

Universal Dilation of Linear Itô SDEs: Quantum Trajectories and Lindblad Simulation of Second Moments

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We present a universal framework for simulating N -dimensional linear Itô stochastic differential equations (SDEs) on quantum computers with additive or multiplicative noises. Building on a unitary dilation technique, we establish a rigorous mapping from the general linear SDEs

$$dX_t = A(t)X_t dt + \sum_{j=1}^J B_j(t)X_t dW_t^j$$

to stochastic Schrödinger equations (SSE) on a dilated Hilbert space. Crucially, this embedding is pathwise exact in that the classical solution is recovered as a projection of the dilated quantum state for each fixed noise realization. We demonstrate that the resulting SSEs are naturally implementable on digital quantum processors, where the stochastic Wiener increments are encoded directly by preparing the ancillary qubits. Exploiting this physical mapping, we develop two algorithmic strategies: (1) a trajectory-based approach that uses sequential weak measurements to realize efficient stochastic integrators, including a second-order scheme, and (2) an ensemble-based approach that maps moment evolution to a deterministic Lindblad quantum master equation, enabling simulation without Monte Carlo sampling. We provide error bounds based on a stochastic light-cone analysis and validate the framework with numerical experiments.

1 Introduction

Stochastic differential equations (SDEs) are ubiquitous models for dynamical systems subject to fluctuating environments [1]. Their applications span asset pricing in quantitative finance [2], particle dynamics and turbulence in statistical physics [3, 4], continuous-time state estimation via Kalman filters [5, 6], and, more recently, generative modeling in machine learning [7]. In practice, the usefulness of these models relies on efficient numerical integration [8]. As the system dimension N increases, classical methods frequently encounter the curse of dimensionality: the cost is amplified both by the high-dimensional linear algebra and by the need to sample sufficiently many trajectories to resolve statistics of interest. Related stochastic trajectory representations also arise in auxiliary-field

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quantum Monte Carlo: a Hubbard–Stratonovich transformation expresses the interacting imaginary-time propagator as an average over stochastic one-body evolutions in auxiliary fields, with fluctuating walker weights, providing a direct bridge between many-body ground-state calculations and linear stochastic dynamics [9].

Meanwhile, quantum algorithms have made significant progress on deterministic linear dynamics. For linear ODEs and related evolution problems, one can leverage quantum linear systems methods [10] and Hamiltonian simulation techniques [11–15], to design efficient simulation algorithms. More generally, the Schrödingerisation paradigm maps linear ODE systems to time-dependent Schrödinger equations, enabling the use of standard Hamiltonian-simulation primitives [16]. These developments motivate a parallel question: can one obtain an equally native quantum representation for linear *stochastic* dynamics?

A central difficulty is structural. The natural evolution of a quantum system is unitary, or, for Markovian open systems [17], described by Lindblad master equations and their stochastic unravellings, also known as the stochastic Schrödinger equations, where the noise amplitude and dissipation are exactly balanced. General classical SDEs do not inherently satisfy these constraints. In fact, the mismatch is already visible for linear Itô systems: the drift matrix in a classical SDE is typically non-Hermitian without possessing the specific dissipative form required by an SSE.

1.1 Problem setup: linear Itô SDEs

We consider an N -dimensional complex-valued process $X_t \in \mathbb{C}^N$ satisfying the linear Itô SDE

$$dX_t = (A(t)X_t + D(t)) dt + \sum_{j=1}^J (B_j(t)X_t + C_j(t)) dW_t^j, \quad X_0 \in \mathbb{C}^N. \quad (1)$$

Here $\{W_t^j\}_{j=1}^J$ are independent Wiener processes. We assume standard regularity conditions (e.g. Lipschitz continuity and linear growth bounds) guaranteeing existence and uniqueness of a strong solution with finite second moments [1, 8].

Without loss of generality, it suffices to treat the homogeneous case (linear multiplicative noise),

$$dX_t = A(t)X_t dt + \sum_{j=1}^J B_j(t)X_t dW_t^j, \quad (2)$$

since additive terms can be embedded by augmenting the state with an auxiliary variable $X_t^0 \equiv 1$ and lifting (1) to a homogeneous system in dimension $N + 1$.

1.2 Itô SDEs versus stochastic Schrödinger equations

A natural quantum analogue of (2) is the stochastic Schrödinger equation (SSE) describing Markovian quantum trajectories [17]:

$$d|\psi_t\rangle = \left(-iH(t) - \frac{1}{2} \sum_{j=1}^J V_j(t)^\dagger V_j(t)\right) |\psi_t\rangle dt + \sum_{j=1}^J V_j(t) |\psi_t\rangle dW_t^j, \quad (3)$$

where $H(t)$ is Hermitian and $V_j(t)$ are coupling operators. The Itô correction $-\frac{1}{2} \sum_j V_j^\dagger V_j$ is not optional: it enforces the characteristic open-system structure, e.g. norm preservation in expectation for physical unravellings.

Comparing (2) and (3) reveals the obstruction. In general, the drift $A(t)$ in a classical SDE (2) cannot be decomposed into $-iH(t) - \frac{1}{2} \sum_j V_j(t)^\dagger V_j(t)$. Equivalently, Eq. (2) only

has the same structure as Eq. (3) if the Hermitian matrix

$$K(t) := \frac{1}{2} \left(A(t) + A(t)^\dagger + \sum_{j=1}^J B_j(t)^\dagger B_j(t) \right) \quad (4)$$

is zero. Consequently this term quantifies the failure of (2) to be compatible with a standard SSE drift. In general it is neither generically small nor sign-definite, and therefore prevents a direct identification of general linear SDEs with quantum stochastic models.

1.3 A dilation viewpoint: compiling linear SDEs into open quantum dynamics

In this work, we resolve the structural mismatch by treating Eq. (2) as a *template* for linear stochastic dynamics rather than as a physical model. The key idea is a unitary *moment-matching dilation*: we embed the system into a larger Hilbert space $\mathcal{H}_{\text{anc}} \otimes \mathcal{H}_{\text{sys}}$ and construct a dilated SSE whose coefficients are chosen so that the original classical solution is recovered by a fixed projection of the dilated trajectory, *pathwise* for each noise realization.

This work should be viewed as a stochastic extension of the moment-matching dilation framework introduced in the earlier work [18]. There, a general linear ODE was embedded into a unitary evolution on an enlarged Hilbert space, allowing the original non-unitary dynamics to be recovered through a fixed projection. The present paper extends this paradigm from deterministic linear systems to linear Itô SDEs by constructing a dilation into a stochastic Schrödinger equation (SSE), and consequently into a Lindblad evolution for second moments.

This dilation viewpoint has two immediate consequences that shape the rest of the paper. First, it converts generic linear stochastic dynamics Eq. (2) into native primitives of open quantum systems: *quantum trajectories*, which can be simulated via repeated interactions and measurements, and *ensemble* evolution, governed by Lindblad dynamics for second moments. Second, it enables a finite-ancilla implementation with provable control via a light-cone property, supporting long-time simulation through segment-wise evolution and ancilla refresh, which has recently been constructed in [18] for deterministic problems.

1.4 Our Contribution: Quantum Simulation via Moment-Matching Dilation

We resolve the structural mismatch between general linear Itô SDEs (2) and physical quantum evolutions (3) by embedding (2) into a standard stochastic Schrödinger equation (SSE) on a dilated Hilbert space $\mathcal{H}_{\text{anc}} \otimes \mathcal{H}_{\text{sys}}$. Building on moment-matching dilation for deterministic linear systems [18], we construct a dilated Hamiltonian and coupling operators so that the classical solution is encoded in the dilated trajectory $|\psi_t(\omega)\rangle$ satisfying an SSE system (3) and is recovered *pathwise* by a fixed linear readout:

$$X_t(\omega) = (\langle l | \otimes I) |\psi_t(\omega)\rangle. \quad (5)$$

This embedding enables two complementary quantum simulation routes, aimed at different output tasks.

Algorithm I: second-moment weak simulation. Many target quantities are quadratic, e.g., $\mathbb{E}[X_T^\dagger O X_T] = \text{tr}(O \Sigma_T)$ with $\Sigma_T = \mathbb{E}[X_T X_T^\dagger]$. The dilated second moment $\rho_t = \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$ satisfies a deterministic Lindblad master equation on $\mathcal{H}_{\text{anc}} \otimes \mathcal{H}_{\text{sys}}$. As a quantum channel, the solution ρ_t can be efficiently simulated using existing Lindblad simulation

algorithms. Consequently, quadratic statistics can be estimated by simulating this Lindblad dynamics (without sampling trajectories), followed by a single observable estimation on the final state. To reach long times T , we use segmentation of length $\tau = \Theta(1/K_{\max})$ with $K_{\max}$ being the norm of the matrix in Eq. (4), together with ancilla refresh, via OAA on a window projector, and amplitude tracking through segment-wise growth factors.

Algorithm II: pathwise trajectory simulation. For applications requiring sample paths or expectations of general nonlinear functions, we directly simulate the dilated SSE (3) as a repeated-interaction circuit. In each time step, we *presample* a discrete approximation of the Wiener increment, encode this choice into the ancilla state, and apply a fixed interaction unitary. The output is a single (unnormalized) quantum state proportional to $X_T(\omega)$ for the chosen noise realization. As in the Lindblad route, long-time simulation uses segmentation with ancilla refresh and the non-unitary trajectory scaling is tracked by estimating growth factors from each segment.

Throughout the paper we treat Lindblad simulation and trajectory simulation as black-box primitives. The quantities $C_{\mathcal{L},T}$ and $C_{\text{traj},T}$ denote the cost of implementing the corresponding dilated evolution under any chosen simulation framework. Consequently, our complexity estimates characterize the additional overhead introduced by the dilation, segmentation, refresh, and amplitude-tracking procedures, independently of the particular oracle or block-encoding model used to represent the operators $A(t)$ and $B_j(t)$.

Theorem (Informal complexity: Algorithm I (Lindblad/second moments)). *Let $C_{\mathcal{L},T}$ denote the cost of simulating the dilated Lindblad dynamics for total time T (including segmentation and refresh), and let $K_{\max} = \sup_t \|K(t)\|$ so that the number of segments satisfies $L = \Theta(TK_{\max})$. Let $\Lambda_T := \text{tr}(\Sigma_T)$.*

For segment m , define the segment trace-growth factor

$$g_m := \frac{\text{tr}(\Sigma_{t_{m+1}})}{\text{tr}(\Sigma_{t_m})}, \quad \Gamma := \sum_{m=0}^{L-1} \frac{1}{\sqrt{g_m}}. \quad (6)$$

Then one can estimate $\mu = \text{tr}(\Sigma_T O)$ to additive error ε using total cost scaling as

$$\tilde{\mathcal{O}}\left(C_{\mathcal{L},T} \Gamma \frac{\Lambda_T}{\varepsilon}\right),$$

up to polylogarithmic factors and constant refresh overhead.

Theorem (Informal complexity: Algorithm II (trajectory generation)). *Let $C_{\text{traj},T}$ denote the cost of implementing the presampled weak-integrator trajectory circuit up to time T , with $L = \Theta(TK_{\max})$ segments of length $\tau = \Theta(1/K_{\max})$. Let $\mathcal{M}_m^{(\tau)}$ denote the (random, presampled) linear segment map on $[t_m, t_{m+1}]$ acting on the system state.*

Define the segment growth factor as the ratio of squared amplitudes,

$$g_m := \frac{\|\mathcal{M}_m^{(\tau)} |\psi_{t_m}\rangle\|^2}{\| |\psi_{t_m}\rangle \|^2}, \quad \Gamma_1 := \sum_{m=0}^{L-1} \frac{1}{\sqrt{g_m}}, \quad \Gamma_2 := \sum_{m=0}^{L-1} \frac{1}{g_m}. \quad (7)$$

Then the algorithm outputs a single trajectory state proportional to $X_T(\omega)$ for a presampled noise realization ω , together with estimates of its amplitude, with overall cost scaling as

$$\tilde{\mathcal{O}}\left(C_{\text{traj},T} \left[\Gamma_1 + \frac{L}{\varepsilon}\right] C_{\text{traj},T} \left[\Gamma_1 + \frac{L}{\varepsilon}\right]\right).$$

1.5 Related works.

Deterministic linear dynamics. Quantum algorithms for deterministic linear ODE/PDE systems are by now well developed, typically reducing time propagation to block-encodings and Hamiltonian simulation primitives (via LCU/QSVT), or to quantum linear-systems subroutines in time-discretized formulations. Representative examples include [19–23] and more recent refinements that improve precision dependence and broaden the class of implementable (generally non-unitary) linear maps. Closest in spirit to our dilation viewpoint is *Schrödingerisation*, which maps general linear evolution to a time-dependent Schrödinger equation on a larger Hilbert space [16, 24], as well as linear combination of Hamiltonian simulation [25, 26] and moment-matching dilations for non-unitary linear dynamics [18]. These techniques motivate the present work: our goal is to extend such dilations from deterministic linear dynamics to *stochastic* linear Itô systems while retaining a physically standard quantum-mechanical form.

Quantum algorithms for SDEs via time discretization and PDE reformulations. One line of work treats SDE simulation by first discretizing time (e.g. Euler–Maruyama or higher weak schemes) and then reducing the resulting random time-stepping to a deterministic quantum evolution after the Brownian increments are *presampled* [27]. In contrast, our formulation is intrinsically continuous-time: we embed the SDE into a SSEs, i.e. a quantum-trajectory model, making explicit connections to the theory of open quantum systems.

A different route replaces the SDE by a deterministic PDE for a probability density, i.e., the Fokker–Planck (Kolmogorov forward) equation. This approach enables the use of quantum PDE solvers [28], but the resulting complexity typically involves the PDE discretization, and hence can inherit polynomial dependence on the SDE dimension N in generic settings.

Complexity-theoretic perspective and nonlinear/noisy dynamics. Beyond algorithmic constructions, recent work indicates that SDE simulation captures the full power of quantum computation in a precise complexity-theoretic sense. Bravyi *et al.* study quantum simulation of noisy classical nonlinear dynamics and establish BQP-completeness for the SDEs simulation tasks. They propose a bosonic-operator encoding to achieve favorable dimension dependence in structured regimes [29].

Relation to quantum trajectories and unravellings. Our trajectory algorithm also connects to the longstanding quantum-jump/quantum-trajectory literature, where Lindblad evolution is unraveled into stochastic pure-state evolutions (SSEs) implemented by repeated interactions and measurements; see, e.g., the review [30] and foundational developments in wave-function Monte Carlo methods. The key distinction is that we use the SSE as a computational representation of a classical linear SDE via our moment-matching dilation, thereby turning generic linear stochastic dynamics into a standard open quantum system model that is amenable to modern Lindblad simulation and trajectory-generation algorithms.

Quantum algorithms for the Lyapunov equation. The second moment of the SDEs (2) satisfies a Lyapunov equation for the covariance $\Sigma(t)$. Benedetti *et al.* [31] propose a probabilistic quantum algorithm to prepare a mixed state proportional to the steady-state solution of the Lyapunov equation. In contrast, our approach embeds the *time-dependent* second-moment dynamics induced by the SDE into a Lindblad evolution via a moment-matching dilation, enabling transient covariance estimation as well.

The remainder of the paper is organized as follows. In [Section 2](#), we introduce the preliminaries for moment-matching dilation and the setup for mapping to SSEs. In [Section 3](#), we present a specific dilation using a finite-dimensional tight-binding model and

prove the finite-time accuracy. In [Sections 4 and 5](#), we elaborate on the implementations of the algorithms on digital quantum devices. Numerical results that validate the error estimates are provided in [Section 6](#).

