

Shadow Pauli Flow: Characterising Determinism in MBQCs involving Pauli Measurements

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We introduce a new characterisation of determinism in Measurement-Based Quantum Computing (MBQC). The one-way model of computation consists in performing local measurements over a large entangled state represented by a graph. The ability to perform an overall deterministic computation requires a correction strategy because of the non-determinism of each measurement. The existence of such correction strategy depends on the underlying open graph, which is a description of the resource state together with the basis of the performed measurements. GFlow is a well-known graphical characterisation of robust determinism in MBQC when every measurement is performed in some specific planes of the Bloch sphere. While Pauli measurements are ubiquitous in MBQC, GFlow fails to be necessary for determinism when a measurement-based quantum computation involves Pauli measurements. As a consequence, Pauli Flow was designed as a generalisation of GFlow to handle MBQC with Pauli measurements: Pauli Flow guarantees robust determinism, however it has been shown more recently that it fails to be a necessary condition.

Our contribution is twofold. First, we demonstrate that Pauli Flow is actually necessary for robust determinism in a weaker sense: given an open graph, i.e. a resource state, a deterministic computation can be driven if only if it has a Pauli Flow. However, the Pauli Flows do not reflect all the possible correction strategies over a particular resource state, and properties like measurement order or computational depth are not necessarily reflected by a Pauli Flow. Thus, to characterise determinism in full generality, we introduce a further extension called *Shadow Pauli Flow* that we prove necessary and sufficient for robust determinism: An MBQC is robustly deterministic if and only if its correction strategy is consistent with a Shadow Pauli Flow. Furthermore we show that Shadow Pauli Flow can be computed in polynomial time.

1 Introduction

Measurement-Based Quantum Computing (MBQC), also called one-way model, has been introduced by Briegel and Raussendorf in [18, 11, 17, 3]. The key idea is to perform single-qubit measurements over a large resource (entangled) state. It had shown very interesting properties like robustness to noise [20] and is also useful for photonic quantum computing proposals [2] via a variant called FBQC. Another application is to decrease the quantum depth: for instance the Quantum Fourier Transform can be approximated in constant

quantum depth using MBQC [4] and thus provides a logarithmic speed-up compared to quantum circuits.

A fundamental property of MBQC is its probabilistic behavior: every measurement leads to two possible outcomes. In this context, performing an overall “deterministic” computation – e.g. for implementing a unitary evolution – requires a correction strategy, an adaptive computation which depends on the classical outcomes of the measured qubits, so that the output of the overall computation does not depend on the intermediate measurement outcomes. Robust determinism strengthens this requirement by demanding that determinism holds at every stage of the computation, with all measurement outcomes occurring with the same probability and remaining deterministic for any variation of the planes measurements angles.

Robust determinism is in particular well-suited for variational quantum computing. Several recent works [10, 21] points out the necessity of determinism in Measurement-based Variational Quantum Eigensolver. Measurement-based Variational Quantum Eigensolver is particularly promising in terms of physical implementation. Robust determinism guarantees the so-called *deterministic Ansätze*, indeed determinism is preserved when angles, i.e. the parameters, vary.

The existence of a correction strategy essentially depends on two parameters: the initial resource state and the performed measurements. In MBQC, every measurement can be performed in three possible planes of the Bloch sphere (the so-called XY , YZ , and XZ -planes). The resource state and the measurements can be abstracted away into an *open graph*, i.e., a graph that describes the resource state together with a labelling of the vertices with the corresponding measurement plane (e.g. XZ). In [5], it has been shown that a robust deterministic computation is possible if and only if the corresponding open graph has a GFlow. Hence GFlow, an easy to compute graphical property, characterises robust determinism in MBQC. GFlow had several applications in MBQC but also in related topics like ZX-calculus [8] and quantum circuit optimisation [9, 1].

Pauli measurements are ubiquitous in MBQC, for instance in the context of error correcting codes, or for implementing Clifford transformations. Some graphical transformations can be performed using Pauli measurements so that they can be used to locally transform a resource state into another one satisfying for instance some architecture constraints. More generally, it is reasonable to assume that a significant portion of the measurements of a practical MBQC are Pauli measurements. This is for instance the case when using as a resource a constraint initial state like a cluster state or a triangular grid [13]. In terms of determinism, Pauli measurements satisfy some particular properties that are not captured by GFlow. For instance, an MBQC involving an X measurement can be deterministic, even though the corresponding open graph has no GFlow, regardless of whether the X measurement is considered an XY or XZ measurement. Hence a generalisation of GFlow, called Pauli Flow has been introduced in the original paper introducing GFlow [5]. Pauli Flow has been proved to be a sufficient condition for robust determinism. Pauli Flow has been proved to be also a necessary condition for determinism when all measurements are real¹, but is not necessary in general [16]. Notice that recently, Pauli Flow has been used to obtain very efficient circuit optimisations [22] and rewrite rules for MBQC patterns preserving Pauli Flow has been derived in [12].

We introduce an extension of the Pauli Flow called *Shadow Pauli Flow* that we prove to be a necessary and sufficient condition for robust determinism:

¹i.e. measurements described by real observables: X , Z and any measurement in the XZ -plane.

Theorem 1 (Informal) *An MBQC is robustly deterministic if and only if it is consistent with a Shadow Pauli Flow on its open graph.*

To prove this theorem, we first show that when all Pauli measurements are performed at the beginning of the computation, the Pauli Flow is actually necessary and sufficient for robust determinism. The rest of the proof is obtained by showing how Pauli measurements can be moved forwards or backwards in an MBQC, and how such moves preserve robust determinism and/or Shadow Pauli Flow existence.

Notice that it implies that Pauli Flow actually also characterises robust determinism but in a weaker way:

Theorem 3 (Informal) *A resource state, described as an open graph, can be used to perform a robustly deterministic MBQC if and only if it has a Pauli Flow.*

The first theorem is stronger as, in particular, it relates the order in which the measurements are performed – and hence the depth of the MBQC – with the order of the corresponding flow.

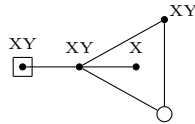
It is common to discard the Pauli measurements in an analysis of MBQC considering that they can be moved first [17]. We strengthen this observation showing that moving the Pauli measurements first can be done in a way that preserves robust determinism. Furthermore, the robustness property applies for the original pattern of measurements which can be relevant in practice as it may be impossible to move some Pauli measurements in some implementations as one can reuse some qubits or have restrictions on the position of the qubits measured at some time step.

We show that given a partial order on measured qubits and an open graph, a Shadow Pauli Flow can be computed in polynomial time.

The paper is structured as follows. We start by recalling the basic definitions of MBQC patterns and their semantics in section 2.1, then in section 2.2 we define the Shadow Pauli Flow. In section 3, we show that Shadow Pauli Flow is sufficient for robust determinism. We demonstrate in Section 4.1 that GFlow is necessary when there are no Pauli measurements (fixing a mistake in the proof published in [5]), then we show in section 4.2 that Pauli Flow is necessary when the Pauli measurements are done first, and that Shadow Pauli Flow is sufficient for a general pattern in section 4.3. Finally, in section 5, we introduce an efficient algorithm for Shadow Pauli Flow.

2 Determinism in MBQC

A resource state for MBQC can be informally represented as follows:



where the underlying graph represents the entangled resource state of the computation, the outputs qubits are represented by white vertices and the input qubits are surrounded by a square. For each measured qubit, a description of the measurement basis is given: it is either a Pauli measurement (X , Y , or Z) or a measurement in a plane of the Bloch sphere (XY , XZ , or YZ). Notice that such a graphical representation provides a partial information of the MBQC: the actual angles describing the measurement basis, the order of the measurements and the correction strategy are not depicted. In this section, we provide a more formal description of MBQC based on the Measurement calculus [7].

2.1 MBQC syntax and semantics

We use the notation $<$ to denote a strict partial order, i.e. a transitive asymmetric binary relation. The reflexive closure of $<$ is denoted $\leq$. Conversely, given a non strict partial order $\leq$, we denote $<$ its irreflexive kernel. We use $\leq$ (resp. $<$) to denote a (resp. strict) total order.

The measurement calculus is a formal framework for describing MBQC using patterns, a pattern being a sequence of commands. There are five kinds of commands: N, E, M, X, Z; describing how the qubits are initialized (N), entangled (E), measured (M), and corrected (X, Z).

A pattern is made of two parts: a description of the initial entanglement and then a description of the measurements and their associated corrections. Given a graph G whose vertices $V(G)$ represent the qubits – including a set I of input qubits – the commands $N_{V(G)\setminus I}$ and E_G describe the initial entanglement of the computation.

The measurement of a qubit u is denoted by $M_u^{\lambda,\alpha}$, where (λ, α) characterizes the measurement basis. λ is either a Pauli observable or a plane defined by two Pauli observables², and α is the measurement angle. Notice that the measurements of the input qubits are restricted to the XY -plane of the Bloch sphere.

The necessary corrections (Pauli operators $X_A^{s_u}, Z_A^{s_u}$ applied to a subset of vertices A depending on the classical outcome s_u of the measurement of u) are also represented in a pattern. Each qubit is measured at most once, and the unmeasured ones form the set O of output qubits.

Definition 1 (Pattern) *A pattern $\mathcal{P} : I \rightarrow O$, with I and O two finite sets, is inductively defined as:*

- for any simple undirected finite graph G , and any $I \subseteq V(G)$,

$$E_G N_{V(G)\setminus I} : I \rightarrow V(G)$$

is a pattern;

- for any pattern $\mathcal{P} : I \rightarrow O$, any qubit $u \in O$, any subsets $A, B \subseteq O \setminus \{u\}$, any $\lambda \subsetneq \{X, Y, Z\}$ s.t. $|\lambda| \geq 1$ and any angle $\alpha \in [0, 2\pi)$

$$X_A^{s_u} Z_B^{s_u} M_u^{\lambda,\alpha} \mathcal{P} : I \rightarrow O \setminus \{u\}$$

is a pattern where $\alpha \in \{0, \pi\}$ when $|\lambda| = 1$, and $\lambda \subseteq \{X, Y\}$ when $u \in I$.

Remark 1 *Some more general form of patterns have been defined in the literature [7]. It has been proved [7] however that, using a standardisation procedure, any pattern can be transformed, preserving the semantics, into a pattern of the form of Definition 1.*

Remark 2 *Notice that Pauli corrections can also be integrated to the upcoming measurements, leading to adaptive measurements instead of Pauli corrections. The two presentations being equivalent [7] we choose w.l.o.g. to represent MBQC patterns with Pauli corrections.*

² λ can be seen as a set of one or two Pauli observables e.g. $\{X\}$ for X -measurement and $\{X, Y\}$ for XY -plane measurement.

We use the following notation to denote an arbitrary pattern:

$$\left(\prod_{u \in O^c}^{\leq} X_{\mathbf{x}(u)}^{s_u} Z_{\mathbf{z}(u)}^{s_u} M_u^{\lambda_u, \alpha_u} \right) E_G N_{I^c} = X_{\mathbf{x}(u_k)}^{s_{u_k}} Z_{\mathbf{z}(u_k)}^{s_{u_k}} M_{u_k}^{\lambda_{u_k}, \alpha_{u_k}} \dots X_{\mathbf{x}(u_1)}^{s_{u_1}} Z_{\mathbf{z}(u_1)}^{s_{u_1}} M_{u_1}^{\lambda_{u_1}, \alpha_{u_1}} E_G N_{I^c}$$

where $<$ is the total order over $O^c := V(G) \setminus O$ s.t. $u_1 < \dots < u_k$.

Each MBQC pattern has an underlying structure that is an extension of graph states taking into account inputs, outputs and measurements that is called an open graph.

Definition 2 (Open graph) *The quadruplet (G, I, O, λ) is the underlying open graph of a pattern $\left(\prod_{u \in O^c}^{\leq} X_{\mathbf{x}(u)}^{s_u} Z_{\mathbf{z}(u)}^{s_u} M_u^{\lambda_u, \alpha_u} \right) E_G N_{I^c}$ with $\lambda : u \mapsto \lambda_u$.*

An open graph is an abstraction of a pattern where three properties are essentially abstracted away: the order (or scheduling) of the measurements, the actual angle of the measurements (only the measurement-plane is kept for non-Pauli measurements), and finally the corrections.

Several patterns can lead to the same open graph, it is however convenient to consider, among all these patterns, those which share the same correction strategy (represented as a function from non output qubits to sets of non input qubits) and have compatible measurement schedulings (represented as a partial order):

Definition 3 (Consistent Pattern) *Given $p : O^c \rightarrow 2^{I^c}$ and $<$, a partial order, a pattern $\left(\prod_{u \in O^c}^{\leq} X_{\mathbf{x}(u)}^{s_u} Z_{\mathbf{z}(u)}^{s_u} M_u^{\lambda_u, \alpha_u} \right) E_G N_{I^c}$ is said to be consistent with $(p, <)$ if $\forall u \in O^c$, $\mathbf{x}(u) = \{v \in p(u) \mid u < v\}$, $\mathbf{z}(u) = \{v \in \text{Odd}(p(u)) \mid u < v\}$, and $\forall v \in O^c$, $u < v \Rightarrow u < v$, where $\text{Odd}(p(u))$ is the set of vertices that have an odd number of neighbours in $p(u)$.*

A pattern involves quantum measurements: each non-output qubit u is measured, which produces a classical outcome $m(u) \in \{0, 1\}$.

$M^{\lambda, \alpha}$ is a measurement in the basis $\{|+\lambda\rangle, |-\lambda\rangle\}$ defined as follows:

- in case of a Pauli measurement i.e. when $|\lambda| = 1$,

$$|+\lambda\rangle = \begin{cases} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \text{if } \lambda = \{X\} \\ \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) & \text{if } \lambda = \{Y\} \\ |0\rangle & \text{if } \lambda = \{Z\} \end{cases} \text{ and } |-\lambda\rangle = \begin{cases} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & \text{if } \lambda = \{X\} \\ \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) & \text{if } \lambda = \{Y\} \\ |1\rangle & \text{if } \lambda = \{Z\} \end{cases} \quad (1)$$

and $|+\lambda\rangle = |-\lambda\rangle$ and $|-\lambda\rangle = |+\lambda\rangle$,

- and when $|\lambda| = 2$, $|+\lambda\rangle = \frac{1}{\sqrt{2}}(|+^P_0\rangle + e^{i\alpha} | -^P_0\rangle)$ and $|-\lambda\rangle = \frac{1}{\sqrt{2}}(|+^P_0\rangle - e^{i\alpha} | -^P_0\rangle)$ where $P \notin \lambda$.

We also define $P(\alpha)$ as the unitary such that $P(\alpha) |+^P_0\rangle = |+^P_0\rangle$ and $P(\alpha) | -^P_0\rangle = e^{-i\alpha} | -^P_0\rangle$.

