity leakage is observed (i.e., $|\langle P \rangle_\rho| < 1$), one may, for example, enforce

a parity-preserving state-preparation circuit so that the prepared state remains within the desired parity sector. Since $[H, P] = 0$, parity is conserved under time evolution, so this calibration can be performed once and reused for subsequent quench spectroscopy runs.

Next, we show that an analogous strategy applies to thermal states. Consider a finite-temperature thermal (quantum Gibbs) state

$$\rho_\beta := \frac{e^{-\beta H}}{\mathcal{Z}}, \quad \mathcal{Z} := \text{tr}(e^{-\beta H}), \quad (26)$$

at inverse temperature β . Since $[H, P] = 0$, it follows that $[\rho_\beta, H] = [P, \rho_\beta] = 0$. However, ρ_β generally does not have definite parity, i.e., it does not satisfy the stronger condition

$$P\rho_\beta = p\rho_\beta, \quad p \in \{\pm 1\}. \quad (27)$$

To recover a fixed-parity setting, we introduce the *symmetric and antisymmetric thermal states*, which correspond to the thermal state projected onto the even and odd parity sectors:

$$\rho_S := \frac{\Pi_S \rho_\beta \Pi_S}{\text{tr}(\rho_\beta \Pi_S)} = \frac{\rho_\beta + \rho_\beta P}{1 + \text{tr}(\rho_\beta P)}, \quad \rho_A := \frac{\Pi_A \rho_\beta \Pi_A}{\text{tr}(\rho_\beta \Pi_A)} = \frac{\rho_\beta - \rho_\beta P}{1 - \text{tr}(\rho_\beta P)}, \quad (28)$$

where $\Pi_S = (I + P)/2$ and $\Pi_A = (I - P)/2$ are two projectors: $\Pi_S^2 = \Pi_S, \Pi_A^2 = \Pi_A$. Both ρ_S, ρ_A are valid quantum states with trace 1.

These satisfy $P\rho_S = \rho_S$ and $P\rho_A = -\rho_A$, and the original thermal state decomposes as

$$\rho_\beta = \frac{1 + \text{tr}(\rho_\beta P)}{2} \rho_S + \frac{1 - \text{tr}(\rho_\beta P)}{2} \rho_A. \quad (29)$$

With these definitions, TQS provides an efficient route to estimating the real component of $C(A, B, t)$, as summarized in the following theorem; see Appendix B.3 for its proof.

Theorem 2. *Let ρ_β , ρ_S , and ρ_A denote the thermal, symmetric, and antisymmetric thermal states of H at inverse temperature β . Under the assumptions Eqs. (17) and (18), for all $t \in \mathbb{R}$,*

$$\begin{aligned} \text{Re}[C(A, B, t)] &= \frac{1 + \text{tr}(\rho_\beta P)}{2} Q_{\rho_S, B} \left(e^{-iHt} U_{\text{Re}} \right) \\ &\quad - \frac{1 - \text{tr}(\rho_\beta P)}{2} Q_{\rho_A, B} \left(e^{-iHt} U_{\text{Re}} \right), \end{aligned} \quad (30)$$

and the imaginary component can be estimated analogously to Theorem 1.

3.2 Preparation of ρ_S, ρ_A

Since thermal-state preparation itself is a challenging task on near-term hardware, we treat the generation of parity-resolved Gibbs states as an input primitive: our TQS protocol applies to any procedure that can prepare ρ_S and ρ_A , exactly or approximately. Nevertheless, we outline two possible approaches. Measuring ρ_β with the two-outcome POVM $\{\Pi_S, \Pi_A\}$ provides a conceptually simple way to obtain ρ_S and ρ_A . However, since P and B anticommute, a direct parity measurement is not compatible with the subsequent measurement in the eigenbasis of B within the same experimental run without additional overhead (e.g., ancilla-assisted parity checks with feed-forward and/or post-selection). Moreover, mid-circuit measurements are typically slow and noisy, making this approach less attractive.

Alternatively, we introduce a tailored *parity-preserving quantum sampler* based on dissipative state preparation, which can directly prepare parity-resolved Gibbs states without requiring mid-circuit measurements. We first briefly review a Lindbladian framework for preparing Gibbs states via dissipative evolution that has recently attracted renewed interest [7, 18, 19, 46, 52]. Let $\mathcal{L}$ be a Lindbladian and consider the master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}[\rho(t)]. \quad (31)$$

Let $\rho(t) = e^{\mathcal{L}t}[\rho(0)]$ denote the solution of the master equation. If the map $e^{\mathcal{L}t}$ is *ergodic* [6], then there exists a unique steady state (in the context of quantum Gibbs sampler, the steady state is ρ_β) such that $\lim_{t \rightarrow \infty} \rho(t) = \rho_\beta$ for any initial state $\rho(0)$. In this way, Gibbs-state preparation is reduced to simulating dissipative dynamics.

A canonical construction is the *Davies generator* [24, 35]. Fix a system operator J and define its *Bohr-frequency components* (i.e., components labeled by energy gaps $\nu = E_m - E_n$) by

$$J_\nu := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dt e^{iHt} J e^{-iHt} e^{-i\nu t}, \quad (32)$$

so that $[H, J_\nu] = \nu J_\nu$, $e^{iHt} J e^{-iHt} = \sum_\nu J_\nu e^{i\nu t}$. Let $\eta(\nu)$ be any function satisfying the relation $\eta(-\nu) = e^{\beta\nu} \eta(\nu)$. Then the Davies generator is defined as

$$\mathcal{L}_D[\cdot] = \sum_\nu \eta(\nu) \left[J_\nu(\cdot) J_\nu^\dagger - \frac{1}{2} \{ J_\nu^\dagger J_\nu, \cdot \} \right]. \quad (33)$$

This Davies generator satisfies the Kubo-Martin-Schwinger (KMS) detailed balance condition [18] with respect to ρ_β :

$$\mathcal{L}_D[\cdot] = \rho_\beta^{1/2} \mathcal{L}_D^\dagger \left[\rho_\beta^{-1/2} \cdot \rho_\beta^{-1/2} \right] \rho_\beta^{1/2}. \quad (34)$$

This implies $\mathcal{L}_D[\rho_\beta] = 0$. A self-consistent derivation is included in Appendix A.3.

To ensure that ρ_β is the *unique* steady state on the full Hilbert space, one imposes an irreducibility condition. One convenient formulation is via reducing projectors: a projector Π *reduces* $\mathcal{L}_D$ if the corresponding subspace is invariant under the dynamics, namely

$$\mathcal{L}_D[\Pi\rho\Pi] = \Pi\mathcal{L}_D[\Pi\rho\Pi]\Pi, \quad \forall \rho. \quad (35)$$

If the only reducing projectors are $\Pi = 0$ and $\Pi = I$, then $\mathcal{L}_D$ is *irreducible* [66]; under additional mild assumptions (e.g., full rank of the stationary state), this implies a unique full-rank stationary state, which must be ρ_β by detailed balance.

In our application, however, we do not need global irreducibility. Instead, we want the dynamics to *preserve parity* so that it does not mix the even and odd sectors, and then be ergodic within each sector. This yields the parity-resolved Gibbs states ρ_S and ρ_A as the long-time limits.

Assuming $[H, P] = 0$, if we choose a jump operator that commutes with parity, $[J, P] = 0$, then the Davies dynamics is block-diagonal in the parity decomposition.

Proposition 1. *Suppose $[H, P] = 0$ and $[J, P] = 0$. Then both Π_S and Π_A reduce the Davies generator $\mathcal{L}_D$.*

Proof. Since $[H, P] = 0$, we have $e^{iHt} P e^{-iHt} = P$. If $[J, P] = 0$, then $[J(t), P] = 0$ for all t , where $J(t) = e^{iHt} J e^{-iHt}$. Taking Fourier components implies $[J_\nu, P] = 0$ for all ν , hence $[J_\nu, \Pi_S] = [J_\nu, \Pi_A] = 0$.

Let σ be supported in the even sector, i.e., $\sigma = \Pi_S \sigma \Pi_S$. Then for each ν ,

$$J_\nu \sigma J_\nu^\dagger = J_\nu (\Pi_S \sigma \Pi_S) J_\nu^\dagger = \Pi_S (J_\nu \sigma J_\nu^\dagger) \Pi_S, \quad (36)$$

$$J_\nu^\dagger J_\nu \sigma = J_\nu^\dagger J_\nu (\Pi_S \sigma \Pi_S) = \Pi_S (J_\nu^\dagger J_\nu \sigma) \Pi_S, \quad (37)$$

and similarly for $\sigma J_\nu^\dagger J_\nu$. Plugging these relations into Eq. (33) yields $\mathcal{L}_D[\sigma] = \Pi_S \mathcal{L}_D[\sigma] \Pi_S$, i.e., Π_S reduces $\mathcal{L}_D$. The same argument applies to Π_A . $\square$

The proposition implies that there is no transition between $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$ in the dissipative dynamics. Because the parity sectors are invariant, the long-time limit of the dynamics depends on the initial parity weights. In particular, if the restriction of $\mathcal{L}_D$ to $\mathcal{H}_{\text{even}}$ is ergodic, then the unique stationary state within that sector is the even-parity Gibbs state ρ_S ; likewise, ergodicity on $\mathcal{H}_{\text{odd}}$ yields ρ_A . Thus, by preparing an initial state supported in the desired sector and evolving under a parity-preserving Davies generator that is ergodic within that sector, we obtain ρ_S or ρ_A without mid-circuit parity measurements. Here we emphasize that the ergodicity condition within the parity sectors is not proved in our parity-preserving Gibbs sampler.