2 Dilation of Stochastic Differential Equations

2.1 Preliminaries: moment-matching dilation for ODEs

We recall the deterministic dilation from our earlier work [\[18\]](#) on non-unitary linear ODEs. Here, we only state what we need for the stochastic extension. Consider the linear ODE on $\mathbb{C}^N$

$$\dot{\mathbf{x}}(t) = L(t)\mathbf{x}(t), \quad L \in \mathbb{C}^{N \times N}.$$

L can be uniquely decomposed as $L(t) = -iH(t) + K(t)$ whereas both H and K are Hermitian.

Definition 1 (Moment-matching dilation). *Let $\mathcal{H}_{\mathcal{A}}$ be a complex ancillary Hilbert space. A triple $(F, |r\rangle, \langle l|)$ with*

$$F : \mathcal{H}_{\mathcal{A}} \rightarrow \mathcal{H}_{\mathcal{A}}, \quad F^\dagger = -F,$$

is called a moment-matching dilation of order ∞ if

$$\langle l| F^k |r\rangle = 1, \quad \forall k \in \mathbb{N}_0. \quad (8)$$

To clarify the notation, let $\mathcal{H}$ be the original Hilbert space, let I be the identity matrix on $\mathcal{H}$, and let $I_{\mathcal{A}}$ denote the identity on the ancilla Hilbert space $\mathcal{H}_{\mathcal{A}}$. Specific examples of moment-fulfilling families can be found in [\[18\]](#).

Theorem 1 (Deterministic moment-matching dilation). *Let $L = -iH + K$ and let $(F, |r\rangle, \langle l|)$ be a moment-matching triple. Define the dilated Hamiltonian on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}$ by*

$$\tilde{H} := I_{\mathcal{A}} \otimes H + iF \otimes K. \quad (9)$$

Then, for all $t \geq 0$,

$$\mathcal{T}e^{\int_0^t L(t')dt'} = (\langle l| \otimes I) \mathcal{T}e^{-i \int_0^t \tilde{H}(t')dt'} (|r\rangle \otimes I). \quad (10)$$

The proof is purely algebraic: expand $\mathcal{T}e^{-i \int_0^t \tilde{H}(t')dt'}$ as Dyson series in t , use that each factor $I \otimes H + iF \otimes K$ is a polynomial in F , and then use (8) to replace each k -fold F and $I_{\mathcal{A}}$ by the scalar 1 in the matrix element $(\langle l| \otimes I)(\cdot)(|r\rangle \otimes I)$, thus recovering the Dyson series of $\mathcal{T}e^{\int_0^t L(t')dt'}$ term by term.

2.2 The exact-mapping theorem for SDEs (2)

We now extend the dilation technique for simulating ODEs to the simulation of SDEs. To guarantee the existence of unique strong solutions and the validity of the higher-order stochastic expansions used in our derivation, we impose the following regularity conditions.

Assumption 1 (Regularity of Coefficients). *The drift operator $A(t)$ and noise operators $B_j(t)$ are uniformly bounded and continuously differentiable functions of time on the interval $t \in [0, T]$. That is, there exists a constant $C > 0$ such that for all t and j :*

$$\|A(t)\| + \|B_j(t)\| \leq C \quad \text{and} \quad \|\dot{A}(t)\| + \|\dot{B}_j(t)\| \leq C.$$

Under [Assumption 1](#), the linear SDE (2) satisfies the standard global Lipschitz and linear growth conditions. This ensures the existence of a unique strong solution X_t adapted to the filtration $\mathcal{F}_t$, satisfying $\sup_{t \in [0, T]} \mathbb{E} \|X_t\|^2 < \infty$ [8, Thm. 4.5.3]. Furthermore, the C^1 -regularity ensures that the stochastic Taylor expansion converges in the mean-square sense.

We summarize the stability properties of the exact solution below [32].

Proposition 1 (Mean-Square Stability and Growth). *Let X_t be the solution to (2) under [Assumption 1](#). Define the Hermitian Lyapunov matrix:*

$$K(t) := \frac{1}{2} \left(A(t) + A(t)^\dagger + \sum_{j=1}^J B_j(t)^\dagger B_j(t) \right). \quad (11)$$

Then, the second moment evolves according to the differential equation

$$\frac{d}{dt} \mathbb{E} [\|X_t\|^2] = 2 \mathbb{E} [\langle X_t, K(t) X_t \rangle]. \quad (12)$$

Consequently, the growth of the system is strictly controlled by the maximal eigenvalue of $K(t)$. If $K(t) \preceq \gamma(t)I$ for a scalar function $\gamma(t)$, we have the a priori bound:

$$\mathbb{E} [\|X_t\|^2] \leq \exp \left(2 \int_0^t \gamma(s) ds \right) \mathbb{E} [\|X_0\|^2]. \quad (13)$$

In particular, if $K(t)$ is uniformly negative definite, the system is exponentially mean-square stable.

We now elaborate on the dilation procedure. Starting from the SDE (2), we fix an ancillary space $\mathcal{H}_A$ and a moment-matching triple $(F, |r\rangle, \langle l|)$ as above. We start by defining,

$$L(t) := A(t) + \frac{1}{2} \sum_{j=1}^J B_j^\dagger(t) B_j(t) \quad (14)$$

from the drift and noise coefficients in the SDE system (2).

Next we split the operator $L(t)$ by defining the Hermitian matrix H and K by,

$$L(t) := -iH(t) + K(t). \quad (15)$$

One can verify that the hermitian part here is the same as that in [Eq. \(4\)](#). Importantly, when $L(t)$ is skew-Hermitian, or equivalently $K(t) \equiv 0$, the original SDE system (2) coincides with the stochastic Schrödinger equation in [Eq. \(61\)](#). To be able to simulate the case when $K(t) \neq 0$, we extend the dilation method, by extending the operators in dilated Hilbert space as follows,

$$\tilde{H} := I_A \otimes H(t) + iF \otimes K(t). \quad (16)$$

We now extend the operators B_j by a direct dilation,

$$V_j(t) := I_A \otimes B_j(t), \quad j = 1, \dots, J. \quad (17)$$

For the convenience of the presentation, we also define,

$$V_0(t) := -i\tilde{H} - \frac{1}{2} \sum_{j=1}^J V_j^\dagger V_j, \quad (18)$$

which will become the non-Hermitian part of the SSE (3).

We will show that under the moment conditions (8), the dilation of these operators yields an SSE of the form,

$$d|\psi_t\rangle = V_0(t)|\psi_t\rangle dt + \sum_{j=1}^J V_j(t)|\psi_t\rangle dW_t^j, \quad (19)$$

which has the same structure as (3).

We set the initial condition of the dilated SSE (19) to

$$|\psi_0\rangle = |r\rangle \otimes X_0, \quad (20)$$

which can be easily prepared as a factored state.

Theorem 2 (Exact recovery of the linear SDE). *Let X_t be the unique strong solution of the linear SDE (2) and $|\psi_t\rangle$ be the strong solution of the dilated SSE (19) with initial data (20). Under the assumptions on the coefficient matrices $A(t)$ and $B_j(t)$ in Assumption 1, and that $(F, |r\rangle, \langle l|)$ is a moment-matching triple according to Eq. (8), for every $t \geq 0$ the following identity holds almost surely:*

$$X_t = (\langle l| \otimes I) |\psi_t\rangle. \quad (21)$$

Equivalently, for each fixed sample path $\omega \in \Omega$, the projected process $t \mapsto (\langle l| \otimes I) \psi_t(\omega)$ is the exact solution of (2) with the same Wiener trajectory.

The proof is presented in Section A.

3 Finite-dimensional tight-binding dilations for SDE systems

The result in Theorem 2 relies on exact moment-matching conditions (8) which are usually fulfilled by an infinite-dimensional ancilla space. Many choices are available [18], but for practical purposes, we consider the following tight-binding type of dilation, which was derived from an infinite-dimensional dilation using a differential operator $F = p\partial_p + \frac{1}{2}$ on the interval $p \in [0, 1]$.

In the continuous setting, this generator F is skew-Hermitian on the Hilbert space $\mathcal{H}_A = H_0^1(0, 1)$, equipped with the inner product

$$\langle f, g \rangle = \int_0^1 f^*(p)g(p) dp,$$

and homogeneous boundary condition $f(1) = g(1) = 0$. The moment-matching construction, however, is formulated on the larger space $H^1(0, 1)$, which contains $H_0^1(0, 1)$ as a subspace. For any $\theta > 0$, θF fulfills the moment condition $\langle l| (\theta F)^k |r\rangle = 1$ for all $k \geq 0$ by choosing the right vector as the eigenfunction $|r\rangle \propto p^\beta$ with $\beta = 1/\theta - 1/2$, which belongs to $H^1(0, 1)$ but not necessarily to $H_0^1(0, 1)$ since it does not satisfy the homogeneous boundary condition. The left functional $\langle l|$ is defined by point evaluation,

$$\langle l| f \propto f(p_*),$$

for some $p_* \in (0, 1]$, acting on functions in $H^1(0, 1)$, where point evaluation is well defined.

To obtain a finite-dimensional realization suitable for digital quantum simulation, we partition $[0, 1]$ into M intervals with grid points $\{p_i\}_{i=0}^M$. The integration by parts that

ensured the skew Hermitian property can be extended to the discrete level using summation by parts (SBP). Following [33, 34], we let $h_j := p_{j+1} - p_j$ and define the SBP trapezoid weights

$$W = \text{diag}(w_0, \dots, w_M), \quad w_0 = \frac{1}{2}h_0, \quad w_j = \frac{1}{2}(h_{j-1} + h_j) \quad (1 \leq j \leq M-1), \quad w_M = \frac{1}{2}h_{M-1}. \quad (22)$$

Let Q be the tridiagonal matrix for a centered difference operator with

$$(Q)_{j,j+1} = \frac{1}{2}, \quad (Q)_{j+1,j} = -\frac{1}{2}, \quad Q_{00} = -\frac{1}{2}, \quad Q_{MM} = +\frac{1}{2}, \quad (23)$$

and set $D := W^{-1}Q$. Then the diagonal-norm SBP identity holds:

$$WD + D^\dagger W = Q + Q^\dagger = B := \text{diag}(-1, 0, \dots, 0, 1).$$

This implies that for all grid functions $\mathbf{u}, \mathbf{v}$, $\langle \mathbf{u}, D\mathbf{v} \rangle_W + \langle D\mathbf{u}, \mathbf{v} \rangle_W = u_M v_M - u_0 v_0$ with weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle_W := \mathbf{x}^\dagger W \mathbf{y}$, thus mimicking the integration by parts property, and thus automatically maintains the skew property after the discretization. Specifically, let $P := \text{diag}(p_0, \dots, p_M)$ and define the (Hamiltonian) split form

$$F_w := \frac{1}{2}(PD + DP) - \frac{1}{2}W^{-1}BP, \quad F_h := W^{1/2}F_w W^{-1/2}. \quad (24)$$

The SBP property automatically guarantees that F_h is skew-Hermitian and tridiagonal, while F_w is skew with respect to the W -inner product.

Explicitly, the SBP discretization on the geometric grid with $\theta = 2$ is given by,

$$p_j = \exp[-h(M - j)], \quad j = 0, 1, \dots, M, \quad (25)$$

where $h > 0$ is a grading parameter (typically $h \approx 1$), yields a tridiagonal matrix F_h with zeros on the diagonal. The off-diagonal entries $f_j := (F_h)_{j,j+1} = -(F_h)_{j+1,j}$ take a uniform value in the bulk of the grid, simplifying the implementation:

$$f_j = \frac{1}{4 \sinh(h/2)} \times \begin{cases} \sqrt{1 + e^{-h}}, & j = 0, \\ 1, & 1 \leq j \leq M-2, \\ \sqrt{1 + e^h}, & j = M-1. \end{cases} \quad (26)$$

This nearest-neighbor connectivity allows F_h to be efficiently mapped to a quantum circuit. For example, the operator iF_h corresponds to a hopping Hamiltonian. It admits a simple 2-local representation using Pauli operators:

$$iF_h = -\frac{1}{2} \sum_{j=0}^{M-1} f_j (X_j Y_{j+1} - Y_j X_{j+1}), \quad (27)$$

where X_j, Y_j are the Pauli matrices acting on the j -th qubit of the register. This structure is amenable to standard Trotterization or block-encoding techniques on digital quantum processors.

For the eigenvectors $|r\rangle$, we choose $\theta = 2$ so $\beta = 0$. In addition, we choose the components according to the weights w_j :

$$|r_h\rangle = \frac{1}{Z_h} \sum_{j=0}^M \sqrt{w_j} |j\rangle, \quad Z_h = \left(\sum_{j=0}^M w_j \right)^{1/2}. \quad (28)$$

One can show that

$$(2F_h |r_h\rangle - |r_h\rangle) \propto |M\rangle. \quad (29)$$

Namely, the residual error is zero for all the interior nodes. This is due to the SBP discretization. For the evaluation operator, we set it to,

$$\langle l_h| = \frac{Z_h}{P_*} \sum_{j=0}^{j_*} \sqrt{w_j} |j\rangle, \quad P_* = \sum_{j=0}^{j_*} w_j,$$

to satisfy $\langle l_h|r_h\rangle = 1$. Here j_* is an index that we pick to post-select the solution, and its choice will be discussed in the next section.

Remark 1. *It is necessary to distinguish three levels of the dilation construction used in this paper. First, [Theorem 2](#) is formulated abstractly in terms of a moment-matching triple $(F, |r\rangle, \langle l|)$ satisfying the condition (8). At this level, no specific realization of the ancillary space or operator F is required. Second, the differential operator $F = p\partial_p + \frac{1}{2}$ provides an infinite-dimensional realization of such a moment-matching triple. This continuous construction serves as a conceptual model that motivates the finite-dimensional approximation. In this setting, F is understood as an unbounded operator acting on a suitable function space satisfying the stated boundary conditions, while the reference state and evaluation functional are chosen so that the moments in [Eq. \(8\)](#) are well defined. Finally, the quantum algorithms developed in [Sections 4](#) and [5](#) use only the finite-dimensional SBP discretization F_h . The matrices F_h , together with the corresponding discrete vectors $|r_h\rangle$ and $\langle l_h|$, constitute the actual objects implemented in the simulation algorithms.*

3.1 Error Analysis and Stochastic Light-Cone Property

While the continuous dilation is exact, the finite-dimensional truncation introduces errors due to the discretization of F and the imposition of artificial boundary conditions at $p = 1$. To quantify this, we first notice that by introducing a boundary impurity potential at the edge of the chain:

$$\hat{F}_h := F_h + \alpha |M\rangle\langle M|, \quad \alpha := \frac{1}{\theta} - \frac{\langle M|F_h|r_h\rangle}{\langle M|r_h\rangle}, \quad (30)$$

we have $\theta\hat{F}_h|r_h\rangle = |r_h\rangle$ is an exact eigenstate. Thus, $\langle l_h|(\theta\hat{F}_h)^k|r_h\rangle = 1, \forall k \geq 0$.

The resulting operator, known as a moment-locking closure (MLC) [\[18\]](#), in light of [Theorem 2](#), leads to an exact dilation of the SDEs. To leverage this, we let the corresponding dilation be,

$$\hat{H} := I_A \otimes H(t) + i\theta\hat{F}_h \otimes K(t). \quad (31)$$

Similarly, we define the modified drift term,

$$\hat{V}_0(t) := -i\hat{H} - \frac{1}{2} \sum_{j=1}^J V_j^\dagger V_j, \quad (32)$$

and let $|\phi_t\rangle$ be the solution of the SDEs

$$d|\phi_t\rangle = \hat{V}_0(t)|\phi_t\rangle dt + \sum_{j=1}^J V_j(t)|\phi_t\rangle dW_t^j, \quad (33)$$

with the same initial condition $\phi(0) = |r_h\rangle \otimes X_0$. Since the dilation using $\hat{F}_h$ is exact, we have

$$|\phi_t\rangle = |r_h\rangle \otimes X_t. \quad (34)$$

As a result, the error $|\chi_t\rangle = |\psi_t\rangle - |\phi_t\rangle$ encodes the error from the finite-dimensional dilation: using $\langle l_h | r_h \rangle = 1$, we have

$$\langle l_h | \otimes I |\chi_t\rangle = \langle l_h | \otimes I |\psi_t\rangle - X_t.$$

Furthermore, we notice that χ_t satisfies $\chi(0) = 0$ and a driven SSE:

$$d|\chi_t\rangle = \tilde{V}_0(t) |\chi_t\rangle dt + \sum_{j=1}^J V_j(t) |\chi_t\rangle dW_t^j + |S_t\rangle dt, \quad (35)$$

where the source term $|S_t\rangle = -\alpha\theta \langle M | r_h \rangle |M\rangle \otimes K(t)X_t$ is localized entirely at the right boundary of the ancilla register (site M).