The action of a pattern on a quantum input state can then be described as a collection of linear maps, one for each possible branch of the computation which corresponds to a sequence of classical outcomes:

Definition 4 (Branch semantics) Given a pattern $\mathcal{P} : I \rightarrow O$ and a sequence of classical outcomes $m : O^c \rightarrow \{0, 1\}$, let $\llbracket \mathcal{P} \rrbracket_m : \mathbb{C}^{2^I} \rightarrow \mathbb{C}^{2^O}$ be inductively defined as

$$\begin{aligned} \llbracket E_G N_{V(G) \setminus I} \rrbracket_m &= |\phi\rangle \mapsto E_G |+\rangle_{I^c} |\phi\rangle_I = \prod_{(u,v) \in G} \Lambda_{Z_{u,v}} \left(\bigotimes_{u \in V(G) \setminus I} \frac{|0_u\rangle + |1_u\rangle}{\sqrt{2}} \right) \otimes |\phi\rangle \\ \llbracket X_A^{s_u} Z_B^{s_u} M_u^{\lambda, \alpha} \mathcal{P} \rrbracket_m &= \begin{cases} \langle +_{\alpha_u} |_u \llbracket \mathcal{P} \rrbracket_{m'}, & \text{if } m(u) = 0 \\ X_A Z_B \langle -_{\alpha_u} |_u \llbracket \mathcal{P} \rrbracket_{m'}, & \text{if } m(u) = 1 \end{cases} \end{aligned}$$

where m' is the restriction of m to $O^c \setminus \{u\}$, $\{|+_{\alpha_u}\rangle, |-_{\alpha_u}\rangle\}$ is the measurement basis defined in previous definition, and ΛZ is the control-Z operator.

The overall semantics of a pattern can then be described as a superoperator:

Definition 5 (Semantics) Given a pattern $\mathcal{P} : I \rightarrow O$, its semantics is the superoperator $\llbracket \mathcal{P} \rrbracket = \rho \mapsto \sum_{m \in \{0,1\}^{O^c}} \llbracket \mathcal{P} \rrbracket_m \rho \llbracket \mathcal{P} \rrbracket_m^\dagger$.

A pattern is deterministic if the overall evolution does not depend on the intermediate classical outcomes. We consider a robust version of determinism which is *strong* (all branches occur with the same probability), *uniform* (for any variation of the measurement angles the pattern remains strongly deterministic), and *stepwise* (every partial computation is also uniform and strongly deterministic).

Definition 6 (Robust determinism) We inductively define robustly deterministic patterns as:

- $E_G N_{V(G) \setminus I}$ is robustly deterministic;
- $X_A^{s_u} Z_B^{s_u} M_u^{\lambda, \alpha} \mathcal{P}$ with $|\lambda| = 1$ is robustly deterministic when $\mathcal{P}$ is robustly deterministic and $\forall \rho, \langle +_{\alpha} |_u \llbracket \mathcal{P} \rrbracket(\rho) | +_{\alpha} \rangle_u = \langle -_{\alpha} |_u Z_B X_A \llbracket \mathcal{P} \rrbracket(\rho) X_A Z_B | -_{\alpha} \rangle_u$
- $X_A^{s_u} Z_B^{s_u} M_u^{\lambda, \alpha} \mathcal{P}$ with $|\lambda| = 2$ is robustly deterministic when $\mathcal{P}$ is robustly deterministic and $\forall \epsilon \in \mathbb{R}, \forall \rho, \langle +_{\alpha+\epsilon} |_u \llbracket \mathcal{P} \rrbracket(\rho) | +_{\alpha+\epsilon} \rangle_u = \langle -_{\alpha+\epsilon} |_u Z_B X_A \llbracket \mathcal{P} \rrbracket(\rho) X_A Z_B | -_{\alpha+\epsilon} \rangle_u$

2.2 Characterising Determinism

A central question in MBQC is to decide whether a deterministic computation can be driven on a given resource, represented as an open graph (G, I, O, λ) . Several sufficient conditions for robust determinism have been introduced: Causal Flow [6] which has been generalized to GFlow and Pauli Flow [5]. GFlow has been proved to be a necessary condition for robust determinism when all measurements are performed in a plane ($\forall u \in O^c, |\lambda_u| = 2$), and Pauli Flow has been proved to be necessary when all measurements are real ($\forall u \in O^c, \lambda_u \subseteq \{X, Z\}$), whereas counter examples are known in the general case.

The correction strategy in MBQC fundamentally relies on some fixed-point properties of graph states: for any subset D of (non input) qubits, the Pauli operator $X_D Z_{\text{Odd}(D)}$ is called a stabilizer, because it leaves the resource state invariant. The local operation, on every qubit $v \in V$, of this stabilizer is called the *action of D on v* , denoted Act_v^D and thus defined as:

$$Act_v^D = \begin{cases} X & \text{if } v \in D \setminus \text{Odd}(D) \\ Z & \text{if } v \in \text{Odd}(D) \setminus D \\ Y & \text{if } v \in D \cap \text{Odd}(D) \\ I & \text{Otherwise.} \end{cases}$$

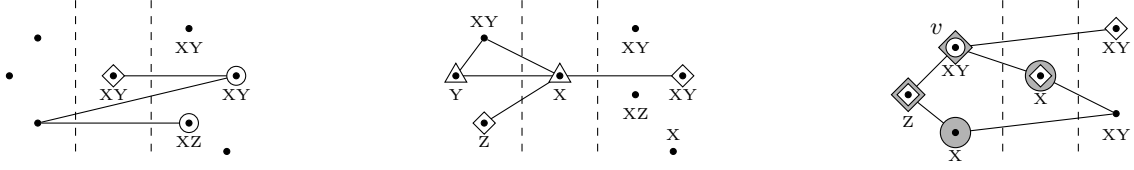


Figure 1: We illustrate the 3 kinds of flow : GFlow on the left, Pauli Flow in the middle, and Shadow Pauli Flow on the right. In each case, the correction required by the measurement of the central vertex involves the vertices with different Paulis represented with white shapes: triangle = $D \cap \text{Odd}(D)$, diamond = $\text{Odd}(D) \setminus D$ and circle = $D \setminus \text{Odd}(D)$. A dependency on the past (vertex v) is allowed in the Shadow Pauli Flow case when it is compensated by a D_v in the past represented by grey shapes.

The action on a vertex can be used to actually correct a measurement: when the action of D on a vertex u anti-commute with each element of λ_u according to which u is measured, the action of D turns one measurement projector into the other, performing the desired correction. Hence, we denote

$$\text{cor}(D) = \{u, \forall P \in \lambda_u, [\text{Act}_u^D, P] \neq 0\}$$

GFlow is based on this idea and consists in finding for every measured qubit u a set D of correctors such that (i) $u \in \text{cor}(D)$ and (ii) D acts trivially on all qubits measured before u to avoid uncontrolled side effects.

The extension from GFlow to Pauli Flow is based on the following additional property: a correction can act on an already measured qubit as long as this qubit has been measured in the appropriate Pauli basis, in this case the qubit is not impacted by the action of a corrector. Hence, we say that the action of D *impacts* a vertex v if the action of D on v anticommutes with one Pauli element of λ_v and denote

$$\text{imp}(D) = \{v, \exists P \in \lambda_v, [\text{Act}_v^D, P] \neq 0\}$$

It leads to the following description of Pauli Flow:

Proposition 1 *Given $p : O^c \rightarrow 2^{I^c}$ and $<$ a strict partial order over O^c , $(p, <)$ is a Pauli Flow of (G, I, O, λ) iff*

$$\forall u \in O^c, u \in \text{cor}(p(u)) \text{ and } \text{ana}_{p, <}(u) = \emptyset$$

where $\text{ana}_{p, <}(u) := \{v \in O^c, \text{ s.t. } u \neq v, \neg(v < u) \text{ and } u \in \text{imp}(p(v))\}$ is the set of vertices corrected after u which correction has an “anachronistic” impact on u .

In the rest of the paper, we will use the Proposition 1 as a definition of Pauli Flow as it is simpler than the original one. The proof is given in the Appendix A.

This notion of *impact* provides a set of constraints on the order according to which the qubits of the pattern can be measured:

Corollary 1 *(G, I, O, λ) has a Pauli Flow iff there exists p s.t. K_p is a DAG³, where $K_p = (V(G), \{(u, v) \in (O^c)^2, v \in \text{imp}(p(u))\})$.*

For such a p , $(p, <)$ will be a Pauli Flow for every partial order $<$ compatible with the DAG.

³Directed Acyclic Graph

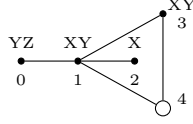


Figure 2: Example of open graph with Shadow Pauli Flow but no Pauli Flow with order $0 < 1 < 2 < 3$ and $p(0) = \{0, 4\}, p(1) = \{4\}, p(2) = \{1\}, p(3) = \{4\}, \text{ana}_{p, <}(1) = \{2, 3\}, D_1 = \{2\}, \text{ana}_{p, <}(0) = \{2\}, D_0 = \{0, 2\}$

While the Pauli Flow condition is based on the fixed-point properties of the underlying graph state and the specific properties of Pauli measurements, we identify a third property, based on *shadow correctors*, which allows deterministic computation in MBQC.

Indeed, an anachronous impact (i.e. an impact on an already measured qubit v) can actually be offset by a shadow corrector D_v which is such that: (i) D_v is correcting v ; (ii) D_v only acts on already measured qubits and (iii) D_v has only impacts on v . Roughly speaking, a pattern $\mathcal{P}$ involving a plane measurement on v , say YZ , with angle α can be decomposed as a linear combination of two patterns $\cos(\alpha)\mathcal{P}_Y + \sin(\alpha)\mathcal{P}_Z$ where v is measured according to Y (resp. Z) in $\mathcal{P}_Y$ (resp. $\mathcal{P}_Z$). If the anachronous impact on v in $\mathcal{P}$ is say Z , then it has actually no impact on v in $\mathcal{P}_Z$, whereas, in $\mathcal{P}_Y$, the action of the shadow corrector can turn the Z action into a Y action which has then no impact on v .

Pauli Flow can thus be extended to *Shadow Pauli Flow*, in order to encounter shadow correctors:

Definition 7 (Shadow Pauli Flow) *An open graph (G, I, O, λ) has a Shadow Pauli Flow $(p, <)$, where $p : O^c \rightarrow 2^{I^c}$ and $<$ partial order, if $\forall v \in O^c$,*

- *v is corrected by $p(v)$, i.e. $v \in \text{cor}(p(v))$ and*
- *if v is anachronistically impacted ($\text{ana}_{p, <}(v) \neq \emptyset$), then v is not a Pauli measurement ($|\lambda_v| = 2$), and it has a set of “shadow correctors” $D_v \subseteq I^c$ such that:*
 - *$v \in \text{cor}(D_v)$ and*
 - *$\forall u \in \text{ana}_{p, <}(v), \forall w \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}, w \notin \text{imp}(D_v)$ and $w \leq u$.*

Note that for any vertex $w \in O^c$, and for any set $D \subseteq V$, if $w \in D \cup \text{Odd}(D)$ and $w \notin \text{imp}(D)$ then $|\lambda_w| = 1$. This means that the new condition for Shadow Pauli Flow implies that when the correction of u impacts v that has already been measured, v must be measured with a plane, and have an extra set of “shadow correctors” D_v such that $D_v \cup \text{Odd}(D_v) \setminus \{v\}$ contains only vertices that are Pauli measured and that are in the past of any u' which measurement has an anachronistic effect on v .

Remark 3 *Similarly to GFlow and Pauli Flow, the existence of a Shadow Pauli Flow implies that any input is measured in XY-plane or a subset of it.*

Figure 2 shows an open graph that has a Shadow Pauli Flow which is not a Pauli Flow.

Theorem 1 (Main result) *A pattern is robustly deterministic iff it is consistent with a Shadow Pauli Flow of its underlying open graph.*

To prove Theorem 1, we first prove in the next Section (Proposition 2) that the existence of a Shadow Pauli Flow is a sufficient condition for robust determinism. The necessity is proved in Proposition 3 section 4.

We also show in Section 5 that Shadow Pauli Flow can be computed efficiently:

Theorem 2 *Given an open graph of order n , and a partial order over the non-output vertices of the graphs, a Shadow Pauli Flow compatible with the partial order, or a certificate of non existence, can be computed in time polynomial in n .*

3 Shadow Pauli Flow is sufficient for Robust Determinism

Proposition 2 *Any pattern consistent with a Shadow Pauli Flow is robustly deterministic.*

The rest of the section is dedicated to the proof of Proposition 2. The robustness of the consistent pattern relies on the fixed-point properties of underlying graph state and also the properties of the particular measurements.

Given a Shadow Pauli Flow $(p, <)$ of an open graph (G, I, O, λ) , the patterns $\mathcal{P}^{<, \alpha}$ consistent with $(p, <)$ are obtained as follows, by refining the partial order $<$ into a total order $<$ and by fixing the angles of measurements α_u for every $u \in O^c$.

Indeed, given a vertex u , the total order $<$ induces the following partition of the set vertices v on which $p(u)$ is acting ($Act_v^{p(u)} \neq I$):

- $A^u := \{v \in p(u) \cup \text{Odd}(p(u)) \mid v < u \wedge |\lambda_v| = 2\}$ the set of correctors which are already measured according to a plane;
- $B^u := \{v \in p(u) \cup \text{Odd}(p(u)) \mid v < u \wedge |\lambda_v| = 1\}$ the set of correctors which are already Pauli-measured;
- $C^u := \{v \in p(u) \cup \text{Odd}(p(u)) \mid u < v \text{ or } v \in O\}$, the set of correctors of u which are not yet measured;
- $F^u := \{u\} \cap (p(u) \cup \text{Odd}(p(u)))$ completes the partition of $p(u) \cup \text{Odd}(p(u))$. Notice that $F^u = \{u\}$ since the Shadow Pauli Flow conditions guarantees that $u \in p(u) \cup \text{Odd}(p(u))$.

Moreover, for any set $S \in \{A^u, B^u, C^u, F^u\}$, we define $S_X := S \cap p(u)$ and $S_Z := S \cap \text{Odd}(p(u))$. Notice that $S_X \cup S_Z = S$.

The consistent pattern is then $\mathcal{P}^{<, \alpha} = \left(\prod_{u \in O^c}^< X_{C_X^u}^{s_u} Z_{C_Z^u}^{s_u} M_u^{\lambda_u, \alpha_u} \right) E_G N_{I^c}$.

The fundamental property of graph states gives us the first useful fixed-point property.