In practice, Lindbladian evolution of this type can be implemented using dissipative dynamics and related simulation primitives [16, 17, 64]. While some implementations introduce additional ancilla qubits, they avoid long-range controlled unitaries and deep interference structures, making them potentially suitable for near-term or early fault-tolerance quantum devices.

We now illustrate how to construct a parity-preserving Lindbladian whose unique steady state is the symmetric thermal state. As a concrete example, we consider the $N = 4$ site XXZ model in Eq. (8) with parameters $J_X = 2$, $J_Z = 1$, and $h = 10$. To generate the full thermal state ρ_β , we choose a parity-breaking jump operator $J = X_0$. To generate the symmetric thermal state ρ_S , we instead choose the parity-preserving two-site operator $J = Y_0 Y_1$. We denote the corresponding Davies generators by $\mathcal{L}_D$ (for $J = X_0$) and $\mathcal{L}_{D,S}$ (for $J = Y_0 Y_1$).

To simulate the Lindbladian dynamics, we vectorize the density matrix using the Pauli basis. Let $\mathcal{P}_n = \{P_i\}_{i=0}^{d^2-1}$ be the set of n -qubit Pauli operators (including the identity), where $d = 2^n$. We define the vectorization of ρ as

$$|\rho\rangle\rangle = \frac{1}{d} \sum_{i=0}^{d^2-1} \text{tr}(\rho P_i) \mathbf{e}_i, \quad (38)$$

with $\{\mathbf{e}_i\}_{i=0}^{d^2-1}$ the standard basis of $\mathbb{R}^{d^2}$. Under this mapping, the action of a Lindbladian $\mathcal{L}$ becomes a matrix $\hat{\mathcal{L}}$ acting linearly on $|\rho\rangle\rangle$, i.e.,

$$|\mathcal{L}(\rho)\rangle\rangle = \frac{1}{d} \sum_{i=0}^{d^2-1} \text{tr}(\mathcal{L}(\rho) P_i) \mathbf{e}_i = \frac{1}{d} \sum_{i=0}^{d^2-1} \text{tr}(\rho \mathcal{L}^\dagger(P_i)) \mathbf{e}_i = \hat{\mathcal{L}} |\rho\rangle\rangle, \quad (39)$$

where

$$\hat{\mathcal{L}} = \frac{1}{d} \sum_{i,j=0}^{d^2-1} \text{tr}(\mathcal{L}^\dagger(P_i) P_j) \mathbf{e}_i \mathbf{e}_j^\top = \frac{1}{d} \sum_{i,j=0}^{d^2-1} \text{tr}(P_i \mathcal{L}(P_j)) \mathbf{e}_i \mathbf{e}_j^\top. \quad (40)$$

The Lindbladian equation thus becomes the linear ODE

$$\frac{d}{dt} |\rho(t)\rangle\rangle = \hat{\mathcal{L}} |\rho(t)\rangle\rangle. \quad (41)$$

We initialize $\rho(0)$ as a random state supported in the even-parity subspace and evolve it under the two Davies generators. As shown in Fig. 2, both dynamics converge to their corresponding stationary states, with $\mathcal{L}_{D,S}$ converging to the parity-resolved thermal state ρ_S .

We emphasize that the relative convergence rates in Fig. 2 are specific to this finite-size example and should not be interpreted as a generic property of the parity-preserving construction. Parity preservation ensures invariance of the symmetry sectors, but does not by itself imply faster or slower mixing. The convergence behavior is instead determined by the spectral properties of the corresponding Lindbladian and depends on the choice of jump operators and transition rates.

The crossing of the two curves likewise has no universal significance. In general, the transient convergence curve depends on the overlap of the initial state with the decay modes of the Lindbladian. Since the initial state used here is a particular random state in the even-parity sector, changing this state can modify the transient behavior and hence the location, or even the presence, of such a crossing. The purpose of Fig. 2 is therefore to demonstrate the feasibility of preparing a parity-resolved Gibbs state using parity-preserving Davies dynamics, rather than to establish an improvement in mixing time.

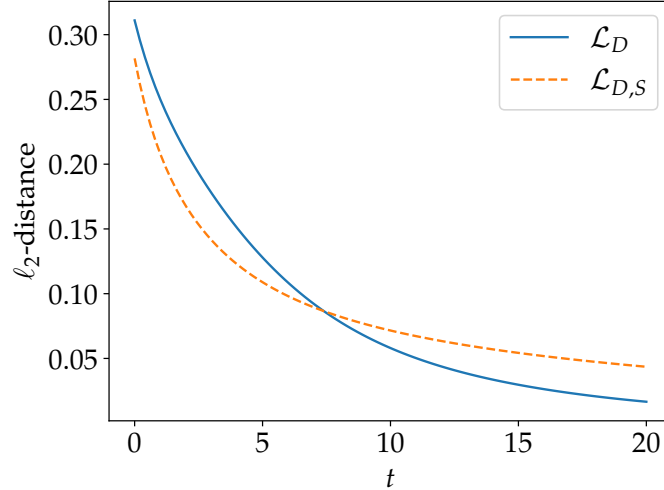


Figure 2: Preparation of thermal states using Davies dynamics for the finite-size XXZ example. The horizontal axis is the Lindbladian evolution time t , and the vertical axis is the ℓ_2 -distance between the vectorized evolved state and the corresponding steady state: $\|v - w\|_2 := (\sum_i |v_i - w_i|^2)^{1/2}$, which equals the Frobenius distance between density operators divided by $\sqrt{d}$. The blue solid curve corresponds to $\mathcal{L}_D$ and convergence to the full Gibbs state, whereas the orange dashed curve corresponds to the parity-preserving generator $\mathcal{L}_{D,S}$ and convergence to the even-parity Gibbs state ρ_S . The initial state is a random state supported in the even-parity sector. The relative convergence rates and the crossing of the curves are specific to this choice of generators and initial state and are not intended to imply a generic mixing-time advantage or disadvantage of the parity-preserving construction.

3.3 Generalization to OTOC

Our approach can be partially extended to a class of n -point correlators [13, 43, 58]. Let $O_0, O_1, \dots, O_{n-1}$ be n observables and consider the correlator

$$\text{tr}(\rho O_0 O_1 \cdots O_{n-1}). \quad (42)$$

Assume the initial state is parity-symmetric, $[\rho, P] = 0$, and that O_0 is parity-odd, $\{O_0, P\} = 0$. As in the two-point setting, we introduce the tailored quench operators

$$U_{\text{Im}} = \frac{I + iO_0}{\sqrt{2}}, \quad U_{\text{Re}} = \frac{P + O_0}{\sqrt{2}}. \quad (43)$$

We focus on the case where the remaining operators $O_1, \dots, O_L$ are Heisenberg evolutions of Pauli observables, i.e., $O_\ell = O_\ell(t_\ell) = e^{iHt_\ell} O_\ell e^{-iHt_\ell}$, and the ordered product is invariant under reversal,

$$O_1 O_2 \cdots O_{n-1} = O_{n-1} \cdots O_2 O_1, \quad (44)$$

which holds, for example, for certain palindromic strings arising in out-of-time-order correlators. Under these conditions, the same TQS measurement primitives can be used to access the desired correlator.

A particularly important example is the out-of-time-order correlator (OTOC), widely used to diagnose quantum chaos and information scrambling [22, 27, 51, 59]. For a thermal state ρ_β and Pauli operators A, B , we define

$$\text{OTOC}(A, B, t) = \text{tr}[\rho_\beta A B(t) A B(t)], \quad (45)$$

where $B(t) = e^{iHt} B e^{-iHt}$. Introduce the operator $\tilde{A} := B e^{-iHt} A e^{iHt} B$. Then its Heisenberg evolution satisfies

$$\tilde{A}(t) = e^{iHt} \tilde{A} e^{-iHt} = B(t) A B(t), \quad (46)$$

and therefore

$$\text{OTOC}(A, B, t) = \text{tr}[\rho_\beta A \tilde{A}(t)] := C(A, \tilde{A}, t), \quad (47)$$

where we use the convention $C(X, Y, t) = \text{tr}(\rho_\beta X Y(t))$. Experimentally, this reduces to estimating the modified quench functions

$$Q_{\rho_\beta, A}(e^{iHt} B e^{-iHt} U_{\text{Im}}), \quad Q_{\rho_A, A}(e^{iHt} B e^{-iHt} U_{\text{Re}}), \quad Q_{\rho_S, A}(e^{iHt} B e^{-iHt} U_{\text{Re}}). \quad (48)$$

The full quantum circuit is shown in Fig. 3.

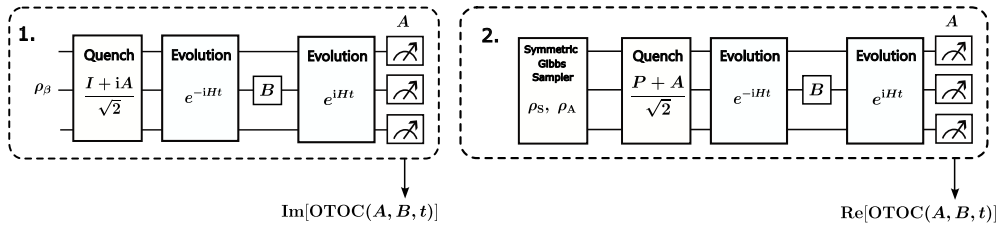


Figure 3: Quantum circuits for estimating $\text{OTOC}(A, B, t)$ using tailored quench spectroscopy.

4 Complexity and error analysis

4.1 Complexity analysis

For a fixed time t , estimating $C(A, B, t)$ (or its real/imaginary parts) reduces to estimating a constant number of quench functions. Therefore, to achieve additive error ε with constant

success probability, the number of shots per time point scales as $\mathcal{O}(\varepsilon^{-2})$ by standard concentration bounds.