The critical observation is that the error propagates from the boundary into the interior solely through the "hopping" term $F_h \otimes K(t)$ in the drift operator. The noise terms $I \otimes B_j$ are diagonal in the ancilla basis and do not induce spatial transport. This leads to a strong light-cone bound that depends principally on the norm of the dissipative coupling $K(t)$. The following theorem establishes a finite propagation speed that depends explicitly on the tight-binding chain with grading h and the dissipative norm $K_{\max}$.

Theorem 3 (Stochastic Light-Cone). *Let $|\chi_T\rangle$ be the error state at time T arising from the boundary truncation of the ancilla. Let j_* be an interior ancilla site, and $m = M - j_*$ be the distance from the boundary $j = M$. Let $K_{\max} = \sup_t \|K(t)\|$. If the parameters h and m satisfy the condition:*

$$\varrho := \frac{e\theta K_{\max} T}{4(M - j_*) \sinh(h/2)} < 1, \quad (36)$$

then the mean-square error projected onto site j_* decays exponentially with distance:

$$\mathbb{E} \left[\|\langle j_* | \otimes I |\chi_T\rangle\|^2 \right] \leq \mathcal{C} \frac{\varrho^{2m}}{(1 - \varrho)^2} \quad (37)$$

where $\mathcal{C}$ is a constant depending on $X(T)$ and grid boundary weights.

We defer the proof to [Section B](#).

Since our bound applies to the mean-square norm of the state error vector $\mathbb{E}[\|\chi_T\|^2]$, this result establishes *strong convergence* of the dilated quantum simulation. This implies that for any single noise trajectory, the output state $|\psi_T\rangle$ is physically close to the exact solution, encoding X_T with high probability, not just consistent in ensemble average. This result establishes a fundamental speed for error propagation in the stochastic dilation framework. Importantly, this speed depends *only* on the norm of the dissipative operator $K(t)$. It is entirely independent of the magnitude of the Hamiltonian drift $H(t)$ or the strength of the noise $B_j(t)$. As a result, the dimension of the ancilla required scales logarithmically with the precision ε and linearly with the "dissipative complexity" $K_{\max}T$. This is analogous to the Lieb-Robinson bounds in many-body physics, where information propagates at a finite velocity determined by the interaction strength.

Remark 2. *The moment-locking closure operator $\hat{F}_h$ is introduced solely as an auxiliary comparison operator. Since it satisfies $\theta \hat{F}_h |r_h\rangle = |r_h\rangle$, it is no longer skew-Hermitian and therefore does not generate the physical Hamiltonian dilation used in the algorithm. Instead, $\hat{F}_h$ defines an exact reference evolution, from which the error equation is derived by comparison with the physical evolution generated by the skew-Hermitian operator F_h . Throughout the remainder of the paper, only the Hermitian dilation constructed from F_h is implemented in the quantum algorithms.*

4 Implementation via Lindblad Simulation Algorithms

A key advantage of the dilation framework is that it enables *ensemble* statistics of the linear SDE (2) to be computed by simulating a *deterministic* quantum master equation, rather than sampling individual trajectories. While Theorem 2 provides a pathwise embedding $X_t(\omega) = (\langle l| \otimes I) |\psi_t(\omega)\rangle$, many quantities of practical interest are *quadratic* in the state; in particular, for any observable O on $\mathcal{H}$,

$$\mathbb{E}[X_t^\dagger O X_t] = \text{tr}(\Sigma_t O), \quad \Sigma_t := \mathbb{E}[X_t X_t^\dagger], \quad (38)$$

where Σ_t is the second-moment (covariance) matrix. Crucially, Σ_t evolves deterministically and satisfies a closed second-moment equation. Our dilation lifts this deterministic evolution to a quantum master equation on an enlarged Hilbert space, which constitutes a completely-positive and trace preserving (CPTP) dynamic map. As a result, the estimation of $\text{tr}(\Sigma_t O)$ can be reduced to the estimation of a single observable on the output of a quantum channel.

Throughout this section we assume $\|X_0\| = 1$ without loss of generality, so that $\text{tr}(\Sigma_0) = \|X_0\|^2 = 1$ and $\sigma_0 = |X_0\rangle\langle X_0|$ is already a normalized rank-one state. Since the second-moment equation and all target observables are linear in Σ_t , the result for a general initial condition X_0 is recovered by multiplying the final estimate by $\|X_0\|^2$.

4.1 The Dilated Master Equation

Let $|\psi_t\rangle$ be the solution of the dilated SSE (19). Define the corresponding density matrix on $\mathcal{H}_A \otimes \mathcal{H}$,

$$\rho_t := \mathbb{E}[|\psi_t\rangle\langle\psi_t|]. \quad (39)$$

A direct application of Itô formula to $|\psi_t\rangle\langle\psi_t|$ yields a Lindblad master equation for ρ_t (see, e.g., [17]), a universal description of CPTP quantum maps [35, 36].

Lemma 1 (Lindblad equation for the dilated second moment). *The density matrix ρ_t satisfies*

$$\frac{d}{dt}\rho_t = \mathcal{L}_t(\rho_t), \quad \mathcal{L}_t(\rho_t) := -i[\tilde{H}(t), \rho_t] + \sum_{j=1}^J \left(V_j(t)\rho_t V_j^\dagger(t) - \frac{1}{2}\{V_j^\dagger(t)V_j(t), \rho_t\} \right), \quad (40)$$

with initial condition $\rho_0 = (|r_h\rangle\langle r_h|) \otimes \sigma_0$, where $\sigma_0 := |X_0\rangle\langle X_0|/\|X_0\|^2$ is the normalized rank-one second-moment seed (Thus, $\text{tr}(\rho_0) = 1$ due to $\|r_h\| = 1$). Here $\tilde{H}(t)$ is the Hermitian dilated Hamiltonian (16) and $V_j(t) = I_A \otimes B_j(t)$ are the dilated noise operators.

Simulating the Markovian quantum dynamics governed by (40) is a central primitive in quantum algorithms. Early approaches had a polynomial dependence on the precision [37], while more recent algorithms achieve near-optimal scaling by exploiting higher-order expansions and block-encoding reductions [38–41]. For our purposes, it is especially convenient to use a simulator that outputs a *purification* of ρ_T (e.g. [39]), since expectation values of observables can be estimated by standard block-encoding/measurement routines [42].

To relate the quadratic statistics of the original SDE (2) to the density matrix ρ_T , recall that $\Sigma_t := \mathbb{E}[X_t X_t^\dagger]$. For the tight-binding dilation, with $\|r_h\| = 1$, one can recover the second moment as follows (up to the light-cone error),

$$\Sigma_t \approx (\langle l_h| \otimes I) \rho_t (|l_h\rangle \otimes I). \quad (41)$$

Hence, for any observable O on the system, one has,

$$\text{tr}(\Sigma_t O) = \text{tr}(\rho_t (\Pi_{l_h} \otimes O)), \quad \Pi_{l_h} := |l_h\rangle\langle l_h|. \quad (42)$$

Estimating this expectation value via amplitude amplification and block encoding requires $\mathcal{O}(1/\varepsilon)$ rounds of preparations of ρ_t [42].

4.2 Segment-wise evolution and ancilla refresh

Due to the finite-dimensional implementation, (42) remains accurate on each segment up to the controlled light-cone error. One can directly extend Theorem 3 to the second moment Σ_t , as follows,

Proposition 2 (Light-Cone for the covariance). *Assume the hypothesis of Theorem 3, i.e., $m = M - j_*$, and*

$$\varrho := \frac{e\theta K_{\max} T}{4m \sinh(h/2)} < 1.$$

Define the j_* -localized blocks from the Lindblad equation Eq. (40) as

$$\rho_T^{(j_*)} := (\langle j_* | \otimes I) \rho_T (|j_* \rangle \otimes I).$$

Writing $\gamma := \langle j_* | r_h \rangle$, then the second-moment error on the interior site obeys the trace-norm bound

$$\|\rho_T^{(j_*)} - |\gamma|^2 \Sigma_T\|_1 \leq 2|\gamma| \sqrt{\text{tr}(\Sigma_T)} \sqrt{C \frac{\varrho^{2m}}{(1-\varrho)^2}} + C \frac{\varrho^{2m}}{(1-\varrho)^2}. \quad (43)$$

We defer the proof to Section C.

The covariance matrix Σ_t associated with Eq. (2) satisfies a matrix differential equation,

$$\frac{d}{dt} \Sigma_t = \mathcal{D}_t(\Sigma_t), \quad \Sigma_0 \succeq 0, \quad (44)$$

where $\mathcal{D}_t$ is a generator for a dynamic map that need not be trace-preserving. Without loss of generality, we assume $\text{tr}(\Sigma_0) = 1$, since Σ_t may be rescaled by the linearity of Eq. (2).

Although the exactness of this dilation is guaranteed for any simulation time T , the light-cone analysis indicates that in order to maintain a finite success probability to post-select out Σ_T using Eq. (41), we must choose T such that

$$T = \mathcal{O}(K_{\max}^{-1}), \quad K_{\max} := \max_{t \in [0, T]} \|K(t)\|. \quad (45)$$

This issue can be circumvented by a segment-wise simulation, and upon the completion of each segment, an oblivious amplitude amplification (OAA) [38, 43] can be applied to restore the ancilla so that the algorithm can be repeated for the following segment. Toward this end, we fix the segment length

$$\tau := \mathcal{O}\left(\frac{1}{K_{\max}}\right), \quad L := \left\lceil \frac{T}{\tau} \right\rceil = \mathcal{O}(TK_{\max}). \quad (46)$$

Define the time segments $t_m := m\tau$ for $m = 0, 1, \dots, L$ (with $t_L \geq T$; one may shorten the last step without affecting the discussion). Let the exact segment evolution from Eq. (44) be

$$\mathcal{E}_m := \mathcal{T} \exp\left(\int_{t_m}^{t_{m+1}} \mathcal{D}_s ds\right), \quad (47)$$

so that the ideal final state can be written as the composition

$$\Sigma_T := (\mathcal{E}_{L-1} \circ \cdots \circ \mathcal{E}_1 \circ \mathcal{E}_0)(\Sigma_0). \quad (48)$$

Notice that Eq. (44) does not necessarily produce a density matrix. Let us introduce the scalar trace

$$\lambda_m := \text{tr}(\Sigma_{t_m}) \text{ for } m = 1, \dots, L, \quad \lambda_0 := 1,$$

and the normalized covariance, which can be regarded as a density matrix, becomes

$$\sigma_m := \frac{\Sigma_{t_m}}{\lambda_m}, \quad \text{tr}(\sigma_m) = 1.$$

To apply a Lindblad simulation algorithm, we define the dilated input at the beginning of segment m ,

$$\rho_{t_m} := |r_h\rangle\langle r_h| \otimes \sigma_m, \quad \text{tr}(\rho_m) = 1, \quad (49)$$

which is the algorithmic working state, without knowing the factors λ_m .

Denote $\tilde{\mathcal{E}}_m$ as the CPTP segment channel induced by the dilated Lindbladian (40) on $[t_m, t_{m+1}]$, and it evolves the system into a *pre-refreshed state*,

$$\rho'_{m+1} := \tilde{\mathcal{E}}_m(\rho_{t_m}). \quad (50)$$

Meanwhile, Theorem 2 ensures that the (unnormalized) covariance update over the segment is extracted by the fixed readout

$$\Sigma'_{m+1} := (\langle l_h| \otimes I) \rho'_{m+1} (|l_h\rangle \otimes I), \quad (51)$$

which (up to the controlled light-cone / simulation errors) satisfies $\Sigma'_{m+1} \approx \mathcal{E}_m(\sigma_m)$. We thus define the *segment trace-growth factor* using the unnormalized projector from Eq. (42)

$$g_m := \text{tr}(\Sigma'_{m+1}) = \text{tr}(\rho'_{m+1} (\Pi_{l_h} \otimes I)), \quad (52)$$

and update the normalized covariance and the scalar trace by

$$\sigma_{m+1} = \frac{\Sigma'_{m+1}}{g_m}, \quad \lambda_{m+1} = \lambda_m g_m, \quad (53)$$

so that the algorithm can proceed to the next time segment. Here g_m can be estimated by repeated preparations of ρ'_{m+1} followed by measuring Π_{l_h} on the ancilla.

Iterating (53) yields

$$\lambda_L = \prod_{m=0}^{L-1} g_m, \quad \Sigma_T = \lambda_L \sigma_L, \quad (t_L = T). \quad (54)$$

Consequently, for any system observable O ,

$$\text{tr}(\Sigma_T O) = \text{tr}(\lambda_L \sigma_L O) = \lambda_L \text{tr}(\sigma_L O) = \left(\prod_{m=0}^{L-1} g_m \right) \text{tr}(\sigma_L O). \quad (55)$$

Before proceeding with the same algorithm to the next time segment, another important step is to restore the ancilla to $|r_h\rangle$ so that the initial density matrix for the next segment takes the same form as Eq. (49). By the light-cone property, the restriction of the ancilla

to the prefront window $\text{win} = \{0, 1, \dots, j_*\}$ remains accurate (up to an ε error) over any segment of length subject to the choice of τ in Eq. (46), the overlap

$$P_{\text{win}} := \sum_{j \in \text{win}} |\langle j | r_h \rangle|^2 \quad (56)$$

is a constant lower bound with appropriate choice of h [18], and without loss of generality we may assume $P_{\text{win}} \geq 1/4$. Let us define the corresponding projector and the truncated ancilla mode, respectively,

$$\Pi_{\text{win}} := \left(\sum_{j \in \text{win}} |j\rangle \langle j| \right) \otimes I, \quad |r_{\text{win}}\rangle := \sum_{j \in \text{win}} |j\rangle \langle j | r_h \rangle. \quad (57)$$

In particular, $P_{\text{win}} \geq 1/4$ ensures that the following trace-decreasing CP map succeeds with constant probability on the relevant states:

$$\mathcal{R}_{\text{win}}(\rho) := \frac{W_{\text{win}} (\Pi_{\text{win}} \rho \Pi_{\text{win}}) W_{\text{win}}^\dagger}{\text{tr}(\Pi_{\text{win}} \rho)}, \quad (58)$$

where W_{win} is any fixed isometry on the ancilla register satisfying $W_{\text{win}} |r_{\text{win}}\rangle / \|r_{\text{win}}\| = |r_h\rangle$.

Operationally, one can realize $\mathcal{R}_{\text{win}}$ either by literal postselection on Π_{win} (repeat-until-success), or coherently via oblivious amplitude amplification (OAA), which restores the ancilla without restarting the segment evolution.