Fact 1 *For any $D \subseteq I^c$, and any $|\phi\rangle \in \mathbb{C}^{2^I}$,*

$$X_D Z_{\text{Odd}(D)} E_G |+\rangle_{I^c} |\phi\rangle_I = \pm E_G |+\rangle_{I^c} |\phi\rangle_I \quad (2)$$

We also have the following standard fixed point property.

Fact 2 *For any $v \in O^c$, there exist $P_v, Q_v \in \lambda_v$, such that*

$$(\cos(\alpha_v)P_v + \sin(\alpha_v)Q_v) \left| +_{\alpha_v}^{\lambda_v} \right\rangle = \left| +_{\alpha_v}^{\lambda_v} \right\rangle \quad (3)$$

Notice that if $|\lambda_v| = 1$ then $P_v = Q_v$ and $\alpha_v = 0 \bmod \pi$, thus $(\cos(\alpha_v)P_v + \sin(\alpha_v)Q_v) = \pm P_v$.

Then, we state a few more almost fixed-point equations which are then used to prove that the existence of a Shadow Pauli Flow is a sufficient condition for robust determinism.

Lemma 1 Given an open graph with a Shadow Pauli Flow, for any $u \in O^c$,

$$\left(\prod_{v \in B^u} \langle +_{\alpha_v}^{\lambda_v} | \right) X_{B_X^u} Z_{B_Z^u} = \pm i^{|B_X^u \cap B_Z^u|} \prod_{v \in B^u} \langle +_{\alpha_v}^{\lambda_v} | \quad (4)$$

The proof is given in the Appendix B.

Lemma 2 Given an open graph (G, I, O, λ) with a Shadow Pauli Flow $(p, <)$, and a vertex v with non empty $\text{ana}_{p, <}(v)$, then for any one-qubit Pauli operator L_v , and any multi-qubit Pauli operator R , $\exists \theta$ s.t.

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \langle +_{\alpha_w}^{\lambda_w} | \right) R E_G |+\rangle_{I^c} |\phi\rangle_I = e^{i\theta} \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \langle +_{\alpha_w}^{\lambda_w} | \right) L_v R E_G |+\rangle_{I^c} |\phi\rangle_I \quad (5)$$

where D_v is as in the definition 7 of the Shadow Pauli Flow.

The proof is given in the Appendix C.

Proof of Proposition 2:

We are now ready to prove by induction that the Shadow Pauli Flow is a sufficient condition for robust determinism of $\mathcal{P}^{<, \alpha}$. By induction hypothesis, there exists $r \in \mathbb{C}$ s.t. the state before the measurement of u is $|\psi\rangle = r \left(\prod_{v < u} \langle +_{\alpha_v}^{\lambda_v} | \right) E_G |+\rangle_{I^c} |\phi\rangle_I$.

The measurement of u leads, up to a renormalisation, to either $\langle +_{\alpha_u}^{\lambda_u} | \psi \rangle$ or $\langle -_{\alpha_u}^{\lambda_u} | \psi \rangle$. In the second case the correction $X_{C_X^u} Z_{C_Z^u}$ is applied. Thus it is enough to show $|\psi_0\rangle = \langle +_{\alpha_u}^{\lambda_u} | \psi \rangle$ and $|\psi_1\rangle = X_{C_X^u} Z_{C_Z^u} \langle -_{\alpha_u}^{\lambda_u} | \psi \rangle$ are equal up to a global phase, to show strong determinism.

The Shadow Pauli Flow condition guarantees $u \in \text{cor}(p(u))$, which implies that for any α_u ,

$$\langle -_{\alpha_u}^{\lambda_u} | \simeq \langle +_{\alpha_u}^{\lambda_u} | X_{F_X^u} Z_{F_Z^u} \quad (6)$$

As a consequence:

$$|\psi_1\rangle \simeq X_{C_X^u} Z_{C_Z^u} \langle +_{\alpha_u}^{\lambda_u} | Z_{F_Z^u} X_{F_X^u} |\psi\rangle \quad (\text{Eq. 6})$$

$$\begin{aligned} &\simeq r X_{C_X^u} Z_{C_Z^u} \left(\prod_{v \leq u} \langle +_{\alpha_v}^{\lambda_v} | \right) Z_{F_Z^u} X_{F_X^u} E_G |+\rangle_{I^c} |\phi\rangle_I \\ &\simeq r X_{C_X^u} Z_{C_Z^u} \left(\prod_{v \leq u} \langle +_{\alpha_v}^{\lambda_v} | \right) X_{B_X^u} Z_{B_Z^u} Z_{F_Z^u} X_{F_X^u} E_G |+\rangle_{I^c} |\phi\rangle_I \end{aligned} \quad (\text{Eq. 4})$$

$$\simeq r X_{C_X^u} Z_{C_Z^u} \left(\prod_{v \leq u} \langle +_{\alpha_v}^{\lambda_v} | \right) X_{A_X^u} Z_{A_Z^u} X_{B_X^u} Z_{B_Z^u} Z_{F_Z^u} X_{F_X^u} E_G |+\rangle_{I^c} |\phi\rangle_I \quad (\text{Eq. 5})$$

$$\begin{aligned} &\simeq r \left(\prod_{v \leq u} \langle +_{\alpha_v}^{\lambda_v} | \right) X_{p(u)} Z_{\text{Odd}(p(u))} E_G |+\rangle_{I^c} |\phi\rangle_I \\ &\simeq r \left(\prod_{v \leq u} \langle +_{\alpha_v}^{\lambda_v} | \right) E_G |+\rangle_{I^c} |\phi\rangle_I \simeq |\psi_0\rangle \end{aligned} \quad (\text{Eq. 2})$$

□

4 Shadow Pauli Flow is necessary for robust determinism

The main result of this Section is the proof of the necessity condition of Shadow Pauli Flow.

Proposition 3 If a pattern $\mathcal{P}$ is robustly deterministic then it is consistent with a Shadow Pauli Flow of its underlying open graph.

To prove this, we start by proving a useful property (Lemma 3) of robustly deterministic computations, that allows us to have necessity conditions for the GFlow when all measurements are performed according to a plane (Proposition 4, which fixes the original proof from [5]), then we prove that for patterns that have the Pauli measurements first, the Pauli Flow is necessary (Proposition 5). Finally, we prove that for general robustly deterministic patterns, Shadow Pauli Flow is a necessary condition.

4.1 Necessity of GFlow for Pauli measurement-free patterns

The GFlow is a particular case of Pauli Flow, when all measurements are performed according to a plane (i.e. $\forall u \in O^c$, $|\lambda_u| = 2$). The original proof from [16] showing the necessity of GFlow used a wrong Lemma (Lemma 4 of [5]) that stated that if two states were the same after any measurement of one qubit in a plane then they were the same. Indeed, as can be seen in Lemma 3 below, there is a case where the measured qubit is separated from the rest and is in a state that is an eigenvector of the observable not in the plane (Z in the XY -plane measurement case). We show in Proposition 4 that the necessity of GFlow still holds in this particular case as by robust determinism it can be corrected: indeed, as the $\langle -|_\alpha$ branch can be changed to the $\langle +|_\alpha$ branch by applying precisely the observable not in the plane, it cannot be an eigenvector.

Lemma 3 *Let $\lambda \subset \{X, Y, Z\}$ a set of two elements and P be the Pauli not in λ . Given $|\phi\rangle$ and $|\phi'\rangle$ two states of a register V of qubits, and $u \in V$, if $\forall \alpha$, $\langle +|_\alpha|_u |\phi\rangle \simeq \langle +|_\alpha|_u |\phi'\rangle$ and $|\langle +|_\alpha|_u |\phi\rangle| = \frac{1}{\sqrt{2}}$ then*

$$\begin{cases} |\phi\rangle \simeq |\phi'\rangle, \text{ or} \\ \exists x \in \{0, 1\}, \exists |\psi\rangle \text{ s.t. } |\phi\rangle \simeq |+_x^P\rangle_u \otimes |\psi\rangle \text{ and } |\phi'\rangle \simeq |-_{x\pi}^P\rangle_u \otimes |\psi\rangle \end{cases}$$

Proof: Since $\{|+_0^P\rangle_u, |-_0^P\rangle_u\}$ is a basis, $|\phi\rangle$ and $|\phi'\rangle$ can be written as $|\phi\rangle = |+_0^P\rangle_u |\phi_+\rangle + |-_0^P\rangle_u |\phi_-\rangle$ and $|\phi'\rangle = |+_0^P\rangle_u |\phi'_+\rangle + |-_0^P\rangle_u |\phi'_-\rangle$ where $|\phi_+\rangle$, $|\phi_-\rangle$, $|\phi'_+\rangle$ and $|\phi'_-\rangle$ are not normalized. Considering $\alpha = 0$, we have $|\phi_+\rangle \simeq |\phi'_+\rangle$ and similarly when $\alpha = \pi$, $|\phi_-\rangle \simeq |\phi'_-\rangle$. So $\exists \theta, \theta'$ s.t. $|\phi'\rangle = e^{i\theta} |+_0^P\rangle_u |\phi_+\rangle + e^{i\theta'} |-_0^P\rangle_u |\phi_-\rangle$. When $\alpha = \pi/2$, it implies $|\phi_+\rangle + i |\phi_-\rangle \simeq e^{i\theta} |\phi_+\rangle + e^{i\theta'} i |\phi_-\rangle$.

- if $|\phi_+\rangle$ and $|\phi_-\rangle$ are not colinear, we have $\theta = \theta'$ and thus $|\phi\rangle \simeq |\phi'\rangle$.
- Otherwise, since $|\langle +|_0|_u |\phi\rangle| = |\langle +|_\pi|_u |\phi\rangle| = \frac{1}{\sqrt{2}}$, there exists γ s.t. $|\phi_+\rangle = e^{i\gamma} |\phi_-\rangle$. Moreover, the condition $|\langle +|_{\frac{\pi}{2}}|_u |\phi\rangle| = \frac{1}{\sqrt{2}}$ implies $\gamma = 0 \bmod \pi$. As a consequence there exist $x \in \{0, 1\}$, and $|\psi\rangle$ s.t. $|\phi\rangle = |+_x^P\rangle_u \otimes |\psi\rangle$. Similarly, there exist $y \in \{0, 1\}$, and $|\psi'\rangle$ s.t. $|\phi'\rangle = |+_y^P\rangle_u \otimes |\psi'\rangle$. If $x = y$ then $|\phi\rangle \simeq |\phi'\rangle$, otherwise $|\phi'\rangle \simeq |-_{x\pi}^P\rangle_u \otimes |\psi\rangle$.

□

Corollary 2 *Let $|\phi\rangle$ a state of a register V of qubits, $\lambda \subset \{X, Y, Z\}$ a set of two elements, $u \in V$, C a multi-qubit Pauli not acting on u and $P \in \{X, Y, Z\} \setminus \lambda$. If $\forall \alpha$, $\langle +|_\alpha|_u |\phi\rangle \simeq C \langle -|_\alpha|_u |\phi\rangle$ then $|\phi\rangle \simeq (P_u \otimes C) |\phi\rangle$.*

Proof: Let $|\phi'\rangle = (P_u \otimes C) |\phi\rangle$. According to Lemma 3, $|\phi'\rangle \simeq |\phi\rangle$ or $\exists x \in \{0, 1\}, \exists |\psi\rangle$ s.t. $|\phi\rangle \simeq |+_x^P\rangle_u \otimes |\psi\rangle$ and $|\phi'\rangle \simeq |-_{x\pi}^P\rangle_u \otimes |\psi\rangle$. In this second case, we would have $|\phi'\rangle = (P_u \otimes C) |\phi\rangle \simeq (P_u \otimes C) |+_x^P\rangle_u \otimes |\psi\rangle \simeq |+_x^P\rangle_u \otimes C |\psi\rangle$ since $P |+_x^P\rangle \simeq |+_x^P\rangle$ which is a contradiction.

□

Proposition 4 *If $\mathcal{P}$ is robustly deterministic and has no Pauli measurement then the underlying open graph has a GFlow.*

Proof: Let $<$ be the order according to which the qubits are measured in the pattern $\mathcal{P}$. For any $u \in O^c$, the robust determinism condition implies the two states obtained after the measurement of u , for any angle α , and the corresponding corrections are equal up to a global phase:

$$\forall \vec{\beta} \in [0, 2\pi)^{\{v \in O^c \mid v < u\}}, \left\langle +_{\alpha}^{\lambda_u} \right| \left\langle +_{\vec{\beta}}^{\vec{\lambda}} \right| E_G N_{V \setminus I} |\phi\rangle_I \simeq X_{C_X^u} Z_{C_Z^u} \left\langle -_{\alpha}^{\lambda_u} \right| \left\langle +_{\vec{\beta}}^{\vec{\lambda}} \right| E_G N_{V \setminus I} |\phi\rangle_I$$

where $\left\langle +_{\vec{\beta}}^{\vec{\lambda}} \right|$ is the tensor product of $\left\langle +_{\beta_v}^{\lambda_v} \right|$ over the $v \in O^c$ s.t. $v < u$.

According to corollary 2, we have

$$\forall \vec{\beta} \in [0, 2\pi)^{\{v \in O^c \mid v < u\}}, \left\langle +_{\vec{\beta}}^{\vec{\lambda}} \right| E_G N_{V \setminus I} |\phi\rangle_I \simeq \left\langle +_{\vec{\beta}}^{\vec{\lambda}} \right| (X_{C_X^u} Z_{C_Z^u} \otimes P_u) E_G N_{V \setminus I} |\phi\rangle_I$$

where $P \in \{X, Y, Z\} \setminus \lambda_u$.

Let $|\psi\rangle = E_G N_{V \setminus I} |\phi\rangle_I$ and $|\psi'\rangle = (X_{C_X^u} Z_{C_Z^u} \otimes P_u) E_G N_{V \setminus I} |\phi\rangle_I$. By iterative applications of Lemma 3, there exists $F \subseteq \{v \in O^c \mid v < u\}$ and $\forall w \in F, x_w \in \{0, 1\}$, s.t. $|\psi\rangle = (\otimes_{w \in F} |+\frac{P_w}{x_w \pi}\rangle_w) \otimes |\Psi\rangle$ and $|\psi'\rangle = (\otimes_{w \in F} |-\frac{P_w}{x_w \pi}\rangle_w) \otimes |\Psi\rangle$, with $P_w \notin \lambda_w$.