To reconstruct the Green's function, we first estimate the active transition frequencies from the measured time-domain correlator. Suppose the correlator contains χ distinct transition frequencies $\{\Omega_k\}_{k=1}^\chi$ with nonzero spectral weights. Define their minimum separation by

$$\Delta_{\text{spec}} = \min_{k \neq l} |\Omega_k - \Omega_l|. \quad (49)$$

For uniform sampling with interval δt , the dimensionless frequencies are $f_k = \Omega_k \delta t \pmod{2\pi}$. The sampling interval must be chosen to avoid aliasing of the relevant frequency range.

In the standard separation-limited regime, a sufficiently long observation window scales as

$$T_{\text{max}} = O(\Delta_{\text{spec}}^{-1}), \quad (50)$$

under suitable separation and noise assumptions. The corresponding number of sampling times is

$$M = O\left(\frac{1}{\Delta_{\text{spec}} \delta t}\right). \quad (51)$$

Estimating each correlator to additive accuracy ε with bounded measurement outcomes requires $O(\varepsilon^{-2} \log(M/\delta))$ circuit executions per time point to achieve simultaneous accuracy with probability at least $1 - \delta$. Consequently, the total number of circuit executions is

$$O\left(\frac{\varepsilon^{-2}}{\Delta_{\text{spec}} \delta t} \log \frac{M}{\delta}\right). \quad (52)$$

This is a correlator-sampling cost; guaranteeing a specified frequency-estimation accuracy requires additional signal-conditioning assumptions.

For thermal correlators, the active frequencies are Bohr frequencies $E_m - E_n$ with nonzero spectral weights. Their minimum separation is generally distinct from the smallest nonzero energy-level gap.

4.2 Error analysis

In the nonequilibrium setting, an error in the prepared initial state translates directly into an error in the correlator: since $C(A, B, t)$ is linear in ρ , the deviation is bounded by the trace distance $\|\rho - \rho_{\text{ideal}}\|_1$ up to an operator-norm factor. In the equilibrium setting, we can further suppress symmetry-breaking preparation errors by applying a *symmetrization channel*

$$\mathcal{S}(\rho) = \frac{\rho + P\rho P}{2}, \quad (53)$$

which removes coherences between opposite parity sectors.

To illustrate the effect, suppose the ideal initial state is the ground state $|0\rangle$ and the actual prepared state is

$$|\psi\rangle = \sqrt{1 - |\mu|^2} |0\rangle + \mu |1\rangle, \quad |\mu| \ll 1, \quad (54)$$

where $|1\rangle$ has opposite parity to $|0\rangle$. Consider the imaginary component $\text{Im}[C(A, B, t)]$ with inaccurate input state $|\psi\rangle$, which can be expanded as

$$Q_{|\psi\rangle, B}(e^{-iHt} e^{i\pi A/4}) = \frac{1}{2} \langle \psi | B(t) | \psi \rangle + \frac{1}{2} \langle \psi | AB(t) A | \psi \rangle - \frac{1}{2} \langle \psi | i[A, B(t)] | \psi \rangle. \quad (55)$$

Note that $i[A, B(t)]$ is parity-even (symmetry-preserving), whereas $B(t)$ and $AB(t)A$ are parity-odd (symmetry-breaking) under the assumptions $\{A, P\} = \{B, P\} = 0$. Hence, the error between the ideal case and the actual case is

$$\begin{aligned} & Q_{|\psi\rangle, B}(e^{-iHt}e^{i\pi A/4}) - Q_{|0\rangle, B}(e^{-iHt}e^{i\pi A/4}) \\ &= \sqrt{1 - |\mu|^2}(\text{Re}[\mu^* \langle 1|B(t)|0\rangle] + \text{Re}[\mu^* \langle 1|AB(t)A|0\rangle]) \\ &\quad - \frac{|\mu|^2}{2} \langle 1|i[A, B(t)]|1\rangle + \frac{|\mu|^2}{2} \langle 0|i[A, B(t)]|0\rangle. \end{aligned} \quad (56)$$

The leading contribution is $\mathcal{O}(|\mu|)$.

After applying symmetrization, the initial state becomes the incoherent mixture

$$\mathcal{S}(|\psi\rangle\langle\psi|) = (1 - |\mu|^2)|0\rangle\langle 0| + |\mu|^2|1\rangle\langle 1|, \quad (57)$$

which removes the off-diagonal coherence terms responsible for the $\mathcal{O}(|\mu|)$ contribution. As a result, the leading error is reduced to $\mathcal{O}(|\mu|^2)$. The analysis for the real component $\text{Re}[C(A, B, t)]$ is similar.

5 Numerical simulation

In this section, we present numerical experiments validating the TQS protocol and its robustness to state-preparation errors. To simulate real-time dynamics without ancillas, we use a first-order Trotter–Suzuki (TS) product formula [9, 10, 53]. For a time-independent Hamiltonian $H = \sum_{\gamma=1}^{\Gamma} H_{\gamma}$, where each H_{γ} is a sum of mutually commuting Pauli terms, we approximate the time-evolution operator by

$$U_{\text{TS}}(t) := \left[\prod_{\gamma=1}^{\Gamma} \exp(-iH_{\gamma} t/N_T) \right]^{N_T}, \quad (58)$$

where N_T is the number of Trotter steps (circuit layers) and directly controls circuit depth. For fixed t , the first-order TS error typically decreases as $\mathcal{O}(\|H\|t^2/N_T)$. Higher-order product formulas can further improve the asymptotic scaling in t and N_T [9, 10]. In our setting, if each term H_{γ} respects parity, i.e., $[H_{\gamma}, P] = 0$, then $U_{\text{TS}}(t)$ also commutes with P ; consequently, the simulated dynamics preserves parity and the symmetry-based conclusions of the previous sections continue to apply.

Because the initial state is the ground state, both the eigenstate condition and the low-energy state condition help improve the error scaling [47, 63]. Let $U_{\text{TS}}(t)$ denote the TS simulation operator with Trotter steps N_T . Define

$$C_{\text{TS}}(A, B, t) := \langle 0|AU_{\text{TS}}^{\dagger}(t)BU_{\text{TS}}(t)|0\rangle. \quad (59)$$

Therefore,

$$\begin{aligned} |C_{\text{TS}}(A, B, t) - C(A, B, t)| &\leq |\langle 0|AU^{\dagger}(t)BU(t)|0\rangle - \langle 0|AU_{\text{TS}}^{\dagger}(t)BU_{\text{TS}}(t)|0\rangle| \\ &\leq |\langle 0|AU^{\dagger}(t)BU(t)|0\rangle - \langle 0|AU^{\dagger}(t)BU_{\text{TS}}(t)|0\rangle| \\ &\quad + |\langle 0|AU^{\dagger}(t)BU_{\text{TS}}(t)|0\rangle - \langle 0|AU_{\text{TS}}^{\dagger}(t)BU_{\text{TS}}(t)|0\rangle| \\ &\leq \|B\|(\varepsilon_0(t) + \varepsilon_1(t)) = \varepsilon_0(t) + \varepsilon_1(t), \end{aligned} \quad (60)$$

where

$$\varepsilon_0(t) := \|(U_{\text{TS}}(t) - U(t))|0\rangle\|, \quad \varepsilon_1(t) := \|(U_{\text{TS}}(t) - U(t))A|0\rangle\|. \quad (61)$$

Since $|0\rangle$ is an eigenstate, we can use the result of Ref. [63]. In many cases including the two-layer scenario $H = H_{\gamma_1} + H_{\gamma_2}$, we have the condition $\sum_{\gamma > \gamma'} \langle 0 | [H_\gamma, H_{\gamma'}] | 0 \rangle = 0$, which leads to

$$\varepsilon_0(t) = O(t^3/N_T^2) + O(t/N_T), \quad (62)$$

Compared with the worst-case error bound, there is an improvement of order $O(t/N_T)$. Furthermore, suppose $A|0\rangle$ is supported in a low-energy subspace, Ref. [47] gives a constant improvement in the error bound of $\varepsilon_1(t)$.

To model imperfect initial-state preparation, we generate a random mixed-state perturbation as follows. We first draw a random traceless Hermitian matrix E by sampling independent Gaussian entries (and enforcing Hermiticity), and normalize it to $\tilde{E}$ such that $\|\tilde{E}\|_1 = 1$. We then define a full-rank state $\varrho := e^{\tilde{E}}/\text{tr}(e^{\tilde{E}})$. Given a noise strength $\varepsilon \in [0, 1]$, we perturb the target initial state ρ by forming $\rho + \varepsilon\varrho$ and renormalizing to trace one, yielding $\tilde{\rho}$. This construction produces a controlled trace-distance deviation, with $\|\rho - \tilde{\rho}\|_1$ typically of order ε .

For the Green's function demonstrations, we consider the Fermi–Hubbard model (FHM) on a 2×3 lattice with open boundary conditions, as depicted in Fig. 4. Lattice sites are labeled by (x, y) with $x \in \{0, 1\}$ and $y \in \{0, 1, 2\}$, giving 12 spin-orbitals, which we index according to

$$(x, y, \uparrow) \mapsto 6x + 2y, \quad (x, y, \downarrow) \mapsto 6x + 2y + 1. \quad (63)$$

In units of the hopping amplitude, the Hamiltonian reads

$$H_{\text{FH}} = - \sum_{\langle p, q \rangle} \sum_{\sigma \in \{\uparrow, \downarrow\}} \left(c_{p\sigma}^\dagger c_{q\sigma} + c_{q\sigma}^\dagger c_{p\sigma} \right) + h_U \sum_p n_{p\uparrow} n_{p\downarrow}, \quad (64)$$

where $\langle p, q \rangle$ runs over nearest-neighbor pairs of sites and h_U denotes the on-site interaction strength.