More precisely, suppose the Lindblad simulator for segment m outputs a purification of ρ'_{m+1} . There exist an isometry (implemented by the simulator) U_m and an environment register E such that, for some purification $|\Phi_m\rangle$ of ρ_m ,

$$|\psi_{m+1}\rangle := U_m |\Phi_m\rangle \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_E, \quad \text{tr}_E |\psi_{m+1}\rangle \langle \psi_{m+1}| = \rho'_{m+1}.$$

Define the window projector on the full space (acting trivially on E)

$$\Pi_{\text{win}}^{(\text{full})} := \Pi_{\text{win}} \otimes I_E, \quad q_m := \langle \psi_{m+1} | \Pi_{\text{win}}^{(\text{full})} | \psi_{m+1} \rangle = \text{tr}(\Pi_{\text{win}} \rho'_{m+1}).$$

By the light-cone property and the choice of τ , we have $q_m \approx P_{\text{win}}$; in particular, $q_m = \Omega(1)$ (e.g. $q_m \gtrsim 1/4$). Here to distinguish the quantum registers, we use $\mathcal{H}_{\text{sys}}$ for the workspace of the SDEs X_T , and $\mathcal{H}_E$ for the additional ancilla for OAA.

Decompose the post-segment state into its “good” (in-window) and “bad” components:

$$|\psi_{m+1}\rangle = |G_m\rangle + |B_m\rangle, \quad |G_m\rangle := \Pi_{\text{win}}^{(\text{full})} |\psi_{m+1}\rangle, \quad |B_m\rangle := (I - \Pi_{\text{win}}^{(\text{full})}) |\psi_{m+1}\rangle,$$

so that $\langle G_m | B_m \rangle = 0$ and $\|G_m\|^2 = q_m$. A direct postselection on Π_{win} would succeed with probability q_m and produce the normalized in-window state $|G_m\rangle / \sqrt{q_m}$.

A more efficient approach is to use OAA, which implements this postselection *coherently* via two reflections:

$$R_{\text{win}} := I - 2\Pi_{\text{win}}^{(\text{full})}, \quad R_r := 2(|r_h\rangle \langle r_h| \otimes I_{\text{sys}} \otimes I_E) - I.$$

Both reflections are ancilla-controlled, and R_r depends only on the fixed reference mode $|r_h\rangle$. Define the Grover iterate

$$Q_m := -R_r U_m^\dagger R_{\text{win}} U_m.$$

Restricted to the two-dimensional invariant subspace $\text{span}\{|G_m\rangle, |B_m\rangle\}$, Q_m acts as a rotation that amplifies the weight on the “good” subspace. Applying Q_m for $\Theta(1/\sqrt{q_m})$ iterations boosts the in-window amplitude to $\Theta(1)$; since $q_m = \Omega(1)$, this requires only $\mathcal{O}(1)$ uses of U_m and $U_m^\dagger$ per segment.

Finally, once the state is supported in win, we deterministically map the truncated ancilla mode back to the reference by an ancilla-only unitary extension of the isometry. Concretely, let $|r_{\text{win}}\rangle := \Pi_{\text{win}} |r_h\rangle$ and choose any unitary (or isometry extended to a unitary) W_{win} on $\mathcal{H}_{\mathcal{A}}$ such that $W_{\text{win}} |r_{\text{win}}\rangle / \|r_{\text{win}}\| = |r_h\rangle$. Applying $W_{\text{win}} \otimes I_{\text{sys}} \otimes I_E$ completes the refresh and yields an output whose $\mathcal{H}_{\mathcal{A}}$ -marginal is restored to $|r_h\rangle$ (up to the same $\mathcal{O}(\varepsilon)$ light-cone leakage), enabling the next segment to start again from the canonical form (49).

4.3 Amplitude estimation for the growth factors

We now discuss how to estimate the growth factors g_m . Write $\beta := \||l_h\rangle\|$ and define the normalized vector and the associated projector,

$$|\tilde{l}_h\rangle := \frac{1}{\beta} |l_h\rangle, \quad \Pi_{\tilde{l}_h} := |\tilde{l}_h\rangle\langle\tilde{l}_h|.$$

Here we note that $\beta = \||l_h\rangle\| = \sqrt{P_{\text{win}}} = \Omega(1)$.

We define the segment success probability

$$q_m := \text{tr}[\rho'_{t_{m+1}} (\Pi_{\tilde{l}_h} \otimes I)] \in [0, 1], \quad (59)$$

which can be estimated by measuring the dilation ancilla in the basis $\{|\tilde{l}_h\rangle, (|\tilde{l}_h\rangle)^\perp\}$ and recording the $|\tilde{l}_h\rangle$ outcome. By construction of the recovery functional, the covariance trace update on segment m satisfies

$$g_m = \beta^2 q_m, \quad \lambda_{m+1} = \lambda_m g_m. \quad (60)$$

We look for an estimator $\hat{g}_m$ with relative error

$$\frac{|\hat{g}_m - g_m|}{g_m} \leq \delta_g, \quad \delta_g := \frac{\varepsilon}{2L},$$

so that the product $\hat{\lambda}_L = \prod_{m=0}^{L-1} \hat{g}_m$ satisfies $|\hat{\lambda}_L / \lambda_L - 1| \leq e^{L\delta_g} - 1 = \mathcal{O}(\varepsilon)$. Equivalently, we require an additive estimate $\hat{q}_m$ obeying

$$|\hat{q}_m - q_m| \leq \delta_p, \quad \delta_p := \delta_g q_m = \frac{\varepsilon}{2L\lambda_L} q_m.$$

We further scaled the error by λ_L because the expectation $\text{tr}(\Sigma_T O)$ carries a normalizing factor λ_L . The ability to restore the density matrix without rerunning the previous segment leads to the following complexity bound.

A straightforward method for estimating g_m is to apply AA [42] after each segment. Let $C_{\mathcal{L},\tau}$ denote the cost of simulating a single Lindblad segment of duration τ . Consequently, estimating g_m requires $\mathcal{O}(mL/\varepsilon)$ rounds of $C_{\mathcal{L},\tau}$. Because the evolution must be restarted from $t = 0$ for each estimation, the total complexity accumulates to $\mathcal{O}(L^3/\varepsilon)$ times $C_{\mathcal{L},\tau}$.

It is also possible to avoid restarting the evolution from $t = 0$ for each segment by employing a coherent mean-estimation (amplitude estimation) routine [44] applied to the

two-outcome measurement $\{\Pi_{\tilde{I}_h} \otimes I, I - \Pi_{\tilde{I}_h} \otimes I\}$, followed by uncomputation to restore the purification of $\rho'_{t_{m+1}}$.

In contrast, the naive approach estimates the growth factor of segment m by repeatedly preparing $\rho'_{t_{m+1}}$, requiring the evolution to be rerun from $t = 0$ to t_{m+1} for each repetition. The coherent procedure instead performs the estimation reversibly. After estimating the success probability, the ancillary workspace is uncomputed, recovering the purification of $\rho'_{t_{m+1}}$. The restored state can then be used directly as the input to the next segment, rather than being regenerated from the initial condition. Consequently, each segment evolution is performed only once (up to the overhead of amplitude estimation), yielding the complexity stated in [Theorem 4](#).

This latter approach is more efficient, and its overall complexity is summarized as follows,

Theorem 4 (Segment-wise Lindblad complexity). *Fix $\tau = \Theta(1/K_{\max})$, $L = \lceil T/\tau \rceil$ and let $\Lambda_T = \text{tr}(\Sigma_T)$. Assume: (i) for each segment, there is a Lindblad simulator that implements the CPTP map $\tilde{\mathcal{E}}_m$ for time τ with cost $C_{\mathcal{L},\tau}$ and diamond-norm error at most $\mathcal{O}(\varepsilon/(\Lambda_T L))$; (ii) the light-cone/window condition holds so that $P_{\text{win}} \geq 1/4$; (iii) $\|O\| \leq 1$, and we estimate $\nu := \text{tr}(\sigma_L O)$ from the final normalized state σ_L using a standard expectation-estimation routine for observables [\[42\]](#), to additive error $\mathcal{O}(\varepsilon/\Lambda_T)$.*

Then there is an algorithm that outputs an estimate $\hat{\mu}$ of $\mu := \text{tr}(\Sigma_T O)$ satisfying $|\hat{\mu} - \mu| \leq \varepsilon$ with constant success probability, using a number of segment-simulation calls scaling as

$$\tilde{\mathcal{O}}\left(\frac{\Lambda_T L}{\varepsilon} + \frac{\Lambda_T L}{\varepsilon} \sum_{m=0}^{L-1} \frac{1}{\sqrt{q_m}}\right),$$

up to polylogarithmic factors in L , $1/\varepsilon$ and the dimension N . Here q_m is defined in [Eq. \(59\)](#).

The term $\tilde{\mathcal{O}}(\Lambda_T L/\varepsilon)$ comes from estimating the final normalized expectation $\nu = \text{tr}(\sigma_L O)$ to additive error $\mathcal{O}(\varepsilon/\Lambda_T)$, which requires repeated segment-wise Lindblad simulations to prepare the purification of ρ_T . The other term accounts for amplitude tracking: to reconstruct the overall scale $\lambda_L = \prod_{m=0}^{L-1} g_m$ with sufficient accuracy for an ε -additive estimate of $\mu = \lambda_L \nu$, we estimate each segment success probability $q_m = \text{tr}[\rho'_{t_{m+1}}(\Pi_{\tilde{I}_h} \otimes I)]$ to the required precision using coherent mean-estimation with state restoration.

Near-optimal Lindblad simulation algorithms typically achieve $C_{\mathcal{L},\tau} = \tilde{\mathcal{O}}(\|\mathcal{L}\|\tau)$ (up to polylogarithmic factors), and in our dilation setting a coarse bound is

$$\|\mathcal{L}\| \leq \max_{t \in [0, T]} \left(\|A(t)\| + \sum_j \|B_j(t)\|^2 \right).$$

In this case, $C_{\mathcal{L},\tau} L = \|\mathcal{L}\| T$.

5 Implementation by quantum trajectories

Recall that our dilation scheme reduces the linear SDE [\(2\)](#) to an Itô stochastic Schrödinger equation system

$$d|\psi_t\rangle = \left(-i\tilde{H}(t) - \frac{1}{2} \sum_{j=1}^J V_j^\dagger(t) V_j(t) \right) |\psi_t\rangle dt + \sum_{j=1}^J V_j(t) |\psi_t\rangle dW_t^j, \quad (61)$$

where $\tilde{H}(t)$ is from Eq. (16) and is Hermitian, $\{W_t^j\}_{j=1}^J$ are the same independent Wiener processes in the original SDEs (2).

The goal of this section is to address an alternative simulation task: rather than estimating $\text{tr}(\Sigma_T O)$ via a Lindblad simulator, we aim to generate a single sample trajectory (pathwise output) at time T , i.e.: a quantum state proportional to the random vector X_T .

The operational realization we use is a repeated-interaction scheme, in which the system interacts sequentially to create random paths of the SSEs [45–47].

5.1 Quantum trajectories for linear SSEs: a first-order weak scheme

We start with one noise channel (time index suppressed) and no Hamiltonian term:

$$d|\psi_t\rangle = -\frac{1}{2}V^\dagger V |\psi_t\rangle dt + V |\psi_t\rangle dW_t. \quad (62)$$

A first-order weak Itô–Taylor step, also known as the Euler–Maruyama method, is

$$\psi_{t+\Delta t} \approx \left(I - \frac{\Delta t}{2}V^\dagger V + V \Delta W_t\right)\psi_t.$$

For weak order 1, it suffices to replace ΔW_t by any random variable whose mean and variance match those of ΔW_t [8]. We use a Rademacher approximation, i.e.:

$$\xi \in \{+\sqrt{\Delta t}, -\sqrt{\Delta t}\}, \quad \mathbb{P}(\xi = +\sqrt{\Delta t}) = \mathbb{P}(\xi = -\sqrt{\Delta t}) = \frac{1}{2}, \quad (63)$$

so that

$$\psi_{t+\Delta t} \approx \left(I - \frac{\Delta t}{2}V^\dagger V + \xi V\right)\psi_t. \quad (64)$$

A trajectory corresponds to a single run with a specific realization of the discrete noise path ξ . Accordingly, we *presample* the Rademacher signs $s_{n,j} \in \{+1, -1\}$ (equivalently $\xi_{n,j} = s_{n,j}\sqrt{\Delta t}$) *before* running the quantum circuit, and then *coherently implement* the corresponding conditional update at each time step. Operationally, the ancilla measurement serves to select the chosen realization.

The goal of the remaining subsection is to construct a quantum circuit whose effective action reproduces the weak-order-1 update in Eq. (64). Rather than implementing the non-unitary map directly, we realize it as a postselected branch of a larger unitary evolution acting on the system and a single ancilla qubit. The ancilla measurement encodes the presampled stochastic increment, and the resulting Kraus operator coincides with the Euler–Maruyama weak step up to higher-order terms.

5.1.1 One-step interaction realization with a single qubit

We now construct a one-qubit interaction circuit whose effective Kraus operator reproduces the weak-order-1 update Eq. (64). This provides the basic building block for trajectory generation.

Define the anti-Hermitian block generator

$$\Omega := \sqrt{\Delta t} \begin{pmatrix} 0 & -V^\dagger \\ V & 0 \end{pmatrix}, \quad \Omega^\dagger = -\Omega, \quad (65)$$

and the interaction unitary $U := e^\Omega$ acting on a (single-qubit) ancilla and the system. Applied to $|0\rangle \otimes |\psi\rangle$, a second-order expansion gives

$$U \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} (I - \frac{\Delta t}{2}V^\dagger V)\psi \\ \sqrt{\Delta t} V \psi \end{pmatrix} + \mathcal{O}(\Delta t^{3/2}). \quad (66)$$

Starting from $|0\rangle \otimes |\psi\rangle$, apply the interaction unitary $U = e^\Omega$. For $s \in \{+1, -1\}$, define the one-qubit unitary W_s by its action

$$W_s |0\rangle = |s\rangle_x, \quad W_s |1\rangle = |-s\rangle_x, \quad (67)$$

where $|\pm\rangle_x = (|0\rangle \pm |1\rangle)/\sqrt{2}$ are the X -eigenstates. Postselecting on $\langle 0|$ produces a Kraus branch

$$M_s := \langle 0| W_s^\dagger U |0\rangle, \quad |\psi\rangle \mapsto M_s |\psi\rangle, \quad (68)$$

and a second-order expansion of U gives

$$M_s |\psi\rangle \propto \left(I - \frac{\Delta t}{2} V^\dagger V \right) |\psi\rangle + s\sqrt{\Delta t} V |\psi\rangle + \mathcal{O}(\Delta t^{3/2}), \quad (69)$$

which matches (64) exactly with $\xi = s\sqrt{\Delta t}$. Since the target state is $|0\rangle$, we can apply OAA to coherently evolve the system to $\psi_{t_n+\Delta t}$.

Multiple channels and drift. Including the Hamiltonian drift and J channels can be done by operator splitting over one step: apply $e^{-i\tilde{H}(t_n)\Delta t}$ on the system and then apply the above interaction (and X -basis measurement) sequentially for $V_1(t_n), \dots, V_J(t_n)$. This yields the weak order 1 splitting

$$\psi_{t_n+\Delta t} \approx \left(\prod_{j=1}^J \left(I - \frac{\Delta t}{2} V_j^\dagger(t_n) V_j(t_n) + \xi_{n,j} V_j(t_n) \right) \right) e^{-i\tilde{H}(t_n)\Delta t} \psi_{t_n}, \quad (70)$$

up to higher-order weak error terms.

5.1.2 Segmented evolution with ancilla refresh

As in the Lindblad simulation approach in Section 4.2, we exploit the light-cone property: for times up to $\tau = \Theta(1/K_{\max})$, boundary reflections remain outside the prefront window $\text{win} = \{0, 1, \dots, j_*\}$ (up to the controlled light-cone error, achieved by choosing a suitable tight-binding chain length M). We therefore partition the evolution into segments of length τ and perform an ancilla refresh at each segment boundary, in analogy with the Lindblad simulations. Concretely, the refresh is implemented as an isometry that (i) flags whether the ancilla lies in win (i.e., projects with Π_{win}) and (ii) conditionally applies a fixed ancilla-only isometry W_{win} mapping $|r_{\text{win}}\rangle / \|r_{\text{win}}\| \mapsto |r_h\rangle$. OAA involves two reflections: one about the “good” subspace (equivalently, $R_{\text{win}} := I - 2\Pi_{\text{win}}$) and one about the prepared flag/ancilla initialization subspace. Since the light-cone guarantee implies $P_{\text{win}} \geq 1/4$, for the relevant pre-refresh states, only $\mathcal{O}(1)$ OAA iterations are needed to boost the refresh success probability to a constant.