Notice that $|\psi'\rangle \simeq (X_{C_X^u} Z_{C_Z^u} \otimes P_u) |\psi\rangle$, so $(\otimes_{w \in F} |-\frac{P_w}{x_w \pi}\rangle_w) \otimes |\Psi\rangle \simeq (\otimes_{w \in F} |+\frac{P_w}{x_w \pi}\rangle_w) \otimes (X_{C_X^u} Z_{C_Z^u} \otimes P_u) |\Psi\rangle$ since $u \notin F$ and $(C_X^u \cup C_Z^u) \cap F = \emptyset$. As a consequence $F = \emptyset$ and

$$E_G N_{V \setminus I} |\phi\rangle_I \simeq X_{C_X^u} Z_{C_Z^u} P_u E_G N_{V \setminus I} |\phi\rangle_I \quad (7)$$

We show that $C_X^u \cap I = \emptyset$. Indeed, by contradiction, assume there exists $e \in I \cap C_X^u$. We initialize this qubit e in the $|0\rangle_e$ -state so that $E_G N_{V \setminus I} |\phi\rangle_I = |0\rangle_e \otimes |\phi\rangle$ for some $|\phi\rangle$, if $e \neq u$, then $X_{C_X^u} Z_{C_Z^u} P_u |0\rangle_e \otimes |\phi\rangle \simeq |1\rangle_e \otimes X_{C_X^u \setminus \{e\}} Z_{C_Z^u \setminus \{e\}} P_u |\phi\rangle$ which contradicts Equation 7. If $u = e$ then $\lambda_u = \{X, Y\}$, so $P_u = Z_u$ (see remark 3). As a consequence, we have $X_{C_X^u} Z_{C_Z^u} Z_u |0\rangle_e \otimes |\phi\rangle \simeq |1\rangle_e \otimes X_{C_X^u \setminus \{e\}} Z_{C_Z^u \setminus \{e\}} |\phi\rangle$, which also contradicts Equation 7.

Moreover, if $|\phi\rangle_I = |+\rangle_I$, then $E_G N_{V \setminus I} |\phi\rangle_I$ is nothing but the graph state $|G\rangle$. Hence $X_{C_X^u} Z_{C_Z^u} P_u$ is in the stabilizer of $|G\rangle$ so it must be of the form $X_D Z_{\text{Odd}(D)}$ for some D .

We define $g(u)$ to be this set D , i.e. $g(u) := \begin{cases} C_X^u & \text{if } \lambda_u = \{X, Y\} \\ C_X^u \cup \{u\} & \text{otherwise} \end{cases}$. Notice that

$$\text{Odd}(g(u)) = \begin{cases} C_Z^u & \text{if } \lambda_u = \{Y, Z\} \\ C_Z^u \cup \{u\} & \text{otherwise} \end{cases}.$$

We can show that $(g, <)$ is a GFlow. Indeed all vertices of $g(u) \setminus \{u\}$ and $\text{Odd}(g(u)) \setminus \{u\}$ are larger than u . Moreover if $\lambda_u = \{X, Y\}$ then $u \notin g(u)$ and $u \in \text{Odd}(g(u))$; if $\lambda_u = \{Y, Z\}$ then $u \in g(u)$ and $u \notin \text{Odd}(g(u))$; and if $\lambda_u = \{X, Z\}$ then $u \in g(u)$ and $u \in \text{Odd}(g(u))$. $\square$

4.2 Necessity of Pauli Flow for Pauli first measurement patterns

We consider here patterns where the Pauli measurements are done first and we prove that for this family of patterns, the “standard” Pauli Flow is a necessary condition for robust determinism. As u is Pauli measured if and only if $|\lambda_u| = 1$, we formally define Pauli-first measurement patterns by:

Definition 8 *A pattern $(\prod_{u \in O^c}^< X_{\mathbf{x}(u)}^{s_u} Z_{\mathbf{z}(u)}^{s_u} M_u^{\lambda_u, \alpha_u}) E_G N_{I^c}$ is Pauli-first if $|\lambda_u| < |\lambda_v| \Rightarrow u < v$.*

In the following we give a characterisation of the stabilizers of any graph state partially measured using Pauli observables.

Definition 9 Given a set A , and a collection $P_A = \{r_i P_i\}_{i \in A}$ where $P_i \in \{X, Y, Z\}$ and $r_i \in \{-1, 1\}$ of Pauli observables, let $\langle P_A | := \prod_i \langle +_{r_i \pi} |$, $x(P_A) := \{i \in A \mid P_i = X \text{ or } Y\}$ and $z(P_A) := \{i \in A \mid P_i = Y \text{ or } Z\}$.

Lemma 4 Given a graph state $|G\rangle$, a Pauli operator M , and a collection P_A of Pauli observables s.t. $|\langle P_A | G \rangle| = 2^{-\frac{|A|}{2}}$, $M \langle P_A | G \rangle \simeq \langle P_A | G \rangle$ iff

$$\exists S \subseteq V(G), S \cap z(P_A) = \text{Odd}(S) \cap x(P_A) \text{ and } M \simeq X_{S \setminus x(P_A)} Z_{\text{Odd}(S) \setminus z(P_A)}.$$

Remark 4 Notice that the condition $S \cap z(P_A) = \text{Odd}(S) \cap x(P_A)$ is equivalent to say that for all u already measured (i.e. $u \in A$),

$$\begin{cases} \text{if } u \in S \setminus \text{Odd}(S) \text{ then } u \text{ has been } X\text{-measured i.e. } u \in x(P_A) \setminus z(P_A), \\ \text{if } u \in S \cap \text{Odd}(S) \text{ then } u \text{ has been } Y\text{-measured i.e. } u \in x(P_A) \cap z(P_A), \text{ and} \\ \text{if } u \in (\text{Odd}(S) \setminus S) \text{ then } u \text{ has been } Z\text{-measured i.e. } u \in z(P_A) \setminus x(P_A). \end{cases}$$

Proof: ($\Leftarrow$) First notice that $S \cap z(P_A) = \text{Odd}(S) \cap x(P_A)$ implies $S \cap z(P_A) \subseteq x(P_A)$, so $S \setminus x(P_A) \subseteq V \setminus (z(P_A) \cup x(P_A)) = V \setminus A$. Similarly, $\text{Odd}(S) \setminus z(P_A) \subseteq V \setminus A$.

$$\begin{aligned} X_{S \setminus x(P_A)} Z_{\text{Odd}(S) \setminus z(P_A)} \langle P_A | G \rangle &= \langle P_A | X_{S \setminus x(P_A)} Z_{\text{Odd}(S) \setminus z(P_A)} | G \rangle \\ &\simeq \langle P_A | X_{S \cap (x(P_A) \setminus z(P_A))} Z_{\text{Odd}(S) \cap (z(P_A) \setminus x(P_A))} \\ &\quad (ZX)_{S \cap \text{Odd}(S) \cap x(P_A) \cap z(P_A)} X_{S \setminus x(P_A)} Z_{\text{Odd}(S) \setminus z(P_A)} | G \rangle \\ &\simeq \langle P_A | X_S Z_{\text{Odd}(S)} | G \rangle \end{aligned}$$

($\Rightarrow$) By induction on the size of A . If $A = \emptyset$ the property is true. Assume the property is true for A of size k , and let $u \in V \setminus A$. W.l.o.g. assume u is X -measured, and $|\langle +_u | \langle P_A | G \rangle| = 2^{-\frac{(|A|+1)}{2}}$. As $|\langle +_u | \langle P_A | G \rangle| = \frac{|\langle P_A | G \rangle|}{\sqrt{2}}$, $\langle P_A | G \rangle$ is not an eigenvector of X_u , so there exists a Pauli operator M' s.t. $M' \langle P_A | G \rangle \simeq \langle P_A | G \rangle$ and M' anticommutes with X_u . By induction hypothesis, there exists S_0 s.t. $S_0 \cap z(P_A) = \text{Odd}(S_0) \cap x(P_A)$ and $M' = X_{S_0 \setminus x(P_A)} Z_{\text{Odd}(S_0) \setminus z(P_A)}$. Moreover, since M' and X_u anticommute, we have $u \in \text{Odd}(S_0)$.

Notice that $\langle P_A | G \rangle = |+\rangle_u \otimes \langle +_u | \langle P_A | G \rangle + |-\rangle_u \otimes \langle -_u | \langle P_A | G \rangle$. Moreover, $\langle -_u | \langle P_A | G \rangle \simeq \langle -_u | X_{S_0 \setminus x(P_A)} Z_{\text{Odd}(S_0) \setminus z(P_A)} \langle P_A | G \rangle \simeq X_{S_0 \setminus x(P_A) \cup u} Z_{\text{Odd}(S_0) \setminus z(P_A) \cup u} \langle +_u | \langle P_A | G \rangle$, since $u \in \text{Odd}(S_0)$.

As a consequence,

$$\langle P_A | G \rangle = |+\rangle_u \otimes \langle +_u | \langle P_A | G \rangle + e^{i\theta} |-\rangle_u \otimes X_{(S_0 \setminus x(P_A)) \cup u} Z_{(\text{Odd}(S_0) \setminus z(P_A)) \cup u} \langle +_u | \langle P_A | G \rangle$$

Let M be a Pauli operator s.t. $M \langle +_u | \langle P_A | G \rangle \simeq \langle +_u | \langle P_A | G \rangle$. Notice that $\langle P_A | G \rangle$ is a eigenvector of $X_u^m \otimes M$ where

$$m = \begin{cases} 0 & \text{if } M \text{ and } X_{(S_0 \setminus x(P_A)) \cup u} Z_{(\text{Odd}(S_0) \setminus z(P_A)) \cup u} \text{ commute} \\ 1 & \text{if they anticommute} \end{cases}.$$

Thus by induction hypothesis there exists S_1 s.t. $X_u^m \otimes M \simeq X_{S_1 \setminus x(P_A)} Z_{\text{Odd}(S_1) \setminus z(P_A)}$. Notice that $m = 1 \iff u \in S_1$ so $M = X_{S_1 \setminus x(P_A)} Z_{\text{Odd}(S_1) \setminus z(P_A)} X_u^m = X_{S_1 \setminus x(P_A \cup \{u\})} Z_{\text{Odd}(S_1) \setminus z(P_A \cup \{u\})}$. □

This allows to prove the following proposition (see appendix D):

Proposition 5 *A robust deterministic Pauli-first pattern has a Pauli Flow.*

A direct corollary of Proposition 5 is that an open graph can be used to performed a deterministic computation if and only if it admits a Pauli Flow:

Theorem 3 *An open graph can be used to perform a robustly deterministic MBQC if and only if it has a Pauli Flow.*

4.3 Necessity of Shadow Pauli Flow

4.3.1 Pushing the Pauli measurements first in robust deterministic patterns

In early papers [19], it is mentioned that Pauli measurements can be done at the beginning of the computation, however it has never been proved that this transformation preserves robust determinism. We discuss in this section how Pauli measurement can be pushed to the beginning in robustly deterministic MBQC patterns preserving the robust determinism.

First we define the relation $\triangleright_u$, such that $\mathcal{P}_0 \triangleright_u \mathcal{P}_1$ if $\mathcal{P}_1$ can be obtained from $\mathcal{P}_0$ by bringing one step forward the Pauli measurement of u in $\mathcal{P}_0$.

Definition 10 *The relation $\triangleright_u$ over patterns is inductively defined as*

- $X_A^{s_v} Z_B^{s_v} M_v^{\lambda, \alpha} \mathcal{P}_0 \triangleright_u X_A^{s_v} Z_B^{s_v} M_v^{\lambda, \alpha} \mathcal{P}_1$ if $\mathcal{P}_0 \triangleright_u \mathcal{P}_1$ $v \neq u$
 - $C^{s_u} M_u^{P, \alpha} \sigma_u^{s_v} R^{s_v} M_v^{\{Q, Q'\}, \theta} \mathcal{P} \triangleright_u R'^{s_v} M_v^{\{Q, Q'\}, \theta} C'^{s_u} M_u^{P, \alpha} \mathcal{P}$
- with $\begin{cases} \text{If } \sigma P = -P\sigma & R' = CR \text{ and } C' = C \\ \text{Otherwise if } RC = CR \text{ xor } \sigma = \tilde{P} & R' = R \text{ and } C' = C \\ \text{Otherwise} & R' \in \{R, \emptyset\} \text{ and } C' \in \{\widetilde{Q}_v C, \widetilde{Q}'_v C\} \end{cases}$
- where R, C are (multi-qubit) Pauli correctors which do not act on u ; $P, Q, Q' \in \{X, Y, Z\}$; $Q \neq Q'$; $\sigma \in \{I, X, Z, XZ\}$ ⁴ and $\tilde{Q} := \begin{cases} XZ & \text{if } Q = Y \\ Q & \text{otherwise} \end{cases}$.

We say that $\mathcal{P} \triangleright \mathcal{P}'$ if there exist $u_1, \dots, u_k$ such that $\mathcal{P} \triangleright_{u_1} \dots \triangleright_{u_k} \mathcal{P}'$.

Lemma 5 *For any pattern $\mathcal{P}$, if there exists no patterns $\mathcal{P}'$ such that $\mathcal{P} \triangleright \mathcal{P}'$, then $\mathcal{P}$ is Pauli-first.*

Proof: Let u be the last vertex with a Pauli measurement following a non Pauli measurement on a vertex v , on some pattern $\mathcal{P}$, following the second case of definition 10 one can always find R' and C' to define $\mathcal{P}'$ satisfying $\mathcal{P} \triangleright_u \mathcal{P}'$. $\square$

Proposition 6 *If $\mathcal{P}_0 \triangleright \mathcal{P}_1$ and $\mathcal{P}_0$ is robustly deterministic so is $\mathcal{P}_1$. Moreover, $[\mathcal{P}_0] = [\mathcal{P}_1]$.*

The proof of this proposition is given in Appendix E.

Notice that the converse of Proposition 6 is not true: it is possible to turn a non robustly deterministic pattern into a robustly deterministic one by doing the Pauli measurements first. Indeed, the patterns $Z_3^{s_2} M_2^Z Z_2^{s_1} M_1^{Y, Z} E_{1,2} E_{2,3} N_1 N_2 N_3$ and $Z_3^{s_2} M_2^Z Z_3^{s_1} Z_2^{s_1} M_1^{Y, Z} E_{1,2} E_{2,3} N_1 N_2 N_3$ both lead to $M_1^{Y, Z} Z_1^{s_2} Z_3^{s_2} M_2^Z E_{1,2} E_{2,3} N_1 N_2 N_3$ which is robustly deterministic when the Pauli measurement is performed first, however only the former is robustly deterministic.

⁴with a slight abuse of notation $\sigma = I$ means that the correctors of v do not act on u .

4.3.2 Robust deterministic measurement patterns have Shadow Pauli Flow

The proof that having a Shadow Pauli Flow is necessary, is given in Appendix F and relies on the fact that if all patterns obtained by pushing forward a Pauli measurement have a Shadow Pauli Flow, so does the original pattern and we can build a compatible Shadow Pauli Flow using the ones where the Pauli measurements are pushed forward.