We work with the rescaled Hamiltonian $H := \gamma H_{\text{FH}}$ and decompose it for TS simulation as $H = \sum_{k=1}^4 H_{\gamma_k}$, where

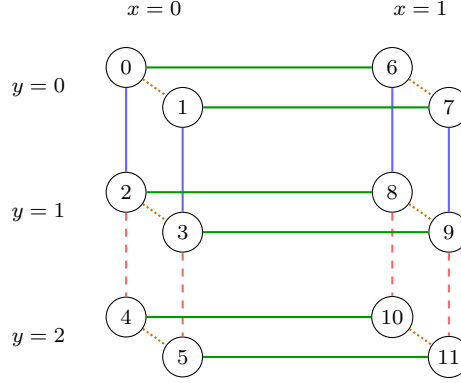
$$\begin{aligned} H_{\gamma_1} &= -\gamma \sum_{j \in \{0, 1, 6, 7\}} \left(c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j \right), \\ H_{\gamma_2} &= -\gamma \sum_{j \in \{2, 3, 8, 9\}} \left(c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j \right), \\ H_{\gamma_3} &= -\gamma \sum_{j=0}^5 \left(c_j^\dagger c_{j+6} + c_{j+6}^\dagger c_j \right), \\ H_{\gamma_4} &= \gamma h_U \sum_{j=0}^5 n_{2j} n_{2j+1}. \end{aligned} \quad (65)$$

The terms within each layer H_{γ_k} act on disjoint sets of modes (or, for $k = 4$, are mutually commuting number operators). Hence each $e^{-itH_{\gamma_k}}$ factorizes exactly into a product of local unitaries.

The rescaling factor is chosen as

$$\gamma := \frac{\pi}{\Lambda}, \quad \Lambda \geq \|H_{\text{FH}}\|, \quad (66)$$

where $\Lambda = \|H_{\text{FH}}\|$ if the spectral norm is known exactly. Otherwise Λ is any efficiently computable upper bound, e.g., one obtained from the triangle inequality applied to the terms of H_{FH} . This choice guarantees $\|H\| \leq \pi$, i.e., $\text{spec}(H) \subseteq [-\pi, \pi]$. Consequently,



H_{γ_1} : solid blue bonds; H_{γ_2} : dashed red bonds; H_{γ_3} : solid green spin-preserving bonds; H_{γ_4} : dotted orange on-site interactions.

Figure 4: Fermi–Hubbard model on a 2×3 lattice. Each physical site (x, y) contains two spin orbitals, mapped to fermionic mode indices as $(x, y, \uparrow) \mapsto 6x + 2y$ and $(x, y, \downarrow) \mapsto 6x + 2y + 1$. Blue and red bonds describe vertical hopping, green bonds describe horizontal hopping, and orange dotted links represent on-site density–density interactions between opposite spins.

each eigenphase $\theta \in \text{spec}(H)$ is uniquely determined by $e^{i\theta}$, up to the identification of the endpoints $\theta = \pm\pi$. This ambiguity is excluded whenever the bound is strict, $\Lambda > \|H_{\text{FH}}\|$.

After estimating the time-domain correlator $C(A, B, t)$ on a discrete time grid, we reconstruct its frequency content using MUSIC, which builds a Hankel matrix from the sampled signal and decomposes it into a signal subspace and an orthogonal noise subspace. From the noise subspace one forms the MUSIC noise-subspace correlation function:

$$R(\omega) := \|U_{\text{noise}}^\dagger a(\omega)\|_2, \quad (67)$$

where $a(\omega) = [e^{-i\omega n}]_{n=1}^N$ and U_{noise} spans the noise subspace; see Appendix C for more details. In the ideal noiseless case and with sufficient sampling, $U_{\text{noise}}^\dagger a(\omega) = 0$ at the true frequencies. As plotted in our numerical results, $R(\omega)$ therefore provides a direct and visually stable diagnostic of the recovered spectral support and its robustness under imperfections. Compared with directly Fourier transforming finite-time data, the MUSIC noise-subspace correlation function can resolve closely spaced frequencies beyond the naive Fourier limit when the signal is well-approximated by a sparse sum of exponentials.

Once the dominant frequencies $\{\hat{f}_k\}$ are identified from the local minima of $R(\omega)$, one can estimate the corresponding amplitudes $\{\hat{c}_k\}$ by a linear least-squares fit of the model $s_n \approx \sum_{k=1}^K \hat{c}_k e^{-i\hat{f}_k n}$ to the measured time samples. Finally, the many-body Green’s function can be reconstructed by substituting the recovered frequencies and weights into the Lehmann form, i.e., as a sum of simple poles at the transition frequencies with residues determined by the fitted amplitudes.

5.1 Ground states

In Fig. 5 and Fig. 6 we evaluate the ground-state time correlator $C(A, B, t)$ that underlies the corresponding Green’s function and benchmark against both exact evolution and first-order TS simulation. We consider two interaction regimes: a strongly correlated case ($h_U = 6$) and a weakly interacting case ($h_U = 0.1$), with $A = B = c_0 + c_0^\dagger$. Across both regimes, the reconstructed real and imaginary parts of the correlator remain stable under moderate initial-state preparation error and under Trotterization at the chosen step size. Moreover, the noise-subspace correlation function $R(\omega)$ computed from the noisy data

closely matches its exact counterpart, indicating that the signal-processing stage is robust to the state-preparation perturbations and Trotter errors considered here.

5.2 Finite-temperature thermal states

Figures 7 and 8 extend the same analysis to finite temperature with $\beta = 1$. In contrast to the ground-state setting, the correlator receives contributions from multiple energy differences weighted by thermal populations, which generally increases spectral congestion. Nevertheless, the TQS estimator continues to yield accurate real and imaginary components of $C(A, A, t)$ in both interaction regimes. These results support the practicality of the symmetry-enabled protocol for probing equilibrium dynamics beyond zero temperature, provided that the parity symmetry condition is satisfied and the thermal-state preparation error remains controlled.

5.3 Finite-temperature OTOCs

Figure 9 illustrates the extension of TQS to out-of-time-order correlators (OTOCs) for the one-dimensional XXZ chain defined in Eq. (8). We set $J_X = 1$, $J_Z = 2$, and $h = 1$, corresponding to an anisotropy parameter $\Delta = J_Z/J_X = 2 > 1$ in the presence of a uniform longitudinal field. At zero field, a chain with $\Delta > 1$ lies in the gapped Néel phase. A longitudinal field exceeding the lower critical field h_{c1} , however, closes the gap and drives the system into a partially magnetized, gapless Luttinger liquid phase, which persists up to the saturation field h_{c2} [49]. For the parameters above, h lies well within this window ($h_{c1} < h < h_{c2}$). Because the longitudinal field commutes with the total magnetization $\sum_i S_i^z$, the model remains Bethe-ansatz integrable. Its dynamics are nevertheless genuinely interacting and nontrivial.

This model is chosen because gapless systems are generally demanding to simulate classically. Their nonequilibrium dynamics generate rapidly growing entanglement, which limits tensor-network methods. This makes the model a suitable benchmark for demonstrating the potential advantage of quantum hardware should the protocol be implemented experimentally. It also provides a representative setting in which to assess the robustness of OTOC estimation against Trotterization errors and imperfections in state preparation.

6 Conclusion and outlook

We introduced a symmetry-enabled direct quantum protocol, tailored quench spectroscopy (TQS), for computing both the real and imaginary parts of two-point time correlators without controlled unitaries or ancilla qubits in the correlator-estimation circuit. The central observation is that when a parity operator P commutes with the Hamiltonian and the observables of interest are parity-odd, symmetry enforces selection rules that simplify quench spectroscopy readout. For fixed-parity initial states (including energy eigenstates) the correlator can be obtained from only two quench functions, and for thermal states we showed how to combine parity-resolved correlators obtained from symmetric and antisymmetric Gibbs states. Together with classical signal processing (e.g., MUSIC), these correlators yield the Lehmann representation of the Green's function. Our error analysis highlights that symmetry-based symmetrization can quadratically suppress parity-breaking state-preparation errors. Numerical simulations on Fermi–Hubbard and XXZ instances support the robustness of the approach under Trotterization and imperfect state preparation. Finally, we demonstrated that the same symmetry-based measurement primitive extends to a class of multi-point correlators, including OTOCs.

This work opens several promising directions. First, it is natural to ask how far symmetry-enabled direct protocols can be extended beyond linear two-point response. One avenue is the measurement of higher-order and nonlinear response functions [41], where parity constraints might reduce the overhead of accessing specific operator orderings or nested commutators. In terms of the symmetric thermal states, other Lindbladians, such as [7, 18], can also be adapted. A related question is whether analogous ideas can be adapted to non-Hermitian [23, 42] or effective open-system settings, e.g., when the dynamics is generated by a non-Hermitian Hamiltonian or by a Lindbladian, where Green's functions and response theory take modified forms and symmetry constraints may still yield simplifications.

Second, a near-term priority is experimental demonstration. Because TQS requires only native time evolution under a time-independent Hamiltonian and end-of-circuit measurements, it is compatible with both digital platforms and analog simulators, where quench-style protocols are natural. A promising first milestone is a hardware demonstration of parity-assisted recovery of $\text{Re}[C(A, B, t)]$ and $\text{Im}[C(A, B, t)]$ in small Hubbard or spin-chain instances, followed by reconstruction of low-lying spectral features via classical signal analysis. Benchmarking the protocol under realistic noise and comparing against other available approaches would clarify the practical advantage of symmetry-enabled direct measurement.