5.1.3 Estimating amplitudes

Because the linear SSE is not norm-preserving pathwise, we represent the unnormalized trajectory state by a normalized quantum state and a classical weight:

$$|\psi_{t_n}\rangle = \sqrt{\lambda_n} |\phi_n\rangle, \quad \|\phi_n\rangle\| = 1, \quad \lambda_n \geq 0. \quad (71)$$

Estimating the stepwise factors g_n at every fine step Δt is expensive. Instead, we *decouple* the estimation timescale from the integration timescale by estimating *products* of growth factors over coarse blocks. For notational alignment with the ancilla refresh in the previous

section, we take the block length to be one refresh segment, τ subject to (46). For simplicity, we choose $\tau = k\Delta t$.

Let $\mathcal{M}_n$ denote the one-step linear map of the chosen weak-1 integrator on $[t_n, t_{n+1}]$ for the presampled increment(s) at step n . Define the *segment map*

$$\mathcal{M}_m^{(\tau)} := \mathcal{M}_{mk\Delta t} \cdots \mathcal{M}_{((m-1)k+1)\Delta t},$$

and the *segment growth factor*

$$g_m := \|\mathcal{M}_m^{(\tau)} |\phi_{(m-1)k}\rangle\|^2. \quad (72)$$

Thus, g_m is exactly the product of the normalized stepwise growth factors inside segment m , but it can be estimated once per segment. In particular, the m th segment admits an implementation

$$U_m(|0\rangle \otimes |\phi_{(m-1)k}\rangle) = |0\rangle \otimes (\mathcal{M}_m^{(\tau)} |\phi_{(m-1)k}\rangle) + |1\rangle \otimes (\cdots), \quad (73)$$

where the flag success probability is exactly g_m . Applying (fixed-point) OAA to (73) with the known projector $|0\rangle\langle 0|$ prepares the normalized post-segment state

$$|\phi_{mk}^-\rangle = \frac{\mathcal{M}_m^{(\tau)} |\phi_{(m-1)k}\rangle}{\sqrt{g_m}}$$

with failure probability exponentially small in the number of OAA rounds [15, 48, 49]. The corresponding overhead is $\tilde{\mathcal{O}}(1/\sqrt{g_m})$ uses of U_m and $U_m^\dagger$ (typically constant when g_m fluctuates around 1).

Theorem 5 (Segmented trajectory generation with amplitude tracking). *Let Y_T denote the random output at time T produced by weak order 1 integrator with presampled Rademacher increments, implemented segment-wise, where each segment contains k inner steps.*

Assume:

(i) *For each segment m , it is implemented as,*

$$U_m(|0\rangle \otimes |\phi_{t_m}\rangle) = |0\rangle \otimes (\mathcal{M}_m^{(\tau)} |\phi_{t_m}\rangle) + |1\rangle \otimes (\cdots),$$

whose success probability is $g_m := \|\mathcal{M}_m^{(\tau)} |\phi_{t_m}\rangle\|^2$, and each call to U_m (or $U_m^\dagger$) costs $C_{\text{traj},\tau}$. The per-segment simulation error in the normalized post-segment state is at most $\mathcal{O}(\varepsilon/L)$.

(ii) *The light-cone/window condition holds so that $P_{\text{win}} \geq 1/4$, and the ancilla refresh at each segment boundary has failure probability and induced state error at most $\mathcal{O}(\varepsilon/L)$ (using $\mathcal{O}(1)$ rounds of fixed-point OAA).*

(iii) *Each segment growth factor g_m is estimated in-line (with coherent state restoration) to relative accuracy $\mathcal{O}(\varepsilon/L)$.*

Then the algorithm outputs, with constant success probability, a normalized final state $|\tilde{\phi}_T\rangle$ and a scalar $\tilde{\lambda}_T$ such that the reconstructed unnormalized trajectory

$$\tilde{Y}_T := \sqrt{\tilde{\lambda}_T} (\langle l_h | \otimes I) |\tilde{\phi}_T\rangle$$

satisfies the algorithmic pathwise error bound

$$\|\tilde{Y}_T - Y_T\| = \mathcal{O}(\varepsilon),$$

and, for any sufficiently smooth test functional f with polynomial growth, the total weak error obeys

$$|\mathbb{E}[f(\tilde{Y}_T)] - \mathbb{E}[f(X_T)]| \leq C_T \Delta t + \mathcal{O}(\varepsilon),$$

where C_T is the standard weak-1 constant.

Moreover, the total number of calls to the segment primitives $\{U_m, U_m^\dagger\}$ scales as

$$\tilde{\mathcal{O}}\left(\sum_{m=0}^{L-1} \frac{1}{\sqrt{g_m}} + \frac{L}{\varepsilon} \sum_{m=0}^{L-1} \frac{1}{g_m}\right),$$

and hence the total gate/query complexity is

$$\tilde{\mathcal{O}}\left(C_{\text{traj},\tau} \left[\sum_{m=0}^{L-1} \frac{1}{\sqrt{g_m}} + \frac{L}{\varepsilon} \sum_{m=0}^{L-1} \frac{1}{g_m}\right]\right),$$

up to an additional additive overhead $\tilde{\mathcal{O}}(L)$ for the L refresh operations (constant-factor in the regime $P_{\text{win}} = \Omega(1)$).

The first term $\sum_m \tilde{\mathcal{O}}(1/\sqrt{g_m})$ is the cost of *state propagation*: to realize the non-unitary segment map $\mathcal{M}_m^{(\tau)}$ coherently and output the *normalized* post-segment state $|\phi_{t_{m+1}}^-\rangle \propto \mathcal{M}_m^{(\tau)} |\phi_{t_m}\rangle$ without restarting, we apply fixed-point OAA to the segment heralding flag, which costs $\tilde{\mathcal{O}}(1/\sqrt{g_m})$ uses of $\{U_m, U_m^\dagger\}$ on segment m . The second term $\frac{L}{\varepsilon} \sum_m \mathcal{O}(1/g_m)$ is the cost of *weight tracking*.

5.2 Second-Order Weak Scheme via Weak Measurement

We develop a *weak order-2* one-step approximation for quantum trajectories of the linear Itô SSE with *scalar noise*

$$d|\psi_t\rangle = A(t)|\psi_t\rangle dt + V(t)|\psi_t\rangle dW_t, \quad t \in [0, T]. \quad (74)$$

Again, we consider a single noise channel, where $A(t) = -\frac{1}{2}V^\dagger(t)V(t)$. The coherent term from $\tilde{H}$ and multiple jump operators can be treated by generalizing Eq. (70) to a symmetric trotter splitting.

We emphasize that the goal is *weak* accuracy: the one-step map should reproduce expectations of smooth functionals up to $\mathcal{O}(\Delta t^3)$ *local* weak error (and hence $\mathcal{O}(\Delta t^2)$ *global* weak error).

Besides the Brownian increment $\Delta W_n := W_{t_{n+1}} - W_{t_n}$, weak order 2 requires the second Itô integral

$$\Delta Z_n := \int_{t_n}^{t_{n+1}} (W_s - W_{t_n}) ds = \int_{t_n}^{t_{n+1}} \int_{t_n}^s dW_r ds, \quad (75)$$

and the related iterated integrals [8]

$$I_{11,n} := \int_{t_n}^{t_{n+1}} \int_{t_n}^s dW_r dW_s = \frac{1}{2}((\Delta W_n)^2 - \Delta t), \quad I_{10,n} := \int_{t_n}^{t_{n+1}} \int_{t_n}^s dr dW_s = \Delta t \Delta W_n - \Delta Z_n. \quad (76)$$

The identity for $I_{11,n}$ follows from Itô isometry, while the relation for $I_{10,n}$ is a direct Itô integration by parts.

A weak Itô–Taylor expansion of order 2.0 [50] gives the one-step local expansion

$$\begin{aligned} |\psi_{n+1}\rangle = & \left[I + \Delta t A(t_n) + V(t_n) \Delta W_n + V(t_n)^2 I_{11,n} \right. \\ & + (\dot{V}(t_n) + V(t_n)A(t_n)) I_{10,n} + (A(t_n)V(t_n)) \Delta Z_n \\ & \left. + \frac{\Delta t^2}{2} (\dot{A}(t_n) + A(t_n)^2) \right] |\psi_n\rangle + \mathcal{O}_w(\Delta t^3), \end{aligned} \quad (77)$$

where $\mathcal{O}_w(\Delta t^3)$ denotes a remainder whose contribution to *weak* local error is $\mathcal{O}(\Delta t^3)$ under standard boundedness and regularity assumptions on $A(\cdot), V(\cdot)$. In particular, for smooth test functionals f one obtains a global weak error bound of the form

$$|\mathbb{E}[f(|\psi_T\rangle)] - \mathbb{E}[f(|\hat{\psi}_N\rangle)]| \leq C_T \Delta t^2, \quad (78)$$

with C_T depending on T and on uniform bounds for A, V and the derivatives required by the weak-2 theory [50].

To streamline subsequent circuit constructions, we evaluate coefficients at the midpoint $t_{n+\frac{1}{2}} := t_n + \Delta t/2$:

$$A_{\text{mid}} := A(t_{n+\frac{1}{2}}), \quad V_{\text{mid}} := V(t_{n+\frac{1}{2}}), \quad \dot{V}_{\text{mid}} := \dot{V}(t_{n+\frac{1}{2}}). \quad (79)$$

For example, for smooth $A(\cdot)$ we have the Taylor relation

$$A_{\text{mid}} = A(t_n) + \frac{\Delta t}{2} \dot{A}(t_n) + \mathcal{O}(\Delta t^2),$$

so the combination $\Delta t A(t_n) + \frac{\Delta t^2}{2} \dot{A}(t_n)$ appearing in (77) is *absorbed* into $\Delta t A_{\text{mid}}$ up to $\mathcal{O}(\Delta t^3)$. Moreover, replacing $A(t_n)$ by A_{mid} in the quadratic term $A(t_n)^2$ changes it only by $\mathcal{O}(\Delta t)$, hence contributes $\mathcal{O}(\Delta t^3)$ after multiplication by Δt^2 . Therefore, (77) can be rewritten (without an explicit $\dot{A}$ term) as the midpoint weak-2 expansion

$$\begin{aligned} |\psi_{n+1}\rangle = & \left[I + \Delta t A_{\text{mid}} + V_{\text{mid}} \Delta W_n + V_{\text{mid}}^2 I_{11,n} \right. \\ & + (\dot{V}_{\text{mid}} + V_{\text{mid}} A_{\text{mid}}) I_{10,n} + (A_{\text{mid}} V_{\text{mid}}) \Delta Z_n + \frac{\Delta t^2}{2} A_{\text{mid}}^2 \left. \right] |\psi_n\rangle + \mathcal{O}_w(\Delta t^3). \end{aligned} \quad (80)$$

This is the form we will discretize and then implement via weak measurement.

The pair $(\Delta W_n, \Delta Z_n)$ forms a centered Gaussian vector with the following covariance structure:

$$\text{Cov} \begin{pmatrix} \Delta W_n \\ \Delta Z_n \end{pmatrix} = \mathbb{E} \left[\begin{pmatrix} \Delta W_n \\ \Delta Z_n \end{pmatrix} \begin{pmatrix} \Delta W_n & \Delta Z_n \end{pmatrix} \right] = \begin{pmatrix} \Delta t & \frac{1}{2} \Delta t^2 \\ \frac{1}{2} \Delta t^2 & \frac{1}{3} \Delta t^3 \end{pmatrix}.$$

A convenient representation is obtained by Cholesky factorization and introducing independent standard normal variables $\xi_{1,n}, \xi_{2,n} \sim \mathcal{N}(0, 1)$ and setting

$$\Delta W_n = \sqrt{\Delta t} \xi_{1,n}, \quad \Delta Z_n = \frac{\Delta t^{3/2}}{2} \left(\xi_{1,n} + \frac{1}{\sqrt{3}} \xi_{2,n} \right). \quad (81)$$

As a result,

$$\begin{aligned} I_{10,n} &= \Delta t \Delta W_n - \Delta Z_n = \frac{\Delta t^{3/2}}{2} \left(\xi_{1,n} - \frac{1}{\sqrt{3}} \xi_{2,n} \right), \\ I_{11,n} &= \frac{1}{2} ((\Delta W_n)^2 - \Delta t) = \frac{\Delta t}{2} (\xi_{1,n}^2 - 1). \end{aligned} \quad (82)$$

Substituting (81)–(82) into (80) and collecting powers of Δt yields the compact local form

$$\begin{aligned} |\psi_{n+1}\rangle &= F_n^{(2)} |\psi_n\rangle, \\ F_n^{(2)} &:= I + \Delta t A_{\text{mid}} + \frac{\Delta t^2}{2} A_{\text{mid}}^2 + \sqrt{\Delta t} \xi_{1,n} V_{\text{mid}} + \frac{\Delta t}{2} (\xi_{1,n}^2 - 1) V_{\text{mid}}^2 \\ &\quad + \frac{\Delta t^{3/2}}{2} \xi_{1,n} (A_{\text{mid}} V_{\text{mid}} + \dot{V}_{\text{mid}} + V_{\text{mid}} A_{\text{mid}}) + \frac{\Delta t^{3/2}}{2\sqrt{3}} \xi_{2,n} (A_{\text{mid}} V_{\text{mid}} - \dot{V}_{\text{mid}} - V_{\text{mid}} A_{\text{mid}}). \end{aligned} \quad (83)$$

It is convenient to package the $\Delta t^{3/2}$ operators as

$$\begin{aligned} B_{\text{mid}} &:= \frac{1}{2} (A_{\text{mid}} V_{\text{mid}} + \dot{V}_{\text{mid}} + V_{\text{mid}} A_{\text{mid}}), \\ C_{\text{mid}} &:= \frac{1}{2\sqrt{3}} (A_{\text{mid}} V_{\text{mid}} - \dot{V}_{\text{mid}} - V_{\text{mid}} A_{\text{mid}}). \end{aligned} \quad (84)$$

On the other hand, for weak order 2 it is sufficient to replace the Gaussian $\xi_{1,n}, \xi_{2,n}$ by discrete random variables that match moments up to order 4. We therefore use the Kloeden–Platen three-point law:

$$\xi_{k,n} \in \{0, \pm\sqrt{3}\}, \quad \mathbb{P}(\xi_{k,n} = 0) = \frac{2}{3}, \quad \mathbb{P}(\xi_{k,n} = \pm\sqrt{3}) = \frac{1}{6}, \quad k \in \{1, 2\}, \quad (85)$$

so that $\mathbb{E}[\xi] = 0$, $\mathbb{E}[\xi^2] = 1$, $\mathbb{E}[\xi^3] = 0$, $\mathbb{E}[\xi^4] = 3$. Under (85), the squared variable $\xi_{1,n}^2$ takes only two values:

$$\xi_{1,n}^2 \in \{0, 3\}, \quad \mathbb{P}(\xi_{1,n}^2 = 0) = \frac{2}{3}, \quad \mathbb{P}(\xi_{1,n}^2 = 3) = \frac{1}{3}.$$

For circuit design it is convenient to treat this as an explicit “control variable” for the V_{mid}^2 term. We therefore introduce a mean-zero random variable,

$$\xi_{3,n} := \xi_{1,n}^2 - 1 \in \{-1, 2\}. \quad (86)$$

Note that $\xi_{3,n}$ is *centered* and *pairwise uncorrelated* with $\xi_{1,n}$: $\mathbb{E}[\xi_{3,n}] = 0$ and $\mathbb{E}[\xi_{1,n}\xi_{3,n}] = \mathbb{E}[\xi_{1,n}^3] - \mathbb{E}[\xi_{1,n}] = 0$, although $(\xi_{1,n}, \xi_{3,n})$ are not independent.