Indeed, combining Proposition 6 and Lemma 5, from any robustly deterministic pattern one can build a robustly deterministic pattern in which the Pauli measurements are done before the general measurements.

Then, given a robustly deterministic pattern $\mathcal{P}$, Lemma 5 guarantees that by pushing the Pauli measures in $\mathcal{P}$ one obtains Pauli-first patterns $\mathcal{P}'$ with $\mathcal{P} \triangleright \mathcal{P}'$. Proposition 6 ensures that each $\mathcal{P}'$ is robustly deterministic, and therefore is consistent with a Pauli Flow by Proposition 5. By induction we show in Appendix F that $\mathcal{P}$ is consistent with a Shadow Pauli Flow.

5 Computing Shadow Pauli Flow

Like GFlow and Pauli Flow, Shadow Pauli Flow can be computed efficiently:

Theorem 2 *Given an open graph of order n , and a partial order over the non-output vertices of the graphs, a Shadow Pauli Flow compatible with the partial order, or a certificate of non existence, can be computed in time polynomial in n .*

To prove Theorem 2, we consider the following algorithm: Given an open graph (G, I, O, λ) and a partial order $<$ of O^c , first it is convenient to extend $<$ to V s.t. any vertex in O is not smaller than any vertex in O^c . The algorithm consists in pre-processing all the shadow correctors: for each vertex v such that $|\lambda(v)| = 2$, find, if it exists a shadow corrector D_v s.t. $v \in \text{cor}(D_v)$ and for any $u \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}$, $\lambda(u) = \text{Act}_u^{D_v}$, so in particular $|\lambda(u)| = 1$. Such a D_v is a subset of $\lambda^{-1}(\{X\}) \cup \lambda^{-1}(\{Y\}) \cup \{v\}$ s.t. $\text{Odd}(D_v) \setminus \lambda^{-1}(\{Z\}) \subseteq \{v\}$, the action on v depending on its measurement plane, e.g. $v \in D_v$ and $v \in \text{Odd}(D_v)$ when $\lambda(v) = \{X, Z\}$. Thus, following the algebraic approach for flow [14, 15], for a fixed v , D_v can be found in $O(n^3)$ time, if it exists, by solving the linear system $M.D = b$ with the additional linear constraint $D[v] = 1$ (i.e. $v \in D$) when $Z \in \lambda(v)$ and $D[v] = 0$ (i.e. $v \notin D$) otherwise, where the column vector b and the matrix M is obtained from the adjacency matrix Γ of G by adding 1 to the diagonal on Y -measured vertices and then keeping only the columns that correspond to v and to the vertices that are X or Y measured; and only the rows that are not Z measured, thus $M = (\Gamma + \text{Id}_{\lambda^{-1}(\{Y\})})[\lambda^{-1}(\{X\}) \cup \lambda^{-1}(\{Y\}) \cup \{v\}, V \setminus \lambda^{-1}(\{Z\})]$. b is the zero vector when $\lambda(v) = YZ$ and has a single non-zero entry $b[v] = 1$ otherwise. One can double check that D_v is a shadow corrector iff it is a solution to this linear system.

Thus in time $O(n.n^3)$ one can associate with all plane-measured vertex a shadow corrector when it exists. Notice that if (G, I, O, λ) has a Shadow Pauli Flow then each vertex has at most one shadow corrector. Indeed if D and D' are two distinct shadow correctors of v , then $X_{\tilde{D}}Z_{\text{Odd}(\tilde{D})}$ is a stabilizer of the graph where $\tilde{D} = D \Delta D'$, moreover for each $u \in \tilde{D} \cup \text{Odd}(\tilde{D})$, $\lambda(u) = \text{Act}_u^{\tilde{D}}$, thus the last qubit of this set to be measured will lead to a deterministic outcome which contradicts that each measurement is balanced.

In the second part of the algorithm, the corrector function $p : O^c \rightarrow I^c$ is constructed vertex by vertex. For any u in O^c , the set of correctors $p(u)$ is obtained as the solution of the linear system $N.D = b$, where N is the matrix $N = (\Gamma + \text{Id}_{\lambda^{-1}(Y)})[\lambda^{-1}(X) \cup \lambda^{-1}(Y) \cup \{v\} \cup F_u \cup S_u, V \setminus (\lambda^{-1}(Z) \cup F_u \cup S_u)]$, where F_u is the future of u , i.e. the set

$F_u := \{w \mid u < w\}$ of vertices that are larger than u , and S_u is the set of vertices that have shadow correctors when u is measured, i.e. $S_u = \{v \mid \forall w \in D_v, w < u \text{ or } w = u\}$. The column vector b is the zero vector when $\lambda(v) = YZ$ and has a single non-zero entry $b[v] = 1$ otherwise. Solving such a linear system find correctors $p(u)$ that freely uses unmeasured qubits at this step of the computation, together with shadow corrected ones, and possibly also Pauli measured qubits when their action is compatible with their Pauli basis.

The second part of the algorithm also has a $O(n^4)$ complexity. Notice that this algorithm can be seen as a generalisation of Pauli Flow algorithms, see for instance [15], with a pre-processing to identify shadow corrected vertices, and then a layer by layer algorithm that treat shadow corrected vertices as output as long as there shadow correctors are actually in the measured subset of vertices.

6 Conclusion

In this paper, we provide a necessary and sufficient condition for robust determinism by defining the Shadow Pauli Flow, that can be efficiently computed. We have also shown that any Shadow Pauli Flow can be changed to a Pauli Flow by pushing forward the Pauli measurements.

Future work could be to consider the case where multiple qubits are measured simultaneously. Indeed, robust determinism implies that qubits are measured one by one.

Finally another perspective is to investigate the applications of the Shadow Pauli Flow in the context of graphical languages like the ZX-calculus where GFlow and Pauli Flow are already extensively used.

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A Proof of Proposition 1

Proposition 1 Given $p : O^c \rightarrow 2^{I^c}$ and $<$ a strict partial order over O^c ,
 $(p, <)$ is a Pauli Flow of (G, I, O, λ) iff

$$\forall u \in O^c, u \in \text{cor}(p(u)) \text{ and } \text{ana}_{p, <}(u) = \emptyset$$

where $\text{ana}_{p, <}(u) := \{v \in O^c, \text{ s.t. } u \neq v, \neg(v < u) \text{ and } u \in \text{imp}(p(v))\}$ is the set of vertices corrected after u which correction has an “anachronistic” impact on u .

From the proof of property 1 in [16] we know that $(p, <)$ is a Pauli Flow for (G, I, O, λ) if the following conditions hold

$$\begin{aligned} (c_X) \quad X \in \lambda_v &\Rightarrow v \in \text{Odd}(p(v)) \setminus \left(\bigcup_{\substack{\neg(u < v) \\ u \notin O \cup \{v\}}} \text{Odd}(p(u)) \right) \\ (c_Y) \quad Y \in \lambda_v &\Rightarrow v \in (\text{Odd}(p(v)) \Delta p(v)) \setminus \left(\bigcup_{\substack{\neg(u < v) \\ u \notin O \cup \{v\}}} \text{Odd}(p(u)) \Delta p(u) \right) \\ (c_Z) \quad Z \in \lambda_v &\Rightarrow v \in p(v) \setminus \left(\bigcup_{\substack{\neg(u < v) \\ u \notin O \cup \{v\}}} p(u) \right) \end{aligned}$$

Where Δ is the symmetric difference.

Notice that $v \in \text{Odd}(p(v)) \setminus \left(\bigcup_{\substack{\neg(u < v) \\ u \notin O \cup \{v\}}} \text{Odd}(p(u)) \right)$ means that $v \in \text{Odd}(p(v))$ and $\forall u \in O^c \setminus \{v\}$ s.t. $\neg(u < v), v \notin \text{Odd}(p(u))$.

Therefore, satisfying (c_X) , (c_Y) and (c_Z) is equivalent to satisfy (i) and (ii), where

- (i) $\forall u \in O^c$
 - If $X \in \lambda_u$ then $\text{Act}_u^{p(u)} \in \{Y, Z\}$ and
 - If $Y \in \lambda_u$ then $\text{Act}_u^{p(u)} \in \{X, Z\}$ and
 - If $Z \in \lambda_u$ then $\text{Act}_u^{p(u)} \in \{X, Y\}$

This corresponds to $u \in \text{cor}(p(u)) = \{v, \forall P \in \lambda_u, [\text{Act}_v^{p(u)}, P] \neq 0\}$.

- (ii) $\forall (u, v) \in (O^c)^2; \neg(u < v) \text{ and } v \neq u,$
 - If $X \in \lambda_v$ then $\text{Act}_v^{p(u)} \notin \{Z, Y\}$ and
 - If $Y \in \lambda_v$ then $\text{Act}_v^{p(u)} \notin \{X, Z\}$ and
 - If $Z \in \lambda_v$ then $\text{Act}_v^{p(u)} \notin \{X, Y\}$

This corresponds to $\text{ana}_{p, <}(u) = \emptyset$.

B Proof of lemma 1

Lemma 1 *Given an open graph with a Shadow Pauli Flow, for any $u \in O^c$,*

$$\left(\prod_{v \in B^u} \left\langle +_{\alpha_v}^{\lambda_v} \right| \right) X_{B_X^u} Z_{B_Z^u} = \pm i^{|B_X^u \cap B_Z^u|} \prod_{v \in B^u} \left\langle +_{\alpha_v}^{\lambda_v} \right| \quad (4)$$

Recall that $B^u := \{v \in p(u) \cup \text{Odd}(p(u)) \mid v < u \wedge |\lambda_v| = 1\}$ is the set of correctors which are already Pauli-measured, $B_X^u := B \cap p(u)$ and $B_Z^u := B \cap \text{Odd}(p(u))$.

For any $v \in B^u$, the Shadow Pauli Flow condition implies $v \notin \text{imp}(p(u))$. So,

- if $\lambda_v = X$ then $v \notin \text{Odd}(p(u))$, and then $v \in B_X^u \setminus B_Z^u$,
- if $\lambda_v = Y$ then $v \notin \text{Odd}(p(u)) \Delta p(u)$, and then $v \in B_X^u \cap B_Z^u$,
- if $\lambda_v = Z$ then $v \notin p(u)$, and then $v \in B_Z^u \setminus B_X^u$.

As a consequence, the Pauli operator $X_{B_X^u} Z_{B_Z^u} = (-i)^{|B_X^u \cap B_Z^u|} X_{B_X^u \setminus B_Z^u} Y_{B_Z^u \cap B_X^u} Z_{B_Z^u \setminus B_X^u}$ can be absorbed by the projectors:

$$\left(\prod_{v \in B^u} \left\langle +_{\alpha_v}^{\lambda_v} \right| \right) X_{B_X^u} Z_{B_Z^u} = \pm i^{|B_X^u \cap B_Z^u|} \prod_{v \in B^u} \left\langle +_{\alpha_v}^{\lambda_v} \right|$$

C Proof of lemma 2

Lemma 2 *Given an open graph (G, I, O, λ) with a Shadow Pauli Flow $(p, <)$, and a vertex v with non empty $\text{ana}_{p, <}(v)$, then for any one-qubit Pauli operator L_v , and any multi-qubit Pauli operator R , $\exists \theta$ s.t.*

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) R E_G |+\rangle_{I^c} |\phi\rangle_I = e^{i\theta} \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) L_v R E_G |+\rangle_{I^c} |\phi\rangle_I \quad (5)$$

where D_v is as in the definition 7 of the Shadow Pauli Flow.

Given a vertex v with non-empty $\text{ana}_{p, <}(v)$, the definition of the Shadow Pauli Flow condition implies that for any $w \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}$, $|\lambda_w| = 1$. Moreover, $w \notin \text{imp}(D_v)$, therefore:

- $\lambda_w = X \Rightarrow w \in D_v \setminus \text{Odd}(D_v)$,
- $\lambda_w = Y \Rightarrow w \in D_v \cap \text{Odd}(D_v)$,
- $\lambda_w = Z \Rightarrow w \in \text{Odd}(D_v) \setminus D_v$.

As a consequence,

$$\begin{aligned} & \left(\prod_{w \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) X_{D_v \setminus \{v\}} Z_{\text{Odd}(D_v) \setminus \{v\}} = \\ & \pm i^{|(D_v \setminus \{v\}) \cap (\text{Odd}(D_v) \setminus \{v\})|} \prod_{w \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}} \left\langle +_{\alpha_w}^{\lambda_w} \right| \end{aligned} \quad (8)$$

Like in Eq. 4, the complex number in Eq. 8 witnesses the fact that when $\lambda_w = Y$, we use $X_w Z_w$ rather than $Y_w = i X_w Z_w$.

Now, the Shadow Pauli Flow condition guarantees also that $v \in \text{cor}(D_v)$, which means that:

- $\lambda_v = \{X, Y\} \Rightarrow v \in \text{Odd}(D_v) \setminus D_v$,
- $\lambda_v = \{X, Z\} \Rightarrow v \in \text{Odd}(D_v) \cap D_v$,
- $\lambda_v = \{Y, Z\} \Rightarrow v \in D_v \setminus \text{Odd}(D_v)$.

Therefore, the previous fixed-point property (Eq. 8) can be completed to get:

$$\left(\prod_{w \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) X_{D_v} Z_{\text{Odd}(D_v)} = \pm \left(\prod_{w \in (D_v \cup \text{Odd}(D_v)) \setminus \{v\}} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) \lambda_v^\perp \quad (9)$$

where $\lambda_v^\perp = \begin{cases} Z_v & \text{if } \lambda_v = \{X, Y\} \\ Y_v = iX_v Z_v & \text{if } \lambda_v = \{X, Z\} \\ X_v & \text{if } \lambda_v = \{Y, Z\} \end{cases}$.

Notice that the term $i^{|(D_v \setminus \{v\}) \cap (\text{Odd}(D_v) \setminus \{v\})|}$ of Eq. 8 is not present in Eq. 9 has it is absorbed by $\lambda_v^\perp$. Indeed, for any D , $|D \cap \text{Odd}(D)| = 0 \bmod 2$, thus $|(D_v \setminus \{v\}) \cap (\text{Odd}(D_v) \setminus \{v\})| = 1 \bmod 2$ iff $v \in D_v \cap \text{Odd}(D_v)$, i.e. $\lambda_v = \{X, Z\}$.