Finally, several technical questions remain open. On the theory side, sharper sample-complexity guarantees for signal reconstruction under realistic noise (including state-preparation and measurement noise) would strengthen the end-to-end performance picture. On the implementation side, developing more efficient and hardware-tailored preparation methods for parity-resolved thermal states, possibly avoiding dissipative overhead in some settings, would broaden the applicability of finite-temperature TQS. We expect that exploring these directions will further establish symmetry-enabled direct protocols as practical tools for probing finite-temperature dynamics of strongly correlated quantum matter.

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Author contributions

Both C.Y. and C.Z. conceived the core idea, discussed the results, and contributed to writing the manuscript. C.Y. performed the numerical simulations. ChatGPT was used to improve the clarity of the writing and assist in identifying mathematical errors.

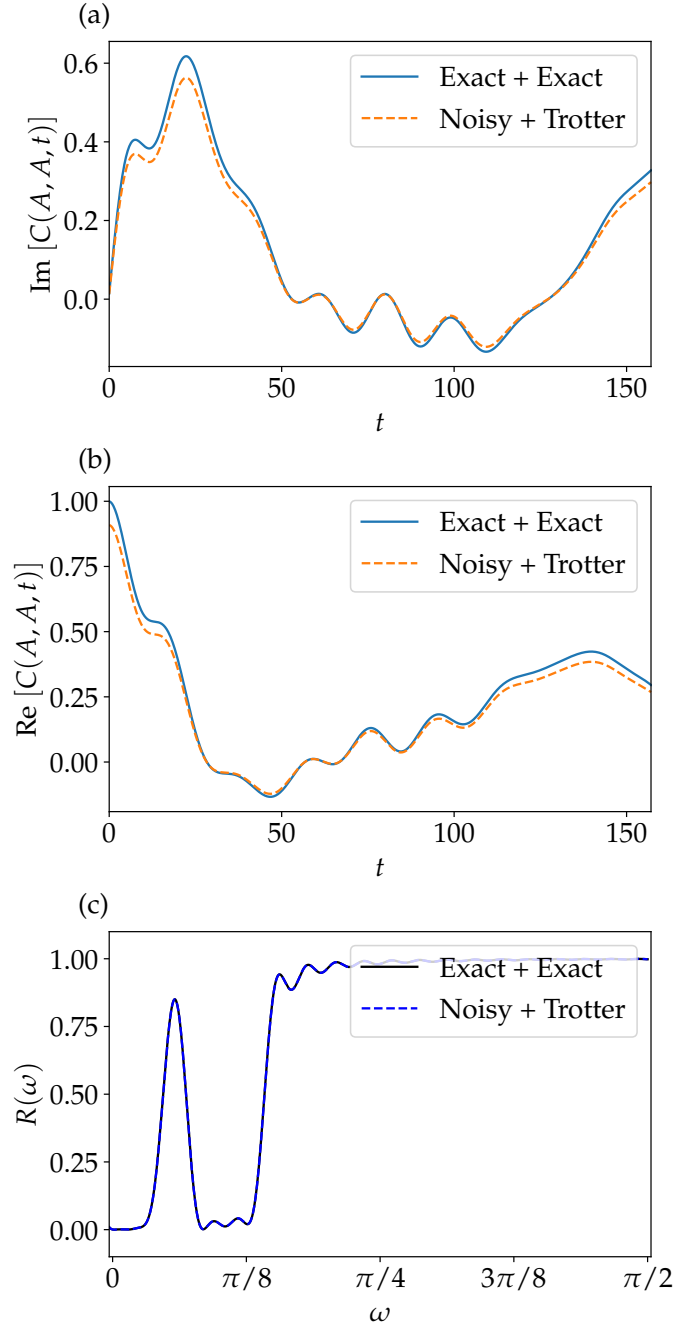


Figure 5: Ground-state numerical simulations for the 2D Fermi-Hubbard model on a 2×3 lattice with open boundary conditions. The interaction strength is $h_U = 6$. The observable is $A = B = c_0 + c_0^\dagger$. The system is evolved up to time $T = 200\pi$ with $N_T = 4 \times 10^3$, so that the Trotter step is $\pi/20$ (only the region $t \in [0, 50\pi]$ is shown). Initial-state preparation error is modeled by perturbing the initial state once as $\tilde{\rho} = (\rho + \varepsilon \varrho) / \text{tr}(\rho + \varepsilon \varrho)$ with $\varepsilon = 0.1$ and a fixed random state ϱ . In the legend, the first “Exact” denotes an exact initial state, the second “Exact” denotes exact time evolution; “Noisy” corresponds to using the perturbed initial state $\tilde{\rho}$; and “Trotter” refers to simulation via the TS decomposition. (a) Imaginary part of $C(A, B, t)$ as a function of time. (b) Real part of $C(A, B, t)$. (c) Noise-subspace correlation function $R(\omega)$ for $\omega \in [0, \pi/2]$, constructed from the sampled time signal.

Data availability

The data and code supporting the findings of this work are available at [62].

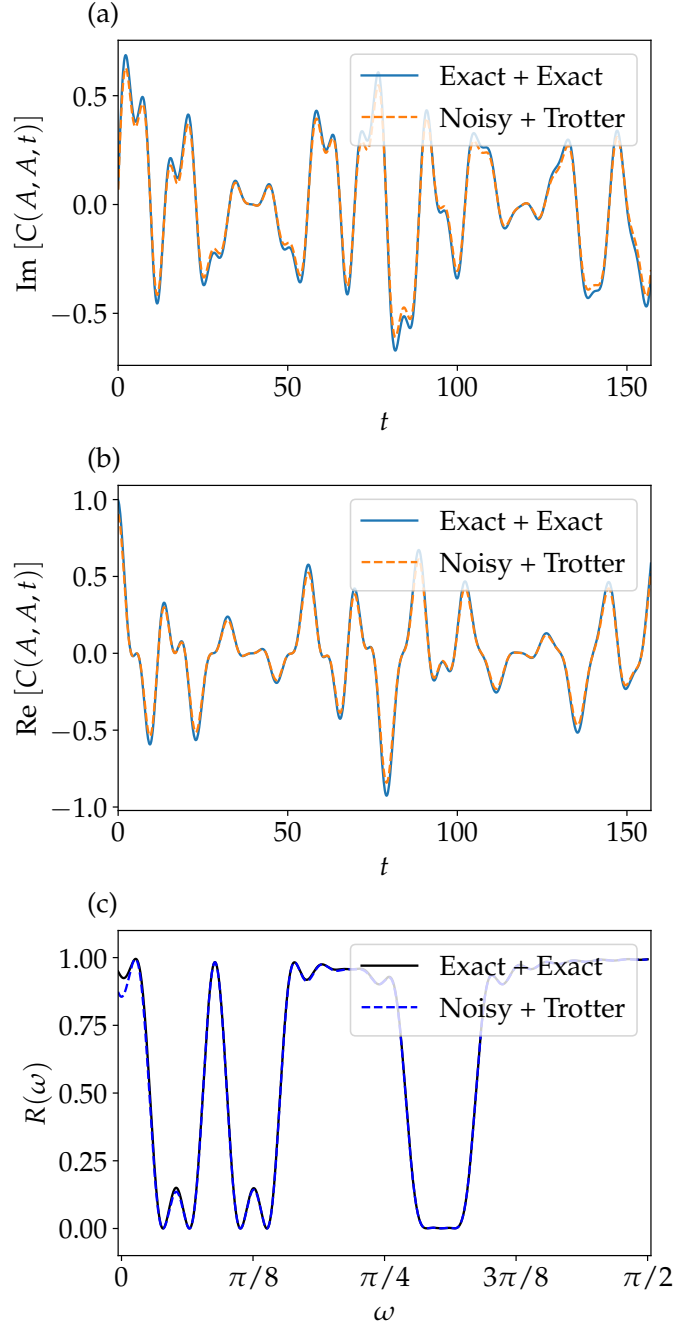


Figure 6: Ground-state numerical simulations for the 2D Fermi-Hubbard model on a 2×3 lattice with open boundary conditions. The setup and the meaning of the subplots are the same as Fig. 5, except the interaction strength is $h_U = 0.1$.