With this notation, the weak-2 one-step update can be written in a compact form:

$$F_n^{(2)} = I + \Delta t A_{\text{mid}} + \frac{\Delta t^2}{2} A_{\text{mid}}^2 + \sqrt{\Delta t} \xi_{1,n} (V_{\text{mid}} + \Delta t B_{\text{mid}}) + \Delta t^{3/2} \xi_{2,n} C_{\text{mid}} + \frac{\Delta t}{2} \xi_{3,n} V_{\text{mid}}^2, \quad (87)$$

where $B_{\text{mid}}, C_{\text{mid}}$ are given in (84). Eq. (87) is the target algebraic form for our second-order weak-measurement step.

The following theorem, as motivated by the construction in [40], summarizes the implementation of Eq. (87) using a repeated interaction scheme with two ancilla qubits.

Theorem 6 (Two-qubit weak-measurement realization of the weak-2 step). *Consider a two-qubit ancilla with basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$ and define the anti-Hermitian generator*

$$\Omega_n := \sum_{\alpha \in \{10, 01, 11\}} \left(|\alpha\rangle\langle 00| \otimes G_{\alpha,n} - |00\rangle\langle \alpha| \otimes G_{\alpha,n}^\dagger \right), \quad U_n := e^{\Omega_n}. \quad (88)$$

Let $V_{\text{mid}}, A_{\text{mid}}, B_{\text{mid}}, C_{\text{mid}}$ be as in (79) and (84), with the noise-block choice $A_{\text{mid}} = -\frac{1}{2} V_{\text{mid}}^\dagger V_{\text{mid}}$. Choose

$$\begin{aligned} G_{10,n} &\equiv G_{1,n} := \sqrt{\Delta t} V_{\text{mid}} + \Delta t^{3/2} \left(B_{\text{mid}} + \frac{1}{6} V_{\text{mid}} V_{\text{mid}}^\dagger V_{\text{mid}} \right), \\ G_{01,n} &\equiv G_{2,n} := \frac{\Delta t}{\sqrt{2}} V_{\text{mid}}^2, \\ G_{11,n} &\equiv G_{3,n} := \Delta t^{3/2} C_{\text{mid}}. \end{aligned} \quad (89)$$

For each presampled pair $(\xi_{1,n}, \xi_{2,n})$ drawn from (85), set $\xi_{3,n} := \xi_{1,n}^2 - 1$ and define the ancilla state

$$|m(\xi_{1,n}, \xi_{2,n})\rangle := \frac{1}{\alpha(\xi_{1,n}, \xi_{2,n})} \left(|00\rangle + \xi_{1,n} |10\rangle + \frac{\xi_{3,n}}{\sqrt{2}} |01\rangle + \xi_{2,n} |11\rangle \right), \quad \alpha(\xi_{1,n}, \xi_{2,n}) > 0. \quad (90)$$

Let the post-selected effective one-step map be

$$\tilde{F}_n(\xi_{1,n}, \xi_{2,n}) := \frac{\langle m(\xi_{1,n}, \xi_{2,n}) | U_n | 00 \rangle}{\langle m(\xi_{1,n}, \xi_{2,n}) | 00 \rangle}. \quad (91)$$

Then, for any system state $|\psi_n\rangle$,

$$\tilde{F}_n(\xi_{1,n}, \xi_{2,n}) |\psi_n\rangle = F_n^{(2)} |\psi_n\rangle + \mathcal{O}_w(\Delta t^3), \quad (92)$$

i.e., the update reproduces the weak Itô–Taylor step (87) with local weak error $\mathcal{O}(\Delta t^3)$.

We present the proof in [Section D](#).

Multiple channels and drift. Including the Hamiltonian drift and J channels can be done by operator symmetric splitting over one step:

$$\psi_{t_n+\Delta t} \approx e^{-i\tilde{H}(t_n)\Delta t/2} \left(\prod_{j=1}^J F_{n,j}^{(2)} \left(\frac{\Delta t}{2} \right) \right) \left(\prod_{j=0}^{J-1} F_{n,J-j}^{(2)} \left(\frac{\Delta t}{2} \right) \right) e^{-i\tilde{H}(t_n)\Delta t/2} \psi_{t_n}, \quad (93)$$

up to higher-order weak error terms. Here $F_{n,j}^{(2)}$ is the corresponding $F_n^{(2)}$ for each noise channel V_j in [Eq. \(87\)](#).

6 Numerical Simulations

In this section, we present numerical results that verify our quantum simulation framework through three representative tests.

In the first numerical test, we demonstrate the recovery of the non-unitary evolution operator for linear SDEs via moment-matching dilation in [Theorem 2](#). We consider the example in [[51](#), Example 5.2], a three-dimensional SDE system

$$d \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} -1 & 10 & 0 \\ 0 & -1 & 10 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} dt + \sum_{j=1}^3 \begin{pmatrix} \frac{\sigma}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{\sigma}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{\sigma}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} dW_t^j. \quad (94)$$

We apply the dilation using [Eqs. \(26\) and \(27\)](#), and we examine the choice of j_* used to post-select the solution, as highlighted by the condition (36) in [Theorem 3](#). Specifically, we simulate the dilated SDE (19) corresponding to [Eq. \(94\)](#) and then apply the localized readout

$$\frac{1}{\langle j_* | r_h \rangle} (\langle j_* | \otimes I)$$

to $|\psi_t\rangle$ to extract an approximation of X_t . [Fig. 1](#) shows that this localized projection yields a more accurate approximation over a longer time interval when $p_* = p_{j_*}$ is closer to the origin, consistent with the light-cone property encoded in (36).

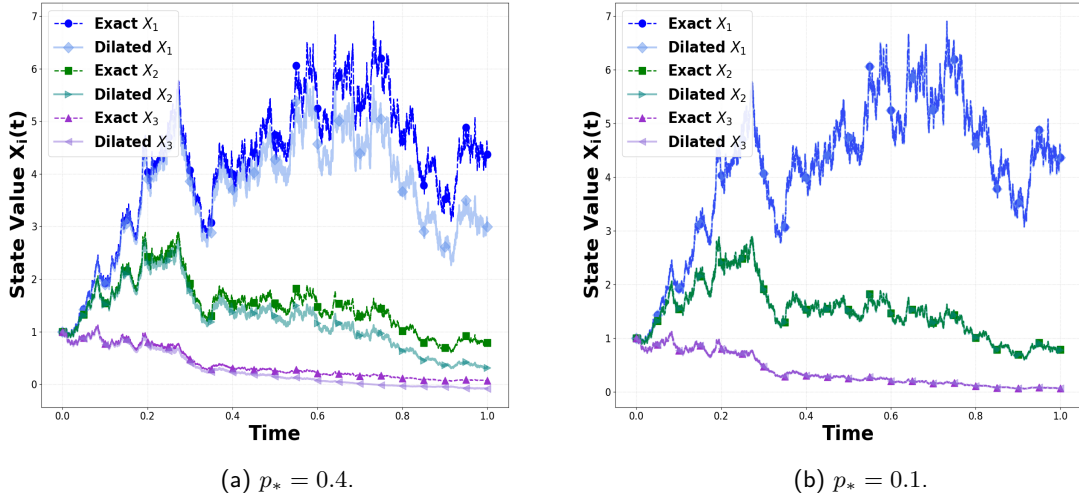


Figure 1: Moment-matching dilation for Eq. (94) with $\sigma = 1$. Both panels compare trajectories from Eq. (94) with the projected dilated SSE Eq. (19), driven by the same Brownian motion. Choosing an evaluation site closer to the origin improves the accuracy and extends the time horizon over which the projected dynamics remain reliable. In particular, the approximation deteriorates earlier for $p_* = 0.4$ (Fig. 1a) than for $p_* = 0.1$ (Fig. 1b).

In the second numerical test, we verify the expected error scaling of our second-order weak trajectory scheme (83) by two examples. We pick $B \in \mathbb{R}^{3 \times 3}$ from a randomly generated matrix,

$$B = \begin{pmatrix} -0.79312248 & 0.24057128 & -1.89632635 \\ 1.39577171 & 0.63829474 & -0.29204749 \\ -0.31194933 & 0.30383537 & -0.2676603 \end{pmatrix}.$$

We then consider the linear SDE,

$$dX_t = -\frac{1}{2}B^\dagger B X_t dt + B X_t dW_t, \quad X(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \quad (95)$$

which was discussed in Sections 5.1 and 5.2.

We do another numerical verification with two noise channels and a small drift term,

$$dX_t = A X_t dt + B_1 X_t dW_t^1 + B_2 X_t dW_t^2, \quad (96)$$

where

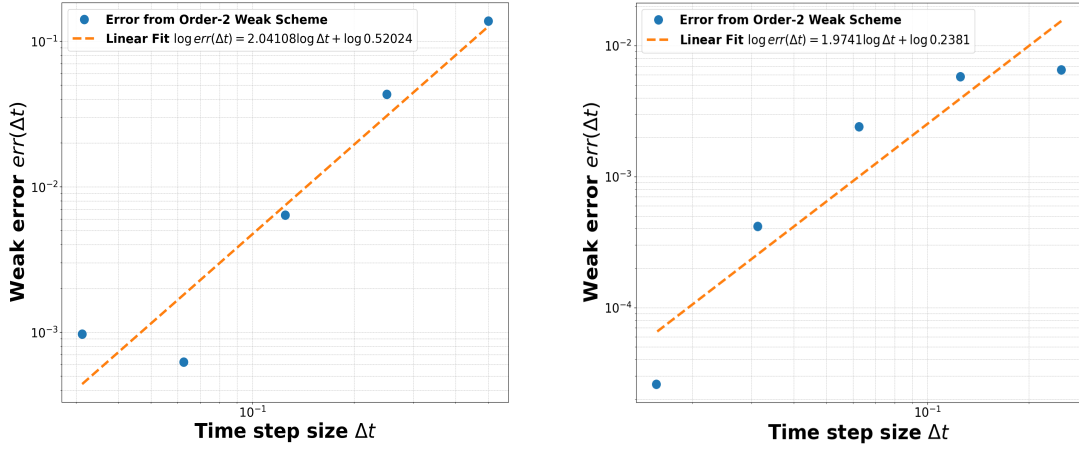
$$B_1 = \begin{pmatrix} -1.9250645 & -3.01879523 & 1.15446315 \\ 1.56278191 & 1.32951737 & -0.4295925 \\ -0.07288832 & -0.29154696 & 1.36824268 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0.87196735 & 0.05837772 & 0.70812363 \\ 1.36030603 & 0.37452505 & -1.08551679 \\ -0.07479491 & -0.42665655 & -1.59184729 \end{pmatrix},$$

$$A = 0.05 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} - \frac{1}{2}B_1^\dagger B_1 - \frac{1}{2}B_2^\dagger B_2.$$

To assess weak convergence, we test the smooth function $f(\mathbf{x}) = \cos(x_1 + x_2 + x_3^2)$ and define the weak error at final time T by

$$\text{err}(\Delta t) := |\mathbb{E}[f(Y_{T,\Delta t}) - \mathbb{E}[f(X_T)]]|,$$



(a) The weak convergence performance for single channel Eq. (95). (b) The weak convergence performance for two noise channels with a small drift term Eq. (96).

Figure 2: Weak error $\text{err}(\Delta t) = |\mathbb{E}[f(Y_{T,\Delta t}) - \mathbb{E}[f(X_T)]]|$ for the second-order scheme applied to Eq. (95) with $f(x) = \cos(x_1 + x_2 + x_3^2)$ at $T = 1.0$. The number of samples is $N_{\text{samp}} = 5000000$. In Fig. 2b, we implement the symmetric splitting for the Hamiltonian term $e^{-i\tilde{H}\Delta t}$, Eq. (93). In the dilation on the geometric grid Eq. (25), we use $M=50$, and $h = 2$.

where $Y_{T,\Delta t}$ is computed by the proposed second-order weak scheme (87) with step size Δt , and X_T is a reference solution computed with the Euler–Maruyama method using a much smaller step $\delta t = 2^{-14}$ over 10^7 realizations. Fig. 2 plots $\text{err}(\Delta t)$ versus Δt on a log–log scale. A linear fit of the data reveals an empirical rate close to 2, confirming the global weak second-order convergence predicted by the analysis.

In the last numerical test, we validate the recovery of the second moment for a stochastic PDE (SPDE) after dilation and transformation into a Lindblad equation (Lemma 1). Specifically, we consider an Itô SPDE, a stochastic advection-diffusion-reaction equation in [52, Example 4.2], on $(0, T] \times (0, 2\pi)$ with periodic boundary conditions:

$$du = \left[\left(\varepsilon + \frac{1}{2} \sigma_1^2 \cos^2(x) \right) \partial_x^2 u + \left(\beta \sin(x) - \frac{1}{4} \sigma_1^2 \sin(2x) \right) \partial_x u \right] dt + \sigma_1 \cos(x) \partial_x u dW_t^1 + \sigma_2 u dW_t^2, \quad u(x, 0) = \sin(x). \quad (97)$$

Let $D_1, D_2 \in \mathbb{R}^{N \times N}$ be the first- and second-order finite difference discretizations using central differences on N grid points. Denoting the semi-discrete solution vector by $X_t \in \mathbb{R}^N$, the SPDE reduces to a linear system of multiplicative-noise SDEs,

$$dX_t = AX_t dt + B_1 X_t dW_t^1 + B_2 X_t dW_t^2, \quad (98)$$

where

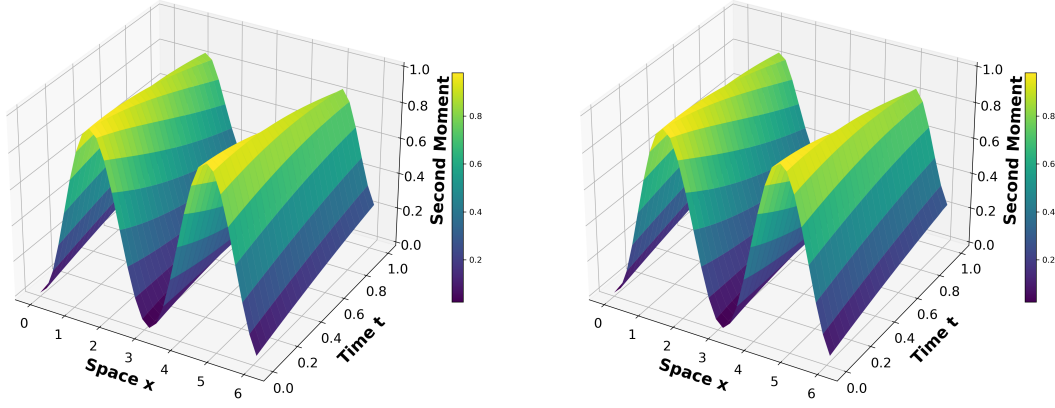
$$A = \text{diag} \left(\varepsilon + \frac{1}{2} \sigma_1^2 \cos^2(x) \right) D_2 + \text{diag} \left(\beta \sin(x) - \frac{1}{4} \sigma_1^2 \sin(2x) \right) D_1, \quad (99)$$

$$B_1 = \text{diag}(\sigma_1 \cos(x)) D_1, \quad (100)$$

$$B_2 = \sigma_2 I. \quad (101)$$

The associated second-moment equation for Eq. (98) is

$$\frac{d\Sigma_t}{dt} = A\Sigma_t + \Sigma_t A^\dagger + B_1 \Sigma_t B_1^\dagger + B_2 \Sigma_t B_2^\dagger, \quad \Sigma_t := \mathbb{E}[X_t X_t^\dagger]. \quad (102)$$



(a) Reference second moment obtained from the closed second-moment equation in Eq. (102). (b) Recovered second moment using the dilation-based Lindblad simulation.