Thus by combining Eq. 9 and 2, we get that for any $v \in A^u$, and any (multi-qubit) Pauli operator R ,

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) RE_G |+\rangle_{I^c} |\phi\rangle_I = \pm \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) \lambda_v^\perp RE_G |+\rangle_{I^c} |\phi\rangle_I \quad (10)$$

We additionally have, according to Eq. 3, for some $P_v, Q_v \in \lambda_v$.

$$\begin{aligned} & \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) RE_G |+\rangle_{I^c} |\phi\rangle_I \\ &= \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) (\cos(\alpha_v) P_v + \sin(\alpha_v) Q_v) RE_G |+\rangle_{I^c} |\phi\rangle_I \\ &= \cos(\alpha_v) \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) P_v RE_G |+\rangle_{I^c} |\phi\rangle_I \\ &+ \sin(\alpha_v) \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) Q_v RE_G |+\rangle_{I^c} |\phi\rangle_I \end{aligned} \quad (11)$$

Thanks to Equation 9, we have

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) Q_v RE_G |+\rangle_{I^c} |\phi\rangle_I = \pm \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) \lambda_v^\perp Q_v RE_G |+\rangle_{I^c} |\phi\rangle_I$$

Moreover, since $\lambda_v = \{P_v, Q_v\}$, we have $\lambda_v^\perp Q = \pm i P$. Thus,

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) \lambda_v^\perp Q_v RE_G |+\rangle_{I^c} |\phi\rangle_I = \pm i \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) P_v RE_G |+\rangle_{I^c} |\phi\rangle_I$$

As a consequence,

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) RE_G |+\rangle_{I^c} |\phi\rangle_I =$$

$$(\cos(\alpha_v) \pm i \sin(\alpha_v)) \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) P_v RE_G |+\rangle_{I^c} |\phi\rangle_I$$

Hence, as $\text{ana}_{p, <}(v) \neq \emptyset$, it can absorb, up to a global phase, one of the Pauli operators of the plane λ_v . Combined with equation 10, any Pauli operator on such a vertex v can be absorbed, and we get Eq 5: for any one-qubit Pauli operator L_v , and any multi-qubit Pauli operator R , $\exists \theta$ s.t.

$$\left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) RE_G |+\rangle_{I^c} |\phi\rangle_I = e^{i\theta} \left(\prod_{w \in D_v \cup \text{Odd}(D_v)} \left\langle +_{\alpha_w}^{\lambda_w} \right| \right) L_v RE_G |+\rangle_{I^c} |\phi\rangle_I$$

D Proof of Proposition 5

Proposition 5 *A robust deterministic Pauli-first pattern has a Pauli Flow.*

Proof:

$$\mathcal{P}' = \left(\prod_{u \in O^c \text{ s.t. } |\lambda_u|=2}^{\prec} C_u^{s_u} M_u^{\lambda_u, \alpha_u} \right) \left(\prod_{u \in O^c \text{ s.t. } |\lambda_u|=1}^{\prec} C_u^{s_u} M_u^{\lambda_u, \beta_u} \right) E_G N_{V \setminus I}$$

Let $\langle \vec{\alpha} | := \bigotimes_{u \in O^c \text{ s.t. } |\lambda_u|=2} \langle +_{\alpha_u}^{\lambda_u} |$. Notice that if $\forall u \in O^c, |\lambda_u| = 1$, then $\langle \vec{\alpha} |$ is the identity.

The proof is by induction. Let u be the largest element according to $<$. For any $|\phi\rangle_I$, uniformly strong determinism implies that

$$\langle \vec{\alpha} | \langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq C_u \langle \vec{\alpha} | \langle P | \bar{\lambda}_u E_G N_{V \setminus I} | \phi \rangle_I$$

where $\bar{\lambda}_u$ is any Pauli operator which is not proportional to a Pauli operator in λ_u (typically $\bar{\lambda}_u = Z$ when $\lambda_u = \{X, Y\}$). So

$$\langle \vec{\alpha} | \langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq \langle \vec{\alpha} | C_u \langle P | \bar{\lambda}_u E_G N_{V \setminus I} | \phi \rangle_I$$

By applying Lemma 3 for each non-Pauli measurement, $\exists L \subseteq \{u \mid |\lambda_u| = 2\}$, $\exists x \in \{0, 1\}^{|L|}$, $\exists |\psi\rangle$ s.t. $\langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq (B |x\rangle_L) \otimes |\psi\rangle$ and $C_u \langle P | \bar{\lambda}_u E_G N_{V \setminus I} | \phi \rangle_I \simeq (B |\bar{x}\rangle_L) \otimes |\psi\rangle$, where $\forall v \in L$,

$$B_v := \begin{cases} I & \text{if } \lambda_v = \{X, Y\} \\ H & \text{if } \lambda_v = \{Y, Z\} \\ \sqrt{X} & \text{if } \lambda_v = \{Z, X\} \end{cases} \text{ where for a Pauli } \sigma, \sqrt{\sigma} = \frac{I+i\sigma}{2}$$

So $\langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq C_u \bar{\lambda}_u (B |\bar{x}\rangle_L \otimes |\psi\rangle)$

- If $u \notin L$, $\langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq B |\bar{x}\rangle_L \otimes C_u \bar{\lambda}_u |\psi\rangle$, so $B |\bar{x}\rangle_L \otimes C_u \bar{\lambda}_u |\psi\rangle \simeq B |x\rangle_L \otimes |\psi\rangle$ which implies $L = \emptyset$.

- If $u \in L$, $\langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq \bar{\lambda}_u B | \bar{x} \rangle_L \otimes C_u | \psi \rangle$, so $\bar{\lambda}_u B | \bar{x} \rangle_L \otimes C_u | \psi \rangle \simeq B | x \rangle_L \otimes | \psi \rangle$. Notice that $\langle x |_L B^\dagger \bar{\lambda}_u B | \bar{x} \rangle_L = 0$, indeed:

- if $\lambda_u = \{X, Y\}$ then $B_u = I$ and $\bar{\lambda}_u \simeq Z$ so $\langle x |_u B_u^\dagger \bar{\lambda}_u B_u | \bar{x} \rangle_u = 0$.
- if $\lambda_u = \{Z, X\}$ then $B_u = \sqrt{X}$ and $\bar{\lambda}_u \simeq Y$ so $\langle x |_u B_u^\dagger \bar{\lambda}_u B_u | \bar{x} \rangle_u = 0$ since $\sqrt{X}^\dagger Y \sqrt{X} \simeq Z$.
- if $\lambda_u = \{Y, Z\}$ then $B_u = H$ and $\bar{\lambda}_u \simeq X$ so $\langle x |_u B_u^\dagger \bar{\lambda}_u B_u | \bar{x} \rangle_u = 0$ since $H X H \simeq Z$.

It implies $L = \emptyset$. As a consequence $\langle P | E_G N_{V \setminus I} | \phi \rangle_I \simeq C_u \langle P | \bar{\lambda}_u E_G N_{V \setminus I} | \phi \rangle_I$

When $|\phi\rangle_I = |+\rangle_I$, we have $\langle P | |G\rangle \simeq C_u \langle P | \bar{\lambda}_u |G\rangle$.

- If $|\lambda_u| = 2$ then $C_u \bar{\lambda}_u \langle P | |G\rangle \simeq \langle P | |G\rangle$, so according to Lemma 4

$$C_u \bar{\lambda}_u \simeq X_{S \setminus x(P)} Z_{\text{Odd}(S) \setminus z(P)} \quad (12)$$

with

$$S \cap z(P) = \text{Odd}(S) \cap x(P). \quad (13)$$

By considering the case $|\phi\rangle_I = |0\rangle_I$, it also implies $S \subseteq I^c$. Moreover C_u should not act on the already measured qubits so we recover in the following the Pauli Flow condition by defining $p(u) := S$. Indeed assume w.l.o.g. that $\lambda_u = \{X, Y\}$, so $\bar{\lambda}_u = Z$, and according to Eq. 12, $u \in \text{Odd}(S) \setminus S$.

Moreover, for any $v < u$, with $v \in S \cup \text{Odd}(S)$,

- if $v \in S \setminus \text{Odd}(S)$, Eq. 12 implies that $v \in x(P)$ and Eq. 13 that $v \notin z(P)$, so $\lambda_v = \{X\}$;
- if $v \in S \cap \text{Odd}(S)$, Eq. 12 implies that $v \in x(P)$ and $v \in z(P)$, so $\lambda_v = \{Y\}$;
- if $v \in \text{Odd}(S) \setminus S$, Eq. 12 implies that $v \in z(P)$ and Eq. 13 that $v \notin x(P)$, so $\lambda_v = \{Z\}$.
- If $|\lambda_u| = 1$, assume $\lambda_u = \{X\}$. Let A be the set of all Pauli measured qubits but u .

$$\langle P_A | |G\rangle = |+\rangle_u \otimes \langle P | |G\rangle + e^{i\theta} |-\rangle_u \otimes \langle P | \bar{\lambda}_u |G\rangle$$

Strongness implies that there exists a stabilizer M of $\langle P_A | |G\rangle$ which anticommutes with λ_u . According to Lemma 4, $\exists S_0$ s.t. $S_0 \cap z(P_A) = \text{Odd}(S_0) \cap x(P_A)$ and $M = X_{S_0 \setminus x(P_A)} Z_{\text{Odd}(S_0) \setminus z(P_A)}$, moreover $u \in \text{Odd}(S_0)$ since M and $\lambda_u = \{X\}$ anticommute.

Since $X_{S_0 \setminus x(P_A)} Z_{\text{Odd}(S_0) \setminus z(P_A)} \langle P_A | |G\rangle \simeq \langle P_A | |G\rangle$, we have

$$\begin{aligned} \langle P | \bar{\lambda}_u |G\rangle &\simeq \langle + |_u \bar{\lambda}_u \langle P_A | |G\rangle \\ &\simeq \langle + |_u Z_u \langle P_A | |G\rangle \\ &\simeq \langle + |_u Z_u X_{S_0 \setminus x(P_A)} Z_{\text{Odd}(S_0) \setminus z(P_A)} \langle P_A | |G\rangle \\ &\simeq X_{S_0 \setminus x(P_A) \setminus u} Z_{\text{Odd}(S_0) \setminus z(P_A)} Z_u \langle P | |G\rangle \end{aligned}$$

So $\langle P | |G\rangle \simeq C_u X_{S_0 \setminus x(P_A) \setminus u} Z_{\text{Odd}(S_0) \setminus z(P_A)} Z_u \langle P | |G\rangle$

As a consequence, according to Lemma 4, $\exists S_1$ s.t. $S_1 \cap z(P) = \text{Odd}(S_1) \cap x(P)$ and

$$C_u X_{S_0 \setminus x(P_A) \setminus u} Z_{\text{Odd}(S_0) \setminus z(P_A)} Z_u \simeq X_{S_1 \setminus x(P) \setminus u} Z_{\text{Odd}(S_1) \setminus z(P)}$$

So $C_u = X_{S_0 \Delta S_1 \setminus x(P)} Z_{\text{Odd}(S_0 \Delta S_1) \setminus z(P)} Z_u$.

Moreover, with $|\phi\rangle_I = |0\rangle_I$, we have $S_0 \Delta S_1 \subseteq I^c$. We recover the Pauli Flow condition by defining $p(u) := S_0 \Delta S_1$.

□

E Proof of Proposition 6

Proposition 6 *If $\mathcal{P}_0 \triangleright \mathcal{P}_1$ and $\mathcal{P}_0$ is robustly deterministic so is $\mathcal{P}_1$. Moreover, $\llbracket \mathcal{P}_0 \rrbracket = \llbracket \mathcal{P}_1 \rrbracket$.*

As $\triangleright$ is defined by a sequence of $\triangleright_u$, we just need to prove by induction that if $\mathcal{P}_0 \triangleright_u \mathcal{P}_1$ and $\mathcal{P}_0$ is robustly deterministic then so is $\mathcal{P}_1$ and $\llbracket \mathcal{P}_0 \rrbracket = \llbracket \mathcal{P}_1 \rrbracket$.

- Let $\mathcal{P}_0 = X_A^{s_v} Z_B^{s_v} M_v^{\lambda, \theta_v} \mathcal{P}'_0$ and $\mathcal{P}_1 = X_A^{s_v} Z_B^{s_v} M_v^{\lambda, \theta_v} \mathcal{P}'_1$ with $\mathcal{P}'_0 \triangleright_u \mathcal{P}'_1$. Since $\mathcal{P}_0$ is robustly deterministic, so is $\mathcal{P}'_0$, by induction $\mathcal{P}'_1$ is robustly deterministic and $\llbracket \mathcal{P}'_0 \rrbracket = \llbracket \mathcal{P}'_1 \rrbracket$ hence $\llbracket \mathcal{P}_0 \rrbracket = \llbracket \mathcal{P}_1 \rrbracket$ and $\mathcal{P}_1$ is robustly deterministic.
- Let $\mathcal{P}_0 = C^{s_u} M_u^{P, \theta_u} \sigma_u^{s_v} R^{s_v} M_v^{\{Q, Q'\}, \theta_v} \mathcal{P}$ and $\mathcal{P}_1 = R^{s_v} M_v^{\{Q, Q'\}, \theta_v} C^{s_u} M_u^{P, \theta_u} \mathcal{P}$.

Let r and c in $\{0, 1\}$ s.t. $R^2 = (-1)^r I$ and $C^2 = (-1)^c I$. Let $|\varphi\rangle$ be the state of the system after the application of $\mathcal{P}$. There exist four (not necessarily normalized) vectors $|\phi_{i,j}\rangle$ s.t.

$$\begin{aligned} |\varphi\rangle = & \left| +_0^{Q, Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle + \left| +_0^{Q, Q'} \right\rangle_v \left| -_0^P \right\rangle_u |\phi_{01}\rangle \\ & + \left| -_0^{Q, Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{10}\rangle + \left| -_0^{Q, Q'} \right\rangle_v \left| -_0^P \right\rangle_u |\phi_{11}\rangle \end{aligned}$$

Let $|\Psi_{s_v}^{\theta_v}\rangle$ be the state of the system after the application of the measurements of v , and appropriate corrections. When $\theta_v = 0$ (i.e. v is Q -measured) and $s_v = 0$, $|\Psi_0^0\rangle = \left| +_0^P \right\rangle_u |\phi_{00}\rangle + \left| -_0^P \right\rangle_u |\phi_{01}\rangle$.