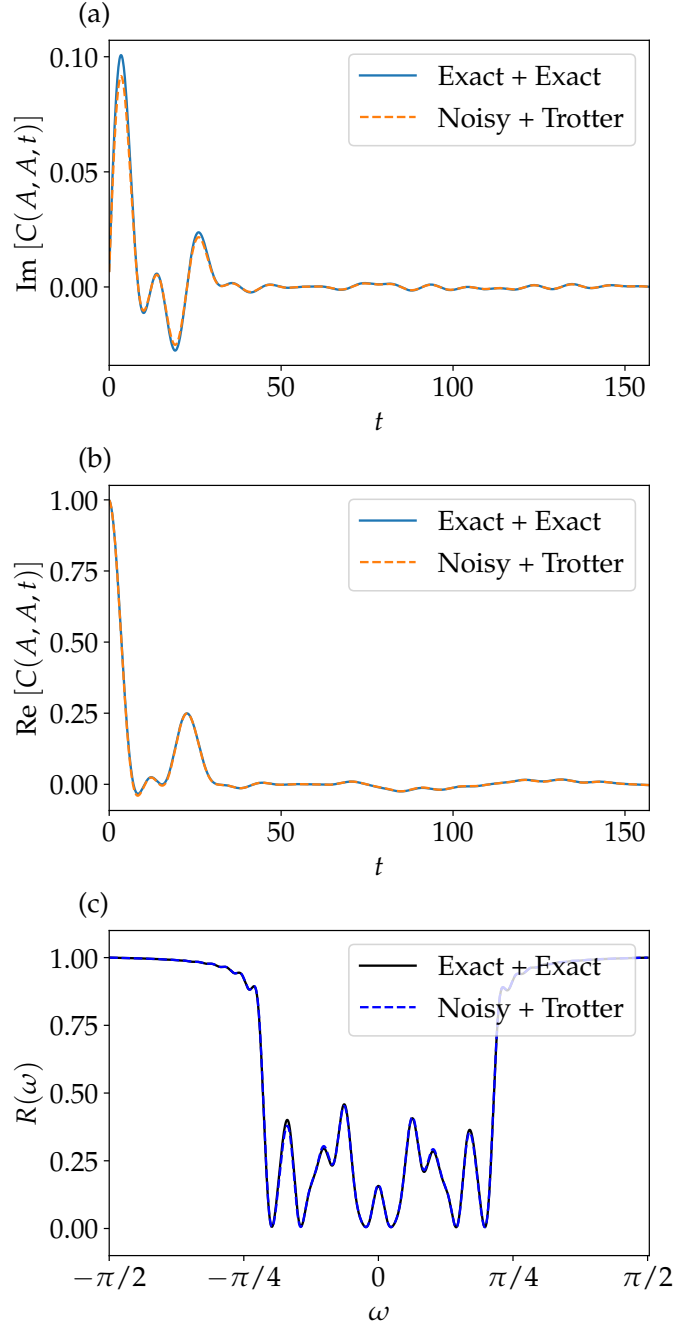


Figure 7: Finite-temperature numerical simulations for the 2D Fermi-Hubbard model on a 2×3 lattice with open boundary conditions. The setup and the meaning of the subplots are the same as Fig. 5, except the initial state is the thermal state ρ_β with $\beta = 1$ and the interaction strength is $h_U = 6$.

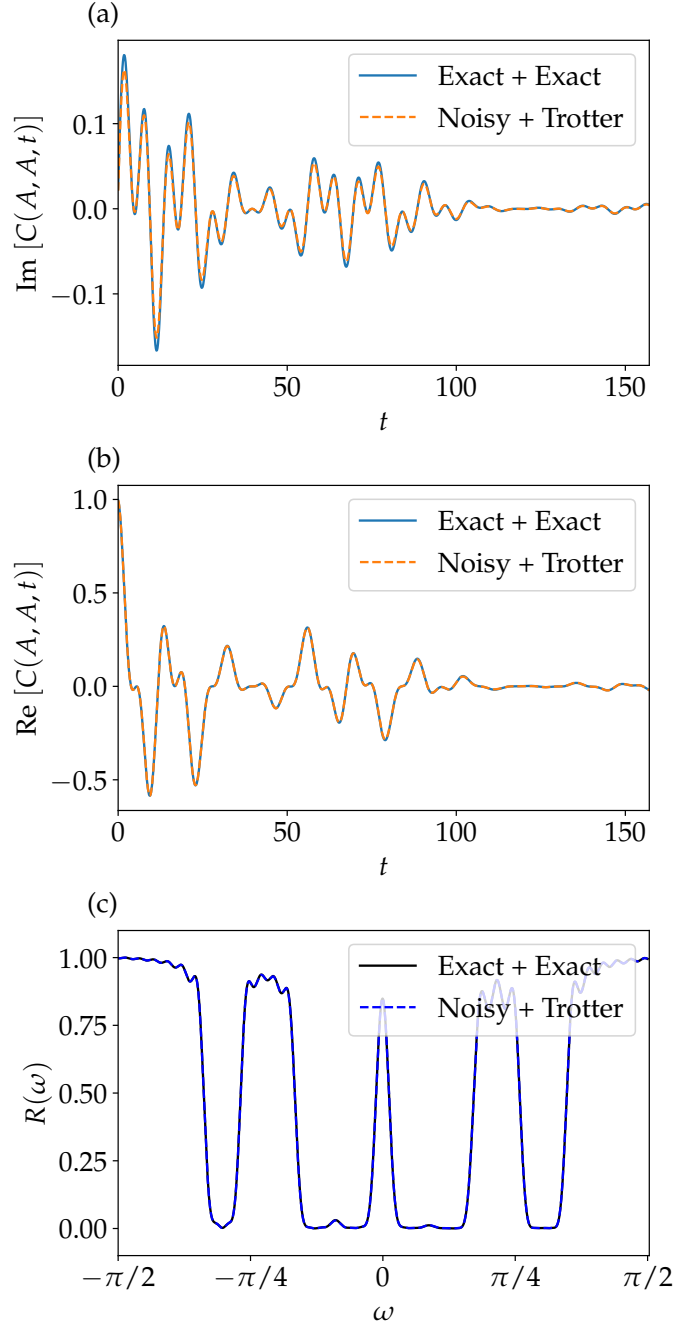


Figure 8: Finite-temperature numerical simulations for the 2D Fermi-Hubbard model on a 2×3 lattice with open boundary conditions. The setup and the meaning of the subplots are the same as Fig. 5, except the interaction strength is $h_U = 0.1$.

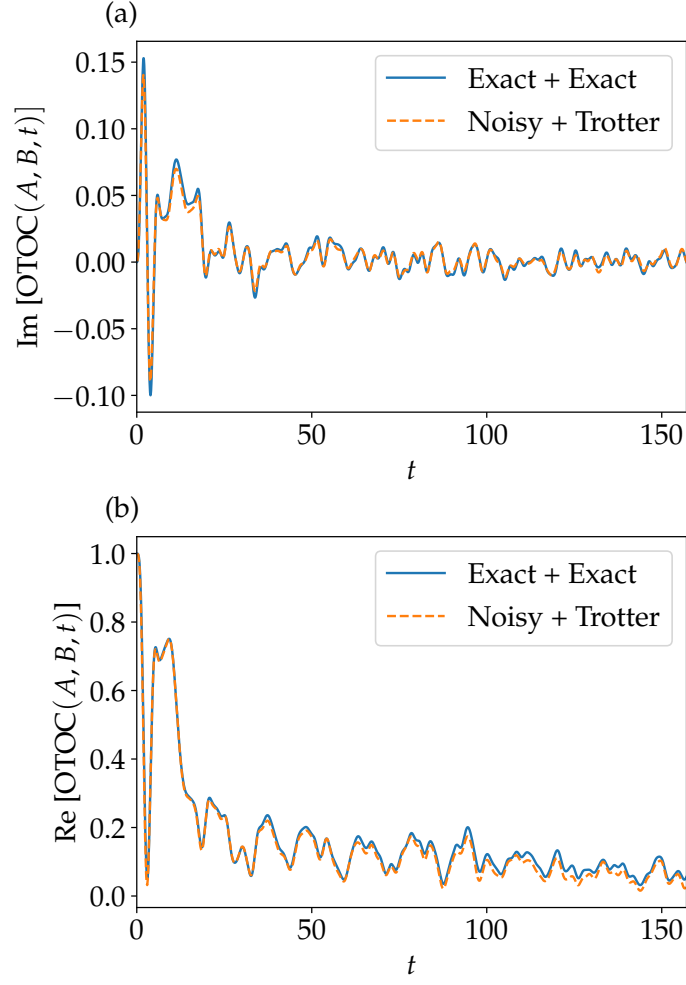


Figure 9: Real and imaginary components of the OTOC for the 1D XXZ model [see Eq. (8)] on $N = 8$ sites with open boundary conditions. The parameters are $J_X = 1$, $J_Z = 2$, and $\hbar = 1$. The local observables are $A = X_0$ and $B = Z_2$. The initial state is the thermal state ρ_β with $\beta = 1$. The remaining setup and the meaning of the subplots are similar to those in Fig. 5.

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A Supplementary background

A.1 Many-Body Green's Function

In this appendix we give an overview of basic facts about many-body Green's function in condensed matter physics. Suppose H is a Hamiltonian with a non-degenerate ground state and has composition $H = \sum_{n=0}^{d-1} E_n |n\rangle\langle n|$. Let O_a, O_b be two linear operators. The many-body Green's function at zero-temperature is defined as:

$$G(z) := \langle 0 | O_a (z - H + E_0)^{-1} O_b | 0 \rangle = \sum_n \frac{\langle 0 | O_a | n \rangle \langle n | O_b | 0 \rangle}{z - E_n + E_0}, \quad \text{Im}(z) > 0. \quad (1)$$

Note that for all ζ with $\text{Re}(\zeta) > 0$, we have $\int_0^\infty e^{-\zeta t} dt = \zeta^{-1}$. Eigenstate decomposition yields

$$C(O_a, O_b, -t) = \sum_n \langle 0 | O_a | n \rangle \langle n | e^{-iHt} O_b e^{iHt} | 0 \rangle = \sum_n \langle 0 | O_a | n \rangle \langle n | O_b | 0 \rangle e^{i(E_0 - E_n)t}. \quad (2)$$

Note that for all z with positive imaginary component, we have

$$\int_0^\infty e^{izt} e^{i(E_0 - E_n)t} dt = \frac{i}{z - E_n + E_0}. \quad (3)$$

Accordingly,

$$G(z) = -i \int_0^\infty e^{izt} C(O_a, O_b, -t) dt. \quad (4)$$

This confirms Eq. (4) in the main text. The purpose of this transformation is to express the Green's function, which is defined in terms of the resolvent $(z - H + E_0)^{-1}$, as the Laplace/Fourier transform of a real-time correlation function. This representation is useful because the time-dependent correlation function can be obtained directly from the real-time evolution generated by H , providing a practical route to estimating the Green's function without explicitly implementing the resolvent.