Figure 3: Recovery of quadratic statistics for the SPDE (97). Parameters are $\varepsilon = 0.1$, $\beta = 0.5$, $\sigma_1 = 0.5$, $\sigma_2 = 0.3$, and $T = 1.0$. The evaluation site is $p_* = 5 \times 10^{-6}$. The recovered second moment closely matches the reference solution over the simulated time interval.

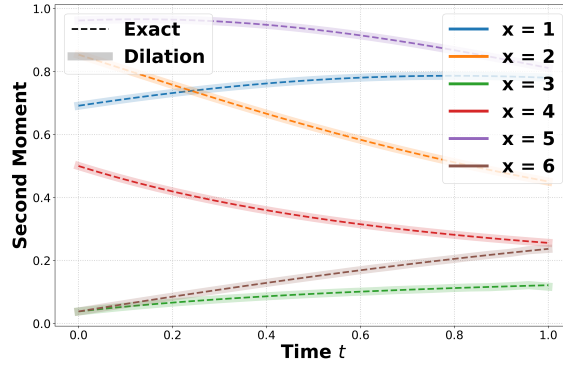


Figure 4: Pointwise comparison of the second moment obtained from the second-moment equation and from the dilation-based recovery. Here $p_* = 5 \times 10^{-6}$. The agreement improves as p_* decreases, consistent with the localization effect of the projection.

We use Eq. (102) as a deterministic reference, and then recover the same quadratic statistics (41) using the dilation-based Lindblad simulation described in Lemma 1. Figs. 3 and 4 show that the second moment of the SPDE can be accurately recovered by solving ρ_t in (40), as expected.

7 Summary and discussions

This work presents an exact mapping from general linear stochastic differential equations (SDEs) to stochastic Schrödinger equations (SSEs) through a finite-dimensional dilation framework. A key structural feature is that the dilation can be realized by a nearest-neighbor (tight-binding) hopping operator on the ancilla register, so the resulting circuits are ancilla-efficient and hardware-friendly: the nontrivial ancilla dynamics reduce to local couplings (plus simple boundary terms), and the overall implementation admits a streamlined gate-level construction.

More broadly, it provides a coherent route for recasting classical stochastic dynamics

into quantum-native primitives. By representing the same underlying SDE either as (i) a deterministic Lindblad evolution governing ensemble moments, or as (ii) a measurement-driven quantum-trajectory process generating pathwise realizations, we obtain complementary algorithmic building blocks for stochastic simulation, filtering, data assimilation, forecasting, and sampling. In regimes where these tasks are bottlenecked by repeated propagation of trajectories or by the evolution of high-dimensional moments, quantum implementations can accelerate the dominant inner loops by enabling long-time propagation through structured dilations and by permitting direct estimation of application-specific observables from the prepared quantum state.

A notable application is the simulation of open quantum systems in the *non-Markovian* regime. Several established trajectory-based methodologies represent non-Markovian effects by embedding the dynamics into an extended (often higher-dimensional) stochastic model, including hierarchical constructions and Markovian embeddings [53–55]. Our dilation framework interfaces naturally with these embeddings: once the dynamics are expressed in an extended linear SDE form, the corresponding second-moment evolution is deterministically captured by a Lindblad equation on the dilated space. This avoids the explicit generation of individual stochastic realizations when only ensemble-level quantities are required, while still retaining the ability to recover trajectory-level information via the SSE route when needed.

The present paper focuses on Brownian-driven dynamics. Extending the framework to SDEs driven by jump processes, or more generally by Lévy noise, is a natural next step and will be pursued in future work. Finally, we note that the trajectory-generation capability (Algorithm II) is particularly relevant to the sampling bottleneck in modern generative models. By mapping the reverse-time SDEs of diffusion models to a dilated quantum evolution, our framework provides a rigorous pathway to accelerate the sampling of high-dimensional distributions beyond the capabilities of classical solvers. To translate this framework into a practical quantum advantage for applications such as diffusion-model sampling, several additional ingredients remain to be developed. These include efficient representations of the problem-specific drift and noise operators, optimized implementations of the resulting Lindblad and trajectory simulators, and a careful end-to-end complexity analysis comparing the full quantum workflow against state-of-the-art classical solvers. We view the current work as providing the mathematical and algorithmic foundation upon which such application-specific quantum accelerations can be built.

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A Proof of the exact dilation

Proof. We begin with the integral form of the SDE for X_t :

$$X_t = X_0 + \int_0^t A(s)X_s ds + \sum_{j=1}^J \int_0^t B_j(s)X_s dW_s^j. \quad (103)$$

Introducing the notation $B_0(t) = A(t)$, $dZ_t^0 = dt$, and $dZ_t^j = dW_t^j$ for $j = 1, \dots, J$, we rewrite this compactly as:

$$X_t = X_0 + \sum_{j=0}^J \int_0^t B_j(s) X_s dZ_s^j. \quad (104)$$

Iterating this integral equation yields the Dyson series expansion:

$$X_t = \sum_{k=0}^{\infty} \left(\sum_{\alpha \in \{0, \dots, J\}^k} \int_0^t \cdots \int_0^{s_2} B_{\alpha_k}(s_k) \cdots B_{\alpha_1}(s_1) X_0 dZ_{s_1}^{\alpha_1} \cdots dZ_{s_k}^{\alpha_k} \right). \quad (105)$$

Under the boundedness assumptions on A, B_j , this series converges in $L^2(\Omega)$ uniformly on compact time intervals [50].

Similarly, the dilated state $|\psi_t\rangle$ evolves according to Eq. (19) starting from $|\psi_0\rangle = |r\rangle \otimes X_0$. Its Dyson expansion is:

$$|\psi_t\rangle = \sum_{k=0}^{\infty} \left(\sum_{\alpha \in \{0, \dots, J\}^k} \int_0^t \cdots \int_0^{s_2} V_{\alpha_k}(s_k) \cdots V_{\alpha_1}(s_1) (|r\rangle \otimes X_0) dZ_{s_1}^{\alpha_1} \cdots dZ_{s_k}^{\alpha_k} \right). \quad (106)$$

We now apply the operator $(\langle l| \otimes I_{\mathcal{A}})$ to the series Eq. (106). By linearity, it enters the sum and integrals. To evaluate the term-wise action, we separate the vector X_0 from the ancilla $|r\rangle$ using the identity $|r\rangle \otimes X_0 = (|r\rangle \otimes I) X_0$:

$$(\langle l| \otimes I) (V_{\alpha_k} \cdots V_{\alpha_1}) (|r\rangle \otimes X_0) = (\langle l| \otimes I) (V_{\alpha_k} \cdots V_{\alpha_1}) (|r\rangle \otimes I) X_0. \quad (107)$$

Now we use the properties of the moment-matching triple. The sandwiching of the dilated operators yields the original operators:

$$(\langle l| \otimes I) (I_{\mathcal{A}} \otimes B_j(t)) (|r\rangle \otimes I) = \langle l| I_{\mathcal{A}} |r\rangle \otimes B_j(t) = B_j(t), \quad (108)$$

$$(\langle l| \otimes I) V_0(t) (|r\rangle \otimes I) = \langle l| (F - I_{\mathcal{A}}) |r\rangle \otimes K + \langle l|r\rangle \otimes A = A(t). \quad (109)$$

Applying this recursively to the product sequence, we obtain,

$$[(\langle l| \otimes I) V_{\alpha_k} \cdots V_{\alpha_1} (|r\rangle \otimes I)] X_0 = (B_{\alpha_k} \cdots B_{\alpha_1}) X_0. \quad (110)$$

Substituting this back into the expansion Eq. (106), we see that the projection of the quantum state series is exactly Eq. (105). $\square$

B Proof of the light cone property

Proof. We proceed in four steps: (1) defining the error dynamics as a driven SSE, (2) expanding the solution using a Dyson series, (3) identifying the non-vanishing terms based on grid locality, and (4) estimating the magnitude of the stochastic integrals.

A crucial property of any SSE (19) is that it preserves the norm of the state *on average*. If $\mathcal{E}(t, s)$ is the propagator for the homogeneous part, then for any state ϕ :

$$\mathbb{E}[\|\mathcal{E}(t, s)\phi\|^2] = \|\phi\|^2. \quad (111)$$

This "mean-square unitarity" simplifies our analysis significantly, as we do not need to worry about the stability of the background evolution.

To isolate spatial propagation, we split the homogeneous drift into a “local” part and the nearest-neighbor hopping term,

$$\tilde{B}_0(t) = \tilde{B}_0^{(0)}(t) + V(t), \quad V(t) := \theta F_h \otimes K(t),$$

where $\tilde{B}_0^{(0)}(t)$ is diagonal in the ancilla basis (and contains the local drift together with the Itô correction), and $V(t)$ is the only term that transports amplitude along the tight-binding chain.

Let $\mathcal{E}_0(t, s)$ denote the propagator of the *local* homogeneous SDE obtained by setting $V \equiv 0$ (i.e., keeping $\tilde{B}_0^{(0)}$ and all noise terms $\tilde{B}_j$). Then the full propagator $\mathcal{E}(t, s)$, using the variation of constants for SDEs [56, Theorem 3.1] repeatedly, admits a Dyson–Duhamel expansion in V , a sum of nested time-ordered integrals

$$|\chi_T\rangle = \sum_{k=0}^{\infty} |\chi_T^{(k)}\rangle, \quad (112)$$

with the k -hop contribution given by the $(k+1)$ -simplex integral

$$|\chi_T\rangle^{(k)} = \int_{0 < s < t_1 < \dots < t_k < T} \mathcal{E}_0(T, t_k) V(t_k) \mathcal{E}_0(t_k, t_{k-1}) \cdots V(t_1) \mathcal{E}_0(t_1, s) |S_s\rangle \, ds \, dt_1 \cdots dt_k. \quad (113)$$

Equivalently, one may write the same expression as an iterated integral:

$$|\chi_T\rangle^{(k)} = \int_0^T \left(\int_s^T \cdots \int_{t_{k-1}}^T \mathcal{E}_0(T, t_k) V(t_k) \mathcal{E}_0(t_k, t_{k-1}) \cdots V(t_1) \mathcal{E}_0(t_1, s) \, dt_k \cdots dt_1 \right) |S_s\rangle \, ds. \quad (114)$$

In Eq. (113)–Eq. (114), $\mathcal{E}_0$ is the propagator of the “local” dynamics (diagonal in the ancilla basis), which includes $\tilde{B}_0^{(0)}(t)$ and all noise terms $\tilde{B}_j(t) dW_t^j$.

We are interested in the projection $(\langle j_* | \otimes I) |\chi_T\rangle$. On our nearest-neighbor grid, the hopping operator $\mathcal{V}$ can move an excitation by at most one site. The source $|S_s\rangle$ starts at site M . To reach site j_* , we must apply $\mathcal{V}$ at least $m = M - j_*$ times. Therefore, all terms in the Dyson series with order $k < m$ vanish identically. We only need to sum terms with $k \geq m$.

For $k \geq m = M - j_*$, recall the simplex representation (113):

$$|\chi_T\rangle^{(k)} = \int_{0 < s < t_1 < \dots < t_k < T} \mathcal{E}_0(T, t_k) V(t_k) \mathcal{E}_0(t_k, t_{k-1}) \cdots V(t_1) \mathcal{E}_0(t_1, s) |S_s\rangle \, ds \, dt_1 \cdots dt_k.$$

Fix $(s, t_1, \dots, t_k)$ and define the random vector

$$|Y(s, t_1, \dots, t_k)\rangle := \mathcal{E}_0(T, t_k) V(t_k) \mathcal{E}_0(t_k, t_{k-1}) \cdots V(t_1) \mathcal{E}_0(t_1, s) |S_s\rangle.$$

We bound $\mathbb{E} [\| |\chi_T\rangle^{(k)} \|^2]$ by applying (i) Cauchy-Schwarz in the time variables and (ii) the mean-square isometry of $\mathcal{E}_0(\cdot, \cdot)$ *stepwise* using conditional expectations.

First, since $\mathcal{E}_0(t, u)$ is the propagator of the homogeneous “local” SSE (with drift in the SSE form and noise terms $\tilde{B}_j$), it is mean-square norm preserving in the following conditional sense: for $u \leq t$ and any $\mathcal{F}_u$ -measurable random vector ζ ,

$$\mathbb{E} [\| \mathcal{E}_0(t, u) \zeta \|^2 \mid \mathcal{F}_u] = \|\zeta\|^2 \quad \text{a.s.} \quad (115)$$

(Equivalently, $\mathbb{E}\|\mathcal{E}_0(t, u)\zeta\|^2 = \mathbb{E}\|\zeta\|^2$ by taking expectations.) Identity (115) follows from Itô's formula applied to $\|\eta_t\|^2$ for the homogeneous local SSE and the fact that the Brownian increments on $(u, t]$ are independent of $\mathcal{F}_u$.

Next, we simplify the nested integral using this isometry. Define

$$\zeta_k := V(t_k)\mathcal{E}_0(t_k, t_{k-1}) \cdots V(t_1)\mathcal{E}_0(t_1, s) |S_s\rangle,$$

which is $\mathcal{F}_{t_k}$ -measurable. Applying (115) with $(t, u) = (T, t_k)$ yields

$$\mathbb{E}\left[\|Y(s, t_1, \dots, t_k)\|^2 \mid \mathcal{F}_{t_k}\right] = \mathbb{E}\left[\|\mathcal{E}_0(T, t_k)\zeta_k\|^2 \mid \mathcal{F}_{t_k}\right] = \|\zeta_k\|^2.$$

Taking expectations gives

$$\mathbb{E}\|Y(s, t_1, \dots, t_k)\|^2 = \mathbb{E}\|\zeta_k\|^2.$$

Now define $\zeta_{k-1} := V(t_{k-1})\mathcal{E}_0(t_{k-1}, t_{k-2}) \cdots V(t_1)\mathcal{E}_0(t_1, s) |S_s\rangle$, which is $\mathcal{F}_{t_{k-1}}$ -measurable. Since $\zeta_k = V(t_k)\mathcal{E}_0(t_k, t_{k-1})\zeta_{k-1}$, taking conditional expectation with respect to $\mathcal{F}_{t_{k-1}}$ gives us

$$\begin{aligned} \mathbb{E}\|\zeta_k\|^2 &= \mathbb{E}\left[\mathbb{E}\left(\|V(t_k)\mathcal{E}_0(t_k, t_{k-1})\zeta_{k-1}\|^2 \mid \mathcal{F}_{t_{k-1}}\right)\right] \\ &\leq \|V(t_k)\|^2 \mathbb{E}\left[\mathbb{E}\left(\|\mathcal{E}_0(t_k, t_{k-1})\zeta_{k-1}\|^2 \mid \mathcal{F}_{t_{k-1}}\right)\right] \\ &= \|V(t_k)\|^2 \mathbb{E}\|\zeta_{k-1}\|^2, \end{aligned}$$

where in the last line we used (115). Iterating this argument yields the “propagator peeling” bound

$$\mathbb{E}\|Y(s, t_1, \dots, t_k)\|^2 \leq \left(\prod_{\ell=1}^k \|V(t_\ell)\|^2\right) \mathbb{E}\|\mathcal{E}_0(t_1, s) |S_s\rangle\|^2. \quad (116)$$

Finally, applying (115) once more with $(t, u) = (t_1, s)$ gives $\mathbb{E}\|\mathcal{E}_0(t_1, s) |S_s\rangle\|^2 = \mathbb{E}\| |S_s\rangle\|^2$. Therefore,

$$\mathbb{E}\|Y(s, t_1, \dots, t_k)\|^2 \leq \left(\prod_{\ell=1}^k \|V(t_\ell)\|^2\right) \mathbb{E}\| |S_s\rangle\|^2. \quad (117)$$

Finally, using $\|V(t)\| \leq \theta \|F_h\| \|K(t)\| \leq \theta 2C_{grid} K_{\max}$, where

$$C_{grid} := \sup_{h \geq 1} \frac{\sqrt{1 + e^h}}{4 \sinh(h/2)} < 1.$$

With direct computation, we can show that $\mathbb{E}\| |S_s\rangle\|^2 \leq 2K_{\max}^2 X(T)$ assuming

$$X(T) := \sup_{0 \leq t \leq T} \mathbb{E}\|X_t\|^2, \quad (118)$$

can come from the stability estimates [Proposition 1](#).