When followed by the P -measurement of u and the corresponding corrections, the two possible states, $|\phi_{00}\rangle \simeq C |\phi_{01}\rangle$ because of the strong determinism of $\mathcal{P}'$. Thus $\exists \alpha$ s.t. $|\phi_{01}\rangle = e^{i\alpha} C |\phi_{00}\rangle$ and

$$|\Psi_0^0\rangle = \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i\alpha} \left| -_0^P \right\rangle_u C |\phi_{00}\rangle.$$

Similarly, for $\theta_v = \pi$, we obtain that $\exists \beta$ s.t. $|\phi_{11}\rangle = e^{i\beta} C |\phi_{10}\rangle$. Thus,

$$\begin{aligned} |\varphi\rangle = & \left| +_0^{Q, Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i\alpha} \left| +_0^{Q, Q'} \right\rangle_v \left| -_0^P \right\rangle_u C |\phi_{00}\rangle \\ & + \left| -_0^{Q, Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{10}\rangle + e^{i\beta} \left| -_0^{Q, Q'} \right\rangle_v \left| -_0^P \right\rangle_u C |\phi_{10}\rangle \end{aligned}$$

[i] Assume $\sigma P = -P\sigma$:

We consider the case $\theta_v = 0$ and $s_v = 1$ which, after the appropriate corrections, produces the state:

$$|\Psi_1^0\rangle \simeq \left| -_0^P \right\rangle_u R |\phi_{10}\rangle + (-1)^d e^{i\beta} \left| +_0^P \right\rangle_u RC |\phi_{10}\rangle$$

with $d \in \{0, 1\}$ ⁵

Since $\mathcal{P}_0$ is robustly deterministic, the possible states of the register after the measurement of v (and the corresponding corrections) should be the same up to a global phase ($|\Psi_1^0\rangle \simeq |\Psi_0^0\rangle$). As a consequence, $\exists \gamma$ s.t. $(-1)^d e^{i\beta} RC |\phi_{10}\rangle = e^{i\gamma} |\phi_{00}\rangle$ and $R |\phi_{10}\rangle = e^{i(\alpha+\gamma)} C |\phi_{00}\rangle$,

⁵According to the definition of the measurement basis $\{ \left| +_{\theta_u}^P \right\rangle, \left| -_{\theta_u}^P \right\rangle \}$ (Equation 1), each $d \in \{0, 1\}$ corresponds to a choice of σ anticommuting with P , more precisely we have $d = 0$ if and only if $\sigma = Z$ or $(\sigma = X \text{ and } P = Z)$.

So $R|\phi_{10}\rangle = (-1)^d e^{i(\alpha+\beta)} C R C |\phi_{10}\rangle = (-1)^{a+d} e^{i(\alpha+\beta)} C^2 R |\phi_{10}\rangle = (-1)^{a+c+d} e^{i(\alpha+\beta)} R |\phi_{10}\rangle$, where $a \in \{0, 1\}$ is s.t. $RC = (-1)^a CR$.

It implies $\beta = -\alpha + (a + c + d)\pi \bmod 2\pi$, $C|\phi_{10}\rangle = (-1)^{d+r} e^{i(\gamma-\beta)} R|\phi_{00}\rangle$, and $|\phi_{10}\rangle = (-1)^r e^{i(\alpha+\gamma)} RC|\phi_{00}\rangle$.

As a consequence,

$$|\varphi\rangle = \left| +_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i\alpha} \left| +_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u C |\phi_{00}\rangle \\ + (-1)^r e^{i(\alpha+\gamma)} \left| -_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u RC |\phi_{00}\rangle + (-1)^{r+d} e^{i\gamma} \left| -_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u R |\phi_{00}\rangle$$

So, with $\delta = \gamma + r\pi$, we get

$$|\varphi\rangle = \left| +_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i(\alpha+\delta)} \left| -_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u RC |\phi_{00}\rangle \\ + e^{i\alpha} \left| +_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u C |\phi_{00}\rangle + (-1)^d e^{i\delta} \left| -_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u R |\phi_{00}\rangle$$

We consider two cases depending whether $|\phi_{00}\rangle$ and $RC|\phi_{00}\rangle$ are independent or not.

- If $|\phi_{00}\rangle$ and $RC|\phi_{00}\rangle$ are not colinear, then we consider the measurement of v according to $\theta_v = \frac{\pi}{2}$,

Since for any $Q \neq Q'$, $\langle +_{\pi/2}^{Q,Q'} | +_0^{Q,Q'} \rangle = \langle -_{\pi/2}^{Q,Q'} | -_0^{Q,Q'} \rangle = \frac{1-i}{2}$ and $\langle +_{\pi/2}^{Q,Q'} | -_0^{Q,Q'} \rangle = \langle -_{\pi/2}^{Q,Q'} | +_0^{Q,Q'} \rangle = \frac{1+i}{2} = \frac{1-i}{2}i$, when the outcome is $s_v = 0$, the resulting state is,

$$\left| \Psi_0^{\frac{\pi}{2}} \right\rangle \simeq \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i(\alpha+\delta')} \left| +_0^P \right\rangle_u RC |\phi_{00}\rangle \\ + e^{i\alpha} \left| -_0^P \right\rangle_u C |\phi_{00}\rangle + (-1)^d e^{i\delta'} \left| -_0^P \right\rangle_u R |\phi_{00}\rangle$$

where $\delta' = \delta + \pi/2$.

When followed by the P -measurement of u and the corresponding corrections, the two possible states:

$$|\phi_{00}\rangle + e^{i(\alpha+\delta')} RC |\phi_{00}\rangle \quad \text{and} \quad (-1)^c e^{i\alpha} |\phi_{00}\rangle + (-1)^{a+d} e^{i\delta'} RC |\phi_{00}\rangle$$

are equal up to a global phase because of the strong determinism of $\mathcal{P}_0$. Thus $\exists \epsilon$ s.t.

$$e^{i\epsilon} (|\phi_{00}\rangle + e^{i(\alpha+\delta')} RC |\phi_{00}\rangle) = (-1)^c e^{i\alpha} |\phi_{00}\rangle + (-1)^{a+d} e^{i\delta'} RC |\phi_{00}\rangle$$

It implies, since $|\phi_{00}\rangle$ and $RC|\phi_{00}\rangle$ are not colinear, that $\epsilon = \alpha + c\pi \bmod 2\pi$ and $\epsilon + \alpha + \delta' = (a + d)\pi + \delta' \bmod 2\pi$, so $2\alpha = (a + c + d)\pi \bmod 2\pi$.

As a consequence,

$$|\varphi\rangle = \left| +_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i(\alpha+\delta)} \left| -_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u RC |\phi_{00}\rangle \\ + (-1)^{a+c+d} e^{-i\alpha} \left| +_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u C |\phi_{00}\rangle \\ + (-1)^d e^{i\delta} \left| -_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u R |\phi_{00}\rangle$$

Hence, the patterns $C^{s_u} M_u^{P, \theta_u} \mathcal{P}$ and $\mathcal{P}_1 = C^{s_v} R^{s_v} \mathbf{M}_v^{\{Q, Q'\}, \theta_v} C^{s_u} \mathbf{M}_u^{P, \theta_u} \mathcal{P}$ are strongly deterministic, so $\mathcal{P}_1$ is robustly deterministic.

Indeed after the measurement of u in $C^{s_u} M_u^{P, \theta_u} \mathcal{P}$ the two possible states after correction are $|\varphi_0^0\rangle = \left| +_0^{Q, Q'} \right\rangle_v |\phi_{00}\rangle + e^{i(\alpha+\delta)} \left| -_0^{Q, Q'} \right\rangle_v RC |\phi_{00}\rangle$ and $|\varphi_0^1\rangle = (-1)^{a+c+d} e^{-i\alpha} \left| +_0^{Q, Q'} \right\rangle_v C^2 |\phi_{00}\rangle + (-1)^d e^{i\delta} \left| -_0^{Q, Q'} \right\rangle_v CR |\phi_{00}\rangle = (-1)^{a+d} e^{-i\alpha} |\varphi_0^0\rangle$, as $RC = (-1)^a CR$. For the full pattern, we check that for any v measurement angle θ_v , the branches (after both measurements) with same s_u and s_v in $\mathcal{P}_0$ and $\mathcal{P}_1$ are equal up to a global phase.

* When $s_u = s_v = 0$ both branches are obviously equal.

* When $s_v = 0$, and $s_u = 1$ the resulting states are respectively $C \langle -_0^P |_u \langle +_{\theta_v}^{Q, Q'} |_v |\varphi\rangle$ for $\mathcal{P}_0$ and $\langle +_{\theta_v}^{Q, Q'} |_v C \langle -_0^P |_u |\varphi\rangle$ for $\mathcal{P}_1$ which are equal.

* When $s_v = 1$ and $s_u = 0$ the resulting states are respectively $\langle +_0^P |_u \sigma_u R \langle -_{\theta_v}^{Q, Q'} |_v |\varphi\rangle$ for $\mathcal{P}_0$ and $CR \langle -_{\theta_v}^{Q, Q'} |_v \langle +_0^P |_u |\varphi\rangle$ for $\mathcal{P}_1$. The strong determinism of $C^{s_u} M_u^{P, \theta_u} \mathcal{P}$ implies that $C \langle -_0^P |_u |\varphi\rangle \simeq \langle +_0^P |_u |\varphi\rangle$ so $\langle -_0^P |_u |\varphi\rangle \simeq C \langle +_0^P |_u |\varphi\rangle$. Moreover $\langle +_0^P |_u \sigma_u \simeq \langle -_0^P |_u$ since σ and P anticommute, as a consequence the two branches are equal up to a global phase.

* Similarly, when $s_u = s_v = 1$ both branches are equal up to a global phase.

Thus the strong determinism of $\mathcal{P}_0$ ensures the strong determinism of $\mathcal{P}_1$, moreover $[\mathcal{P}_0] = [\mathcal{P}_1]$.

– If $|\phi_{00}\rangle$ and $RC |\phi_{00}\rangle$ are colinear, then $|\phi_{00}\rangle$ is an eigenvector of RC , so $\exists k \in \{0, 1, 2, 3\}$ s.t. $RC |\phi_{00}\rangle = i^k |\phi_{00}\rangle$, then $C |\phi_{00}\rangle = i^{2r+k} R |\phi_{00}\rangle$, and

$$\begin{aligned} |\varphi\rangle &= \left| +_0^{Q, Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle + i^k e^{i(\alpha+\delta)} \left| -_0^{Q, Q'} \right\rangle_v \left| +_0^P \right\rangle_u |\phi_{00}\rangle \\ &\quad + i^{2r+k} e^{i\alpha} \left| +_0^{Q, Q'} \right\rangle_v \left| -_0^P \right\rangle_u R |\phi_{00}\rangle + (-1)^d e^{i\delta} \left| -_0^{Q, Q'} \right\rangle_v \left| -_0^P \right\rangle_u R |\phi_{00}\rangle \\ &= \left(\left| +_0^{Q, Q'} \right\rangle_v + i^k e^{i(\alpha+\delta)} \left| -_0^{Q, Q'} \right\rangle_v \right) \left| +_0^P \right\rangle_u |\phi_{00}\rangle \\ &\quad + i^{2r+k} e^{i\alpha} \left(\left| +_0^{Q, Q'} \right\rangle_v + i^{2r-k+2d} e^{i(\delta-\alpha)} \left| -_0^{Q, Q'} \right\rangle_v \right) \left| -_0^P \right\rangle_u R |\phi_{00}\rangle \end{aligned}$$

Strongness implies

$$\begin{aligned} &\left| \left\langle +_{\pi/2}^{Q, Q'} \right| \left(\left| +_0^{Q, Q'} \right\rangle_v + i^k e^{i(\alpha+\delta)} \left| -_0^{Q, Q'} \right\rangle_v \right) \right| = \\ &\left| \left\langle -_{\pi/2}^{Q, Q'} \right| \left(\left| +_0^{Q, Q'} \right\rangle_v + i^k e^{i(\alpha+\delta)} \left| -_0^{Q, Q'} \right\rangle_v \right) \right| \end{aligned}$$

and

$$\begin{aligned} &\left| \left\langle +_{\pi/2}^{Q, Q'} \right| \left(\left| +_0^{Q, Q'} \right\rangle_v + i^{2r-k+2d} e^{i(\delta-\alpha)} \left| -_0^{Q, Q'} \right\rangle_v \right) \right| = \\ &\left| \left\langle -_{\pi/2}^{Q, Q'} \right| \left(\left| +_0^{Q, Q'} \right\rangle_v + i^{2r-k+2d} e^{i(\delta-\alpha)} \left| -_0^{Q, Q'} \right\rangle_v \right) \right|, \end{aligned}$$

so $\exists \ell, m \in \{0, 1\}$ s.t. $i^k e^{i(\alpha+\delta)} = (-1)^\ell i$, $e^{i(\delta-\alpha-r\pi-d\pi-k\frac{\pi}{2})} = (-1)^m i$, which implies $2\delta = (r+m+\ell+d+1)\pi \bmod 2\pi$, and $2\alpha = (r+m+\ell+d+k)\pi \bmod 2\pi$.

As a consequence,

$$\begin{aligned} |\varphi\rangle &= \left(\left| +_0^{Q, Q'} \right\rangle_v + (-1)^\ell i \left| -_0^{Q, Q'} \right\rangle_v \right) \left| +_0^P \right\rangle_u |\phi_{00}\rangle \\ &\quad + i^{2r+k} e^{i\alpha} \left(\left| +_0^{Q, Q'} \right\rangle_v + (-1)^m i \left| -_0^{Q, Q'} \right\rangle_v \right) \left| -_0^P \right\rangle_u R |\phi_{00}\rangle \end{aligned}$$

Thus

$$\left| \Psi_0^{\frac{\pi}{2}} \right\rangle = \sqrt{2} \left(e^{(-1)^\ell \frac{\pi}{4} i} \left| +_0^P \right\rangle_u \left| \phi_{00} \right\rangle + i^{2r+k} e^{i\alpha} e^{(-1)^m \frac{\pi}{4} i} \left| -_0^P \right\rangle_u R \left| \phi_{00} \right\rangle \right)$$

and

$$\left| \Psi_1^{\frac{\pi}{2}} \right\rangle = \sqrt{2} \left(e^{(-1)^\ell \frac{\pi}{4} i} \left| -_0^P \right\rangle_u R \left| \phi_{00} \right\rangle + (-1)^d i^k e^{i\alpha} e^{(-1)^m \frac{\pi}{4} i} \left| +_0^P \right\rangle_u \left| \phi_{00} \right\rangle \right)$$

Stepwise strong determinism implies that $\exists \epsilon$ s.t. $e^{i\epsilon} e^{(-1)^\ell \frac{\pi}{4} i} = (-1)^d i^k e^{i\alpha} e^{(-1)^m \frac{\pi}{4} i}$ and $e^{i\epsilon} i^{2r+k} e^{i\alpha} e^{(-1)^m \frac{\pi}{4} i} = e^{(-1)^\ell \frac{\pi}{4} i}$. As a consequence $2\alpha = (r+k+d)\pi \bmod 2\pi$. Combining with previous equations on 2α , namely $2\alpha = (r+m+\ell+d+k)\pi \bmod 2\pi$, we get $m = \ell \bmod 2$. As a consequence:

$$\left| \varphi \right\rangle = \left(\left| +_0^{Q,Q'} \right\rangle_v + (-1)^\ell i \left| -_0^{Q,Q'} \right\rangle_v \right) \otimes \left(\left| +_0^P \right\rangle_u \left| \phi_{00} \right\rangle + i^{2r+k} e^{i\alpha} \left| -_0^P \right\rangle_u R \left| \phi_{00} \right\rangle \right)$$

Hence, the pattern $C^{s_u} M_u^{P,\theta_u} \mathcal{P}$ is strongly deterministic as $CR \left| \phi_{00} \right\rangle \simeq \left| \phi_{00} \right\rangle$, and the four branches $\mathcal{P}_1 = C^{s_v} R^{s_v} \mathbf{M}_v^{\{Q,Q'\},\theta_v} C^{s_u} M_u^{P,\theta_u} \mathcal{P}$ are equal up to a global phase to the corresponding branches of $\mathcal{P}_0$ which guarantees strong determinism of $\mathcal{P}_1$ and that $[\mathcal{P}_1] = [\mathcal{P}_0]$. Therefore $\mathcal{P}_1$ is robustly deterministic.