Let $\Delta_n = E_n - E_0$ and define

$$w_n = \langle 0 | O_a | n \rangle \langle n | O_b | 0 \rangle. \quad (5)$$

In general, w_n may be complex. For $z = \omega + i\eta$, with $\omega \in \mathbb{R}$ and $\eta > 0$, we have

$$G(\omega + i\eta) = \sum_n w_n \frac{\omega - \Delta_n - i\eta}{(\omega - \Delta_n)^2 + \eta^2}. \quad (6)$$

Define the broadened spectral function

$$S(\omega, \eta) := -2 \text{Im} G(\omega + i\eta). \quad (7)$$

Taking the imaginary part gives

$$S(\omega, \eta) = 2 \sum_n \frac{\eta \text{Re} w_n - (\omega - \Delta_n) \text{Im} w_n}{(\omega - \Delta_n)^2 + \eta^2}. \quad (8)$$

For any z with $\text{Im} z > 0$, the Cauchy transform of S satisfies

$$\int_{-\infty}^\infty \frac{d\omega}{2\pi} \frac{S(\omega, \eta)}{z - \omega} = \sum_n \frac{w_n}{z - \Delta_n + i\eta} = G(z + i\eta). \quad (9)$$

Consequently, the Green's function is recovered in the limit

$$G(z) = \lim_{\eta \rightarrow 0^+} \int_{-\infty}^\infty \frac{d\omega}{2\pi} \frac{S(\omega, \eta)}{z - \omega}, \quad \text{Im} z > 0. \quad (10)$$

A.2 Jordan–Wigner Transformation

Jordan–Wigner transformation maps fermionic creation and annihilation operators to Pauli strings on qubits. For a fermionic lattice with N sites and two spin species, there are $J = 2N$ spin-orbitals, which we index by $j \in \{0, 1, \dots, J-1\}$. The standard Jordan–Wigner mapping is

$$c_j = \left(\prod_{n=0}^{j-1} Z_n \right) \sigma_j^-, \quad \sigma_j^\pm = \frac{X_j \pm iY_j}{2}. \quad (11)$$

From this mapping one obtains the identities

$$c_j c_j^\dagger = \frac{1 - Z_j}{2}, \quad n_j := c_j^\dagger c_j = \frac{1 + Z_j}{2}, \quad (12)$$

and, for nearest neighbors,

$$c_j^\dagger c_{j+1} = -\sigma_j^+ \sigma_{j+1}^-, \quad c_{j+1}^\dagger c_j = -\sigma_{j+1}^+ \sigma_j^-, \quad (13)$$

which imply the familiar hopping relation

$$c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j = -\frac{X_j X_{j+1} + Y_j Y_{j+1}}{2}. \quad (14)$$

Finally, note that fermionic bilinears do not commute in general: for example,

$$[c_i^\dagger c_j, c_j^\dagger c_k] = c_i^\dagger c_k \neq 0 \quad (15)$$

whenever the indices overlap.

A.3 Davies generators and Gibbs-state preparation

We briefly review the Davies generator, a canonical Markovian open-system dynamics describing the weak coupling of a quantum system to a thermal bath. Under the usual weak-coupling, Markovian, and secular approximations, the reduced dynamics of the system is generated by a Lindbladian whose jump operators resolve the energy differences of the system Hamiltonian. When the transition rates satisfy the thermal Kubo–Martin–Schwinger (KMS) condition, the Gibbs state is stationary under the resulting dynamics.

Let

$$H = \sum_n E_n |\psi_n\rangle\langle\psi_n| \quad (16)$$

be the system Hamiltonian. For an arbitrary linear operator J , define its Heisenberg evolution $J(t) = e^{iHt} J e^{-iHt}$. The operator can be decomposed into its Bohr-frequency components as

$$J(t) = \sum_{\nu \in \mathcal{D}(H)} J_\nu e^{i\nu t}, \quad (17)$$

where

$$\mathcal{D}(H) := \{E_m - E_n : m, n\} \quad (18)$$

is the set of Bohr frequencies and

$$J_\nu = \sum_{m,n: E_m - E_n = \nu} \langle\psi_m|J|\psi_n\rangle |\psi_m\rangle\langle\psi_n|. \quad (19)$$

In particular, when J is Hermitian, $J_\nu^\dagger = J_{-\nu}$, these components satisfy

$$[H, J_\nu] = \nu J_\nu, \quad (J_\nu)^\dagger = (J^\dagger)_{-\nu}. \quad (20)$$

Equivalently, J_ν maps an energy eigenspace of energy E into one of energy $E + \nu$.

For simplicity, consider a single Hermitian system coupling operator J . Given non-negative transition rates $\eta(\nu) \geq 0$, the dissipative part of the Davies generator takes the form

$$\mathcal{L}[\rho] = \sum_{\nu \in \mathcal{D}(H)} \eta(\nu) \left(J_\nu \rho J_\nu^\dagger - \frac{1}{2} \{J_\nu^\dagger J_\nu, \rho\} \right). \quad (21)$$

More generally, one may introduce a family of coupling operators $\{J_a\}_a$ and corresponding rates $\eta_a(\nu)$, in which case the generator is obtained by summing over both a and ν . The full Davies generator may also contain a Lamb-shift Hamiltonian term $-i[H_{\text{LS}}, \rho]$, where H_{LS} commutes with H ; since this term does not affect the thermal fixed-point structure discussed below, we omit it here.

The adjoint generator in the Heisenberg picture acts as

$$\mathcal{L}^\dagger[O] = \sum_{\nu \in \mathcal{D}(H)} \eta(\nu) \left(J_\nu^\dagger O J_\nu - \frac{1}{2} \{J_\nu^\dagger J_\nu, O\} \right). \quad (22)$$

By construction, the Davies generator is covariant with respect to the Hamiltonian evolution: its dissipative terms do not mix different Bohr frequencies. This frequency resolution is a consequence of the secular approximation and is one of the characteristic structural properties of Davies dynamics.

Let ρ_β denote the Gibbs state at inverse temperature β . The Bohr-frequency operators obey the modular relations

$$\rho_\beta J_\nu = e^{-\beta\nu} J_\nu \rho_\beta, \quad \rho_\beta^{1/2} J_\nu = e^{-\beta\nu/2} J_\nu \rho_\beta^{1/2}. \quad (23)$$

Thermal equilibrium of the bath imposes the KMS relation on the transition rates,

$$\eta(-\nu) = e^{\beta\nu} \eta(\nu), \quad \nu \in \mathcal{D}(H). \quad (24)$$

Thus transitions that increase and decrease the system energy occur with relative weights determined by the Boltzmann factor.

Under this condition, the Davies generator satisfies KMS detailed balance with respect to ρ_β . In the present convention, this can be written as

$$\mathcal{L}[X] = \rho_\beta^{1/2} \mathcal{L}^\dagger \left[\rho_\beta^{-1/2} X \rho_\beta^{-1/2} \right] \rho_\beta^{1/2}. \quad (25)$$

In particular,

$$\mathcal{L}[\rho_\beta] = 0, \quad (26)$$

so the Gibbs state is a stationary state of the dynamics. If the jump operators connect all relevant energy and symmetry sectors, so that the Davies semigroup is primitive, ρ_β is moreover the unique full-rank stationary state. Without such an ergodicity condition, additional stationary states may exist.

The Davies generator therefore provides a canonical connection between thermal equilibrium and Markovian quantum dynamics: its jump operators are obtained by resolving physical system–bath couplings into the Bohr frequencies of H , while the KMS relation fixes the relative rates of opposite-frequency transitions and guarantees thermal stationarity.

B Proof of Main Results

B.1 Proof of Lemma 1

Proof. Observe that

$$\langle n|A|m\rangle = -\langle n|PAP|m\rangle = -p_m p_n \langle n|A|m\rangle. \quad (27)$$

Hence, if $p_n p_m = 1$, then $\langle n|A|m\rangle = 0$. Conversely, if $\langle n|A|m\rangle \neq 0$, then $p_n p_m = -1$. This completes the proof of Lemma 1. $\square$

B.2 Proof of Theorem 1

Lemma 1. *Suppose ρ commutes with P , and observables $O_1, \dots, O_L$ all anti-commute with P . If L is odd, then*

$$\text{tr}(\rho \prod_{\ell=1}^L O_\ell) = 0. \quad (28)$$

Proof. Note that

$$P \prod_{\ell=1}^L O_\ell = - \prod_{\ell=1}^L O_\ell P. \quad (29)$$

Hence,

$$\text{tr}(\rho \prod_{\ell=1}^L O_\ell) = \text{tr}(P \rho P \prod_{\ell=1}^L O_\ell) = -\text{tr}(\rho \prod_{\ell=1}^L O_\ell P^2) = -\text{tr}(\rho \prod_{\ell=1}^L O_\ell). \quad (30)$$

$\square$

With the above observation, we present the proof of Theorem 1 as follows.