Inequality (117) implies

$$\mathbb{E}\|Y(s, t_1, \dots, t_k)\|^2 \leq (\theta C_{grid} K_{\max})^{2k} K_{\max}^2 X(T).$$

Applying Cauchy–Schwarz in the time variables gives

$$\begin{aligned}\mathbb{E}\|\chi_T^{(k)}\|^2 &= \mathbb{E}\left\|\int_{\Delta_{k+1}(T)} |Y(s, t_1, \dots, t_k)\rangle d(s, t_1, \dots, t_k)\right\|^2 \\ &\leq |\Delta_{k+1}(T)| \int_{\Delta_{k+1}(T)} \mathbb{E}\|Y\|^2 d(s, t_1, \dots, t_k),\end{aligned}$$

where $\Delta_{k+1}(T) = \{0 < s < t_1 < \dots < t_k < T\}$ and $|\Delta_{k+1}(T)| = T^{k+1}/(k+1)!$. Hence

$$\mathbb{E}\|\chi_T^{(k)}\|^2 \leq K_{\max}^2 X(T) (\theta C_{\text{grid}} K_{\max})^{2k} \left(\frac{T^{k+1}}{(k+1)!}\right)^2. \quad (119)$$

Using Stirling’s approximation $(k+1)! \geq (\frac{k+1}{e})^{k+1}$, we obtain the geometric decay for $k \geq m$:

$$\mathbb{E}\|\chi_T^{(k)}\|^2 \leq K_{\max}^2 X(T) \left(\frac{e \theta C_{\text{grid}} K_{\max} T}{k+1}\right)^{2k} \cdot \left(\frac{T}{k+1}\right)^2 \lesssim K_{\max}^2 X(T) \varrho^{2k},$$

where $\varrho := \frac{e \theta C_{\text{grid}} K_{\max} T}{m}$.

The different hopping orders $|\chi_T^{(k)}\rangle$ are not necessarily orthogonal in $L^2(\Omega)$, so we do not sum the squared norms directly. Instead, we apply Minkowski’s inequality before squaring:

$$\left(\mathbb{E}\|\chi_T\|^2\right)^{1/2} = \left(\mathbb{E}\left\|\sum_{k=m}^{\infty} |\chi_T^{(k)}\rangle\right\|^2\right)^{1/2} \leq \sum_{k=m}^{\infty} \left(\mathbb{E}\|\chi_T^{(k)}\|^2\right)^{1/2}.$$

Using the per-order estimate above gives

$$\left(\mathbb{E}\|\chi_T\|^2\right)^{1/2} \lesssim K_{\max} \sqrt{X(T)} \sum_{k=m}^{\infty} \varrho^k = K_{\max} \sqrt{X(T)} \frac{\varrho^m}{1-\varrho}.$$

Therefore,

$$\mathbb{E}\|\chi_T\|^2 \leq \mathcal{C} \frac{\varrho^{2m}}{(1-\varrho)^2},$$

for a constant $\mathcal{C}$ depending on $K_{\max}$, $X(T)$, T , and grid boundary constants. This proves the claimed light-cone decay without assuming orthogonality between different hopping orders. $\square$

C Proof of the light-cone property for the covariance

Proof. Let $|\phi_t\rangle = |r_h\rangle \otimes X_t$ be the ideal dilated state (solution to the SDE embedded in the dilated space). Let $|\psi_t\rangle$ be the actual dilated state evolving under the dilated SSE. The error state is defined as $|\chi_t\rangle = |\psi_t\rangle - |\phi_t\rangle$. We aim to bound the trace distance between the actual projected density matrix, $\rho_T^{(j^*)}$, and the ideal covariance block, $|\gamma|^2 \Sigma_T$.

The full density matrix is defined as $\rho_T = \mathbb{E}|\psi_T\rangle \langle \psi_T|$. Substituting $|\psi_T\rangle = |\phi_T\rangle + |\chi_T\rangle$, we expand the outer product:

$$\begin{aligned}\rho_T &= \mathbb{E}(|\phi_T\rangle + |\chi_T\rangle)(\langle \phi_T| + \langle \chi_T|) \\ &= \underbrace{\mathbb{E}|\phi_T\rangle \langle \phi_T|}_{\text{Ideal Term}} + \underbrace{\mathbb{E}|\phi_T\rangle \langle \chi_T| + \mathbb{E}|\chi_T\rangle \langle \phi_T|}_{\text{Cross Terms}} + \underbrace{\mathbb{E}|\chi_T\rangle \langle \chi_T|}_{\text{Quadratic Error}}\end{aligned} \quad (120)$$

Let $\mathcal{P}_{j_*} := |j_*\rangle\langle j_*| \otimes I$ be the projection operator onto the ancilla site j_* . We analyze the projection of each term separately.

We first look at the ideal term. Recalling that $|\phi_T\rangle = |r_h\rangle \otimes X_T$ and $\gamma := \langle j_* | r_h \rangle$. We have

$$\mathcal{P}_{j_*} \mathbb{E} |\phi_T\rangle \langle \phi_T | \mathcal{P}_{j_*}^\dagger = \mathbb{E} (\langle j_* | r_h \rangle X_T) (\langle j_* | r_h \rangle X_T^\dagger) = |\gamma|^2 \mathbb{E} X_T X_T^\dagger = |\gamma|^2 \Sigma_T.$$

Now we evaluate the bound for the error terms. Let the projected state vectors be denoted as $|\phi\rangle_{j_*} := \mathcal{P}_{j_*} |\phi_T\rangle$ and $|\chi\rangle_{j_*} := \mathcal{P}_{j_*} |\chi_T\rangle$. The error in the projected density matrix is:

$$\rho_T^{(j_*)} - |\gamma|^2 \Sigma_T = \mathbb{E} |\phi\rangle_{j_*} \langle \chi|_{j_*} + \mathbb{E} |\chi\rangle_{j_*} \langle \phi|_{j_*} + \mathbb{E} |\chi\rangle_{j_*} \langle \chi|_{j_*}. \quad (121)$$

We apply the trace norm $\|\cdot\|_1$ and the triangle inequality:

$$\left\| \rho_T^{(j_*)} - |\gamma|^2 \Sigma_T \right\|_1 \leq 2 \left\| \mathbb{E} |\phi\rangle_{j_*} \langle \chi|_{j_*} \right\|_1 + \left\| \mathbb{E} |\chi\rangle_{j_*} \langle \chi|_{j_*} \right\|_1. \quad (122)$$

We obtain the following inequality by using Jensen's inequality and the property that $\| |u\rangle \langle v| \|_1 = \|u\| \|v\|$,

$$\left\| \mathbb{E} |\phi\rangle_{j_*} \langle \chi|_{j_*} \right\|_1 \leq \mathbb{E} \left\| |\phi\rangle_{j_*} \langle \chi|_{j_*} \right\|_1 = \mathbb{E} \left[\left\| |\phi\rangle_{j_*} \right\| \cdot \left\| |\chi\rangle_{j_*} \right\| \right]. \quad (123)$$

We further apply the Cauchy-Schwarz inequality for expectations,

$$\mathbb{E} \left[\left\| |\phi\rangle_{j_*} \right\| \cdot \left\| |\chi\rangle_{j_*} \right\| \right] \leq \sqrt{\mathbb{E} \left\| |\phi\rangle_{j_*} \right\|^2} \cdot \sqrt{\mathbb{E} \left\| |\chi\rangle_{j_*} \right\|^2}. \quad (124)$$

Note that $\left\| |\phi\rangle_{j_*} \right\| = |\gamma| \|X_T\|$, combining with the result from [Theorem 3](#), we get

$$\sqrt{\mathbb{E} \left\| |\psi\rangle_{j_*} \right\|^2} = |\gamma| \sqrt{\mathbb{E} \|X_T\|^2} = |\gamma| \sqrt{\text{tr}(\Sigma_T)}, \quad (125)$$

$$\sqrt{\mathbb{E} \left\| |\chi\rangle_{j_*} \right\|^2} \leq \sqrt{\mathcal{C} \frac{\varrho^{2m}}{(1-\varrho)^2}}. \quad (126)$$

Thus, the contribution of the cross terms is bounded by

$$2 \left\| \mathbb{E} |\phi\rangle_{j_*} \langle \chi|_{j_*} \right\|_1 \leq 2 \cdot |\gamma| \sqrt{\text{tr}(\Sigma_T)} \sqrt{\mathcal{C} \frac{\varrho^{2m}}{(1-\varrho)^2}}. \quad (127)$$

Since $|\chi\rangle_{j_*} \langle \chi|_{j_*}$ is positive semi-definite, its trace norm equals its trace, we obtain

$$\left\| \mathbb{E} |\chi\rangle_{j_*} \langle \chi|_{j_*} \right\|_1 = \text{tr} \left(\mathbb{E} |\chi\rangle_{j_*} \langle \chi|_{j_*} \right) = \mathbb{E} \left[\text{tr} \left(|\chi\rangle_{j_*} \langle \chi|_{j_*} \right) \right] = \mathbb{E} \left[\left\| |\chi\rangle_{j_*} \right\|^2 \right] \leq \mathcal{C} \frac{\varrho^{2m}}{(1-\varrho)^2}. \quad (128)$$

The proof is completed by summing up the bounds in [Eq. \(127\)](#) and [Eq. \(128\)](#). $\square$

D Proof of the weak order 2 dilation

Proof. Write $K := \Omega_n$, $U := U_n$, and abbreviate $G_\alpha := G_{\alpha,n}$. A direct computation using orthogonality of $|10\rangle, |01\rangle, |11\rangle$ yields, for any $|\psi\rangle$,

$$K(|00\rangle \otimes |\psi\rangle) = \sum_{\alpha \in \{10,01,11\}} |\alpha\rangle \otimes (G_\alpha |\psi\rangle), \quad (129)$$

$$K^2(|00\rangle \otimes |\psi\rangle) = -|00\rangle \otimes (R^2 |\psi\rangle), \quad R^2 := \sum_{\alpha \in \{10,01,11\}} G_\alpha^\dagger G_\alpha. \quad (130)$$

Iterating gives the standard even/odd pattern $K^{2q}(|00\rangle \otimes |\psi\rangle) = (-1)^q |00\rangle \otimes (R^{2q} |\psi\rangle)$ and $K^{2q+1}(|00\rangle \otimes |\psi\rangle) = (-1)^q \sum_{\alpha} |\alpha\rangle \otimes (G_{\alpha} R^{2q} |\psi\rangle)$. Therefore the first column of $U = e^K$ admits the truncated series

$$U(|00\rangle \otimes |\psi\rangle) = |00\rangle \otimes \Omega |\psi\rangle + \sum_{\alpha \in \{10,01,11\}} |\alpha\rangle \otimes (G_{\alpha} \Lambda |\psi\rangle) + \mathcal{R} |\psi\rangle, \quad (131)$$

where $\Omega := I - \frac{1}{2}R^2 + \frac{1}{24}R^4$, $\Lambda := I - \frac{1}{6}R^2$, and $\|\mathcal{R}\| = \mathcal{O}(\Delta t^{5/2})$ because $\|G_1\| = \mathcal{O}(\Delta t^{1/2})$, $\|G_2\| = \mathcal{O}(\Delta t)$, $\|G_3\| = \mathcal{O}(\Delta t^{3/2})$.

Now fix presampled (ξ_1, ξ_2) and set $\xi_3 = \xi_1^2 - 1$, $|m\rangle := |m(\xi_1, \xi_2)\rangle$. By construction,

$$\frac{\langle m|10\rangle}{\langle m|00\rangle} = \xi_1, \quad \frac{\langle m|01\rangle}{\langle m|00\rangle} = \frac{\xi_3}{\sqrt{2}}, \quad \frac{\langle m|11\rangle}{\langle m|00\rangle} = \xi_2. \quad (132)$$

Project Eq. (131) with $\langle m|$ and divide by $\langle m|00\rangle$ to obtain

$$\widetilde{K} = \Omega + \xi_1 G_1 \Lambda + \frac{\xi_3}{\sqrt{2}} G_2 \Lambda + \xi_2 G_3 \Lambda + \mathcal{O}_w(\Delta t^3). \quad (133)$$

It remains to expand each term to the weak-2 relevant orders. First, $R^2 = G_1^{\dagger} G_1 + \mathcal{O}(\Delta t^2) = \Delta t V_{\text{mid}}^{\dagger} V_{\text{mid}} + \mathcal{O}(\Delta t^2)$, hence $\Omega = I - \frac{\Delta t}{2} V_{\text{mid}}^{\dagger} V_{\text{mid}} + \mathcal{O}(\Delta t^2) = I + \Delta t A_{\text{mid}} + \mathcal{O}(\Delta t^2)$, and $\Lambda = I - \frac{\Delta t}{6} V_{\text{mid}}^{\dagger} V_{\text{mid}} + \mathcal{O}(\Delta t^2)$. Using the choice (89), we then obtain

$$\begin{aligned} \xi_1 G_1 \Lambda &= \xi_1 \left(\sqrt{\Delta t} V_{\text{mid}} + \Delta t^{3/2} (B_{\text{mid}} + \frac{1}{6} V_{\text{mid}} V_{\text{mid}}^{\dagger} V_{\text{mid}}) \right) \left(I - \frac{\Delta t}{6} V_{\text{mid}}^{\dagger} V_{\text{mid}} \right) + \mathcal{O}(\Delta t^{5/2}) \\ &= \sqrt{\Delta t} \xi_1 V_{\text{mid}} + \Delta t^{3/2} \xi_1 B_{\text{mid}} + \mathcal{O}(\Delta t^{5/2}), \end{aligned}$$

where the $\frac{1}{6} V_{\text{mid}} V_{\text{mid}}^{\dagger} V_{\text{mid}}$ term cancels the $-\frac{1}{6} V_{\text{mid}} V_{\text{mid}}^{\dagger} V_{\text{mid}}$ contribution induced by Λ . Similarly,

$$\frac{\xi_3}{\sqrt{2}} G_2 \Lambda = \frac{\xi_3}{\sqrt{2}} \cdot \frac{\Delta t}{\sqrt{2}} V_{\text{mid}}^2 \cdot (I + \mathcal{O}(\Delta t)) = \frac{\Delta t}{2} \xi_3 V_{\text{mid}}^2 + \mathcal{O}_w(\Delta t^3), \quad \xi_2 G_3 \Lambda = \Delta t^{3/2} \xi_2 C_{\text{mid}} + \mathcal{O}_w(\Delta t^3).$$

Finally, the deterministic $\mathcal{O}(\Delta t^2)$ contribution produced by Ω (coming from the $-\frac{1}{2}R^2$ and $\frac{1}{24}R^4$ terms, and from the $\mathcal{O}(\Delta t^2)$ part of R^2) matches $\frac{\Delta t^2}{2} A_{\text{mid}}^2$ up to an $\mathcal{O}_w(\Delta t^3)$ weak remainder; this is precisely the same weak-2-preserving midpoint simplification used in passing from Eq. (77) to Eq. (80). Collecting all terms gives Eq. (92). $\square$