Proof. By definition, we have

$$Q_{\rho,B}(e^{-iHt}A) = \text{tr}(e^{-iHt}A\rho A e^{iHt}B) = \text{tr}(\rho AB(t)A). \quad (31)$$

Since $\{A, P\} = \{B(t), P\} = [\rho, P] = 0$, in conjunction with Lemma 1, we have $Q_{\rho,B}(e^{-iHt}A) = 0$. Similarly, we have $Q_{\rho,B}(e^{-iHt}P) = Q_{\rho,B}(e^{-iHt}P) = 0$. This confirms that

$$\begin{aligned} Q_{\rho,B}(e^{-iHt}U_{\text{Im}}) &= \text{tr}\left[e^{-iHt}\frac{I+iA}{\sqrt{2}}\rho\frac{I-iA}{\sqrt{2}}e^{iHt}B\right] \\ &= \text{Im}[C(A, B, t)] + \frac{1}{2}Q_{\rho,B}(e^{-iHt}) + \frac{1}{2}Q_{\rho,B}(e^{-iHt}A) \\ &= \text{Im}[C(A, B, t)], \end{aligned} \quad (32)$$

$$\begin{aligned} Q_{\rho,B}(e^{-iHt}U_{\text{Re}}) &= \text{tr}\left[e^{-iHt}\frac{P+A}{\sqrt{2}}\rho\frac{P+A}{\sqrt{2}}e^{iHt}B\right] \\ &= \text{Re}[C(PA, B, t)] + \frac{1}{2}Q_{\rho,B}(e^{-iHt}P) + \frac{1}{2}Q_{\rho,B}(e^{-iHt}A) \\ &= \text{Re}[C(PA, B, t)]. \end{aligned} \quad (33)$$

Because $\rho P = p\rho$ with $p \in \{\pm 1\}$, we further have

$$C(PA, B, t) = \text{tr}(\rho PAB(t)) = p\text{tr}(\rho AB(t)) = pC(A, B, t). \quad (34)$$

This completes the proof of Theorem 1. $\square$

B.3 Proof of Theorem 2

Proof. Recall the decomposition

$$\rho_\beta = \frac{1 + \text{tr}(\rho_\beta P)}{2} \rho_S + \frac{1 - \text{tr}(\rho_\beta P)}{2} \rho_A, \quad (35)$$

where $P\rho_S = \rho_S$ and $P\rho_A = -\rho_A$. By linearity of the correlator in the state, we have

$$C(A, B, t) = \text{tr}(\rho_\beta AB(t)) = \frac{1 + \text{tr}(\rho_\beta P)}{2} \text{tr}(\rho_S AB(t)) + \frac{1 - \text{tr}(\rho_\beta P)}{2} \text{tr}(\rho_A AB(t)). \quad (36)$$

Taking real parts yields

$$\text{Re}[C(A, B, t)] = \frac{1 + \text{tr}(\rho_\beta P)}{2} \text{Re}[\text{tr}(\rho_S AB(t))] + \frac{1 - \text{tr}(\rho_\beta P)}{2} \text{Re}[\text{tr}(\rho_A AB(t))]. \quad (37)$$

Now apply Theorem 1 to each parity sector. Since ρ_S has definite parity $p = +1$ and ρ_A has definite parity $p = -1$, we obtain

$$\text{Re}[\text{tr}(\rho_S AB(t))] = Q_{\rho_S, B}(e^{-iHt} U_{\text{Re}}), \quad \text{Re}[\text{tr}(\rho_A AB(t))] = -Q_{\rho_A, B}(e^{-iHt} U_{\text{Re}}). \quad (38)$$

Substituting into the previous expression gives Eq. (30), completing the proof. $\square$

C Signal analysis algorithm

We use the MULTIPLE SIGNAL CLASSIFICATION (MUSIC) algorithm [34, 48] to reconstruct the frequencies appearing in time-domain correlators.

Consider a signal

$$s_n = \sum_{k=1}^{\chi} c_k e^{-if_k n}, \quad n = 0, \dots, 2N - 2, \quad (39)$$

where $f_k \in [0, 2\pi)$ are distinct frequencies modulo 2π , $c_k \in \mathbb{C} \setminus \{0\}$, and $\chi < N$ is the number of spectral lines.

Construct the Hankel matrix $M \in \mathbb{C}^{N \times N}$,

$$M_{m,n} = s_{m+n}, \quad m, n = 0, \dots, N - 1. \quad (40)$$

Writing

$$V_{m,k} = e^{-if_k m}, \quad D = \text{diag}(c_1, \dots, c_\chi), \quad (41)$$

we obtain the Vandermonde factorization

$$M = V D V^\top, \quad \text{rank}(M) = \chi. \quad (42)$$

Let $M = U \Sigma V_M^\dagger$ be its singular value decomposition, with singular values in non-increasing order. Partition

$$U = [U_{\text{sig}} \quad U_{\text{noise}}], \quad (43)$$

where $U_{\text{noise}} \in \mathbb{C}^{N \times (N-\chi)}$ spans the orthogonal complement of the signal subspace.

Define the normalized steering vector

$$a(\omega) = \frac{1}{\sqrt{N}} [e^{-i\omega n}]_{n=0}^{N-1}. \quad (44)$$

The MUSIC noise-space correlation function is

$$R(\omega) = \|U_{\text{noise}}^\dagger a(\omega)\|_2. \quad (45)$$

In the noiseless case,

$$R(\omega) = 0 \iff \omega \in \{f_1, \dots, f_\chi\} \pmod{2\pi}. \quad (46)$$

We next recall a deterministic stability bound from Ref. [34].

Theorem 1 (MUSIC subspace stability). *Let $\widetilde{M} = M + E$ be the Hankel matrix constructed from noisy signal samples. Denote the nonzero singular values of M by*

$$\sigma_1(M) \geq \dots \geq \sigma_\chi(M) > 0. \quad (47)$$

Suppose

$$\varepsilon_H := \|E\|_2 < \sigma_\chi(M). \quad (48)$$

Let $\widetilde{R}(\omega)$ be the MUSIC noise-space correlation function obtained from the leading χ left singular vectors of $\widetilde{M}$. Then

$$\sup_{\omega \in [0, 2\pi)} |\widetilde{R}(\omega) - R(\omega)| \leq \frac{4\sigma_1(M) + 2\varepsilon_H}{[\sigma_\chi(M) - \varepsilon_H]^2} \varepsilon_H. \quad (49)$$

The bound is deterministic and applies to arbitrary additive errors. Its stability depends on the smallest nonzero singular value of the noiseless Hankel matrix.

For independent, centered sub-Gaussian sample errors with scale parameter σ , standard random-Hankel estimates give, with high probability,

$$\varepsilon_H = O\left(\sigma\sqrt{N \log N}\right). \quad (50)$$

If the frequencies are sufficiently separated and the nonzero amplitudes are bounded above and away from zero so that

$$\sigma_1(M) = O(N), \quad \sigma_\chi(M) = \Omega(N), \quad (51)$$

with constants independent of N , then, in the perturbative regime,

$$\sup_{\omega} |\widetilde{R}(\omega) - R(\omega)| = O\left(\sigma\sqrt{\frac{\log N}{N}}\right). \quad (52)$$

Sufficient quantitative separation conditions are given in Ref. [34].

Importantly, stability of the noise-space correlation does not by itself imply an explicit error bound for the estimated frequencies. Such a bound additionally requires control of the local behavior of $R(\omega)$ near its zeros. Frequency-estimation bounds under further nondegeneracy and separation assumptions are discussed in Ref. [34].

D Computing time correlators via mid-circuit measurements

In this section, we summarize a slightly simplified mid-circuit measurement (MCM) protocol (adapted from [38]) for estimating two-point correlators for general initial states. We include it here for its simplicity and generality. However, unlike our TQS protocol

Algorithm 2 MUSIC (basic form)

Input: Noisy samples $(\tilde{s}_n)_{n=0}^{2N-2}$; model order $\chi < N$.

Output: Frequency estimates $\{\hat{f}_k\}_{k=1}^\chi$.

- 1: Construct the Hankel matrix $\tilde{M}_{m,n} = \tilde{s}_{m+n}$, $m, n = 0, \dots, N-1$.
 - 2: Compute the SVD $\tilde{M} = \tilde{U}\tilde{\Sigma}\tilde{V}^\dagger$.
 - 3: Set $\tilde{U}_{\text{noise}} = [\tilde{u}_{\chi+1}, \dots, \tilde{u}_N]$.
 - 4: Evaluate $\tilde{R}(\omega) = \|\tilde{U}_{\text{noise}}^\dagger a(\omega)\|_2$ on a frequency grid in $[0, 2\pi)$.
 - 5: Return the χ smallest local minima, with optional local refinement.
-

(which uses only end-of-circuit measurements), MCM-based estimators can be particularly sensitive to mid-circuit measurement imperfections such as readout-induced dephasing, crosstalk, and imperfect reset.

Let A and B be Pauli observables with eigenvalues ± 1 . Measuring A corresponds to the projective measurement $\{\Pi_+^A, \Pi_-^A\}$ with

$$\Pi_\pm^A = \frac{I \pm A}{2}. \quad (53)$$

For an initial state ρ , the outcome $o_1 \in \{+1, -1\}$ occurs with probability

$$\Pr(o_1) = \text{tr}(\rho \Pi_{o_1}^A), \quad (54)$$

and the post-measurement state is

$$\rho_{o_1} = \frac{\Pi_{o_1}^A \rho \Pi_{o_1}^A}{\Pr(o_1)}. \quad (55)$$

Conditioned on o_1 , we next measure B using $\Pi_\pm^B = (I \pm B)/2$, obtaining $o_2 \in \{\pm 1\}$. The joint probability is

$$\Pr(o_1, o_2) = \text{tr}(\Pi_{o_2}^B \Pi_{o_1}^A \rho \Pi_{o_1}^A). \quad (56)$$

A direct calculation shows that

$$\mathbb{E}[o_1 o_2] = \sum_{o_1, o_2 \in \{\pm 1\}} o_1 o_2 \Pr(o_1, o_2) = \frac{1}{2} \text{tr}(\rho(AB + BA)) = \text{Re}[\text{tr}(\rho AB)]. \quad (57)$$

Therefore, $\text{Re}[\text{tr}(\rho AB)]$ can be estimated by repeating the two-step measurement and averaging the product $o_1 o_2$ over many shots.