[ii] Otherwise $\sigma P = P\sigma$,

Now, we consider the case $\theta_v = 0$ and $s_v = 1$ which, after the appropriate corrections, produces the state:

$$\left| \Psi_1^0 \right\rangle \simeq \left| +_0^P \right\rangle_u R \left| \phi_{10} \right\rangle + (-1)^{d'} e^{i\beta} \left| -_0^P \right\rangle_u RC \left| \phi_{10} \right\rangle$$

with $d' \in \{0, 1\}$ s.t. $d' = 0$ iff $\sigma = I$.

Since $\mathcal{P}_0$ is stepwise and strong deterministic the possible states of the register after the measurement of v (and the corresponding corrections) should be the same up to a global phase ($\left| \Psi_1^0 \right\rangle \simeq \left| \Psi_0^0 \right\rangle$). As a consequence, $\exists \gamma$ s.t. $R \left| \phi_{10} \right\rangle = e^{i\gamma} \left| \phi_{00} \right\rangle$ and $(-1)^{d'} e^{i\beta} RC \left| \phi_{10} \right\rangle = e^{i(\alpha+\gamma)} C \left| \phi_{00} \right\rangle$, so $e^{i(\beta+\gamma)} = (-1)^{a+d'} e^{i(\alpha+\gamma)}$ where $a \in \{0, 1\}$ is s.t. $RC = (-1)^a CR$. It implies $\beta = \alpha + (a+d')\pi \bmod 2\pi$.

As a consequence,

$$\begin{aligned} \left| \varphi \right\rangle &= \left| +_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u \left| \phi_{00} \right\rangle + e^{i\alpha} \left| +_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u C \left| \phi_{00} \right\rangle \\ &\quad + (-1)^r e^{i\gamma} \left| -_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u R \left| \phi_{00} \right\rangle \\ &\quad + (-1)^r e^{i(\alpha+\gamma+(a+d')\pi)} \left| -_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u CR \left| \phi_{00} \right\rangle \end{aligned}$$

So, with $\delta = \gamma + r\pi$ we get

$$\begin{aligned} \left| \varphi \right\rangle &= \left| +_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u \left| \phi_{00} \right\rangle + e^{i\delta} \left| -_0^{Q,Q'} \right\rangle_v \left| +_0^P \right\rangle_u R \left| \phi_{00} \right\rangle \\ &\quad + e^{i\alpha} \left| +_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u C \left| \phi_{00} \right\rangle + e^{i(\alpha+\delta+(a+d')\pi)} \left| -_0^{Q,Q'} \right\rangle_v \left| -_0^P \right\rangle_u CR \left| \phi_{00} \right\rangle \end{aligned}$$

[ii.1] If $a + d' = 0$ we see that the patterns $C^{s_u} M_u^{P,\alpha} \mathcal{P}$ and $\mathcal{P}_1 = R^{s_v} \mathbf{M}_v^{\{Q,Q'\},\theta} C^{s_u} M_u^{P,\alpha} \mathcal{P}$ are strongly deterministic, so $\mathcal{P}_1$ is robustly deterministic.

Indeed after the measurement of u in $C^{s_u} M_u^{P,\theta_u} \mathcal{P}$ the two possible states after correction are $\left| \varphi_0^0 \right\rangle = \left| +_0^{Q,Q'} \right\rangle_v \left| \phi_{00} \right\rangle + e^{i\delta} \left| -_0^{Q,Q'} \right\rangle_v R \left| \phi_{00} \right\rangle$ and $\left| \varphi_0^1 \right\rangle =$

$e^{i\alpha} \left| +_0^{Q,Q'} \right\rangle_v C^2 |\phi_{00}\rangle + e^{i(\alpha+\delta)} \left| -_0^{Q,Q'} \right\rangle_v C^2 R |\phi_{00}\rangle = (-1)^c e^{i\alpha} |\varphi_0^0\rangle$, as $C^2 = (-1)^c I$. For the full pattern, we check that for any v measurement angle θ_v , the branches (after both measurements) with same s_u and s_v in $\mathcal{P}_0$ and $\mathcal{P}_1$ are equal up to a global phase.

- When $s_u = s_v = 0$ both branches are obviously equal.
- When $s_v = 0$, and $s_u = 1$ the resulting states are respectively $C \langle -_0^P |_u \left\langle +_{\theta_v}^{Q,Q'} \right|_v |\varphi\rangle$ for $\mathcal{P}_0$ and $\left\langle +_{\theta_v}^{Q,Q'} \right|_v C \langle -_0^P |_u |\varphi\rangle$ for $\mathcal{P}_1$ which are equal.
- When $s_v = 1$ and $s_u = 0$ the resulting states are respectively $\langle +_0^P |_u \sigma_u R \left\langle -_{\theta_v}^{Q,Q'} \right|_v |\varphi\rangle$ for $\mathcal{P}_0$ and $R \left\langle -_{\theta_v}^{Q,Q'} \right|_v \langle +_0^P |_u |\varphi\rangle$ for $\mathcal{P}_1$. Since σ and P commute, $\langle +_0^P |_u \sigma_u \simeq \langle +_0^P |_u$ as a consequence the two branches are equal up to a global phase.
- Similarly, when $s_u = s_v = 1$ both branches are equal up to a global phase.

Thus the strong determinism of $\mathcal{P}_0$ ensures the strong determinism of $\mathcal{P}_1$, moreover $[\mathcal{P}_0] = [\mathcal{P}_1]$.

[ii.2] Otherwise, $a + d' = 1$.

By measuring v with $\theta = \pi/2$ one gets:

$$\left| \Psi_0^{\frac{\pi}{2}} \right\rangle \simeq (I + ie^{i\delta} R) \left| +_0^P \right\rangle_u |\phi_{00}\rangle + e^{i\alpha} C (I - ie^{i\delta} R) \left| -_0^P \right\rangle_u |\phi_{00}\rangle$$

Then by measuring u with P , there exists η s.t.

$$(I + ie^{i\delta} R) |\phi_{00}\rangle = e^{i(\alpha+\eta+c\pi)} (I - ie^{i\delta} R) |\phi_{00}\rangle$$

So $|\phi_{00}\rangle$ and $R|\phi_{00}\rangle$ are colinear, hence $|\phi_{00}\rangle$ is an eigenvector of R , and $\exists k \in \{0, 1, 2, 3\}$ s.t. $R|\phi_{00}\rangle = i^k |\phi_{00}\rangle$, and

$$\begin{aligned} |\varphi\rangle &= \left(\left| +_0^{Q,Q'} \right\rangle_v + e^{i(\delta+k\pi/2)} \left| -_0^{Q,Q'} \right\rangle_v \right) \left| +_0^P \right\rangle_u |\phi_{00}\rangle \\ &\quad + e^{i\alpha} \left(\left| +_0^{Q,Q'} \right\rangle_v - e^{i(\delta+k\pi/2)} \left| -_0^{Q,Q'} \right\rangle_v \right) \left| -_0^P \right\rangle_u C |\phi_{00}\rangle. \end{aligned}$$

Strongness implies

$$\left| \left\langle +_{\pi/2}^{Q,Q'} \right| \left(\left| +_0^{Q,Q'} \right\rangle_v + i^k e^{i\delta} \left| -_0^{Q,Q'} \right\rangle_v \right) \right| = \left| \left\langle -_{\pi/2}^{Q,Q'} \right| \left(\left| +_0^{Q,Q'} \right\rangle_v + i^k e^{i\delta} \left| -_0^{Q,Q'} \right\rangle_v \right) \right|$$

so $\exists \ell \in \{0, 1\}$ s.t. $i^k e^{i\delta} = (-1)^\ell$

As a consequence,

$$\begin{aligned} |\varphi\rangle &= \left(\left| +_0^{Q,Q'} \right\rangle_v + (-1)^\ell \left| -_0^{Q,Q'} \right\rangle_v \right) \left| +_0^P \right\rangle_u |\phi_{00}\rangle + \\ &\quad e^{i\alpha} \left(\left| +_0^{Q,Q'} \right\rangle_v - (-1)^\ell \left| -_0^{Q,Q'} \right\rangle_v \right) \left| -_0^P \right\rangle_u C |\phi_{00}\rangle \end{aligned}$$

Hence, for $C' \in \{\widetilde{Q}_v C, \widetilde{Q}'_v C\}$ where $\widetilde{K} := \begin{cases} \text{XZ} & \text{if } K = \text{Y} \\ K & \text{otherwise} \end{cases}$, the pattern $C'^{s_u} M_u \mathcal{P}$ is robustly deterministic.

Moreover, for any $R' \in \{R, \emptyset\}$, the four branches of the pattern $\mathcal{P}_1 = R'^{s_v} M_v^{Q,Q'} C'^{s_u} M_u \mathcal{P}$ are equal up to a global phase to the corresponding branches of $\mathcal{P}_0$ which guarantees strong determinism of $\mathcal{P}_1$ and that $[\mathcal{P}_1] = [\mathcal{P}_0]$. Therefore $\mathcal{P}_1$ is robustly deterministic. □

F Proof of Proposition 3

Proposition 3 *If a pattern $\mathcal{P}$ is robustly deterministic then it is consistent with a Shadow Pauli Flow of its underlying open graph.*

Proof: Let $m_{\mathcal{P}} = \sum_{u \in O^c, |\lambda_u|=1} |\{v \in O^c \mid v < u \text{ and } |\lambda_v| = 2\}|$ the total distance of Pauli measurements to the end of the computation. We show by induction on $m_{\mathcal{P}}$ that $\mathcal{P}$ has a Shadow Pauli Flow $(p, <)$ that induces $\mathcal{P}$ and such that Condition (1) is satisfied, where **Condition (1)** is: If $w_1, w_2 \in O^c, w_2 \in \text{imp}(p(w_1))$ and $w_2 < w_1$ then $|\lambda_{w_1}| = 1$ and after the measurement⁶ of w_2 , either:

- w_1 is an isolated qubit in the state $\left| +_{\delta - \alpha_{w_2}}^{\lambda_{w_2}} \right\rangle$
- the state is of the form $\lambda_{w_1}(\alpha_{w_2}) |\phi\rangle$ where $|\phi\rangle$ does not depend on α_{w_2} .

Where $\delta \in [0, 2\pi)$ is a constant that only depends on w_1 and w_2 (but not on the measurement angles α).

If $m_{\mathcal{P}} = 0$ then all the Pauli measurements are at the end of the computation so according to Proposition 5, since $\mathcal{P}$ is deterministic it has a Pauli Flow which can be seen has an Shadow Pauli Flow that trivially satisfies the induction hypothesis.

Otherwise there exists $u \in O^c$ s.t. $|\lambda_u| = 1$ and its previous measurement is on a qubit v s.t. $|\lambda_v| = 2$. We consider all the patterns $\mathcal{P}'_i$ s.t. $\mathcal{P} \triangleright_u \mathcal{P}'_i$. According to Proposition 6 all $\mathcal{P}'_i$ are robustly deterministic, so by induction hypothesis they have a Shadow Pauli Flow $(p'_i, <')$ satisfying Condition (1) where $<'$ is obtained from $<$ by exchanging u and v . We use the notation $\mathcal{P}'$ and p' when there is a single $\mathcal{P}'_i$. Let C be the corrector of u in $\mathcal{P}$ and $R\sigma_u$ be the corrector of v in $\mathcal{P}$ (where R is not acting on u), and $\lambda_v = \{Q_0, Q_1\}$.

- If $\lambda_u \sigma = -\sigma \lambda_u$, then there is a single $\mathcal{P}'$ s.t. $\mathcal{P} \triangleright_u \mathcal{P}'$. Let C' (resp R') be the corrector of u (resp. v) in $\mathcal{P}'$.

$$\text{Let } p : w \mapsto \begin{cases} p'(w) & \text{if } w \neq v \\ p'(v) \Delta p'(u) & \text{otherwise} \end{cases}$$

We show that $(p, <)$ is a Shadow Pauli Flow that satisfies Condition (1) and that $(p, <)$ induces $\mathcal{P}$. Notice that it is enough to show that $(p, <)$ satisfies the Shadow Pauli Flow conditions for u and v as it has been modified only on those two qubits. First notice that according to Definition 10 the corrector $C' = C$ consistent with $p'(u)$ in $\mathcal{P}'$ does not act on v so $v \notin p'(u) \cup \text{Odd}(p'(u))$ so $v \notin \text{imp}(p'(u))$ and therefore $v \notin \text{imp}(p(u))$ as $p(u) = p'(u)$. Moreover, C is consistent with $p(u)$ and Condition (1) is satisfied for $w_1 = u$.

Regarding v , $R' = CR$ so $R = CR'$, as a consequence $p(v)$ induces R as $p'(u)$ induces C and $p'(v)$ induces R' . Therefore $(p, <)$ induces $\mathcal{P}$. Moreover, regarding Condition (1) in $\mathcal{P}$ for $w_1 = v$, we prove by contradiction that $w_2 \notin \text{imp}(p(v))$ for any $w_2 < v$.

Notice that if there exists $w_2 <' u$ s.t. $w_2 \in \text{imp}(p'(u))$ then according to Condition (1) in $\mathcal{P}'$, either:

- (i) The qubit u is an isolated qubit in the state $\left| +_{\delta - \alpha_{w_2}}^{\lambda_{w_2}} \right\rangle$ (before the measurements of both u and v). Thus in $\mathcal{P}$, the state of u after the measurement of v does not depend on the outcome of the measurement v . Thus by determinism the

⁶and the associated correction

state of u must be an eigenvector of σ_u , therefore $\sigma_u \notin \lambda_{w_2}$ and $\delta - \alpha_{w_2} = 0 \pmod{\pi}$ which contradicts the uniform determinism. So there is no $w_2 <' u$ s.t. $w_2 \in \text{imp}(p'(u))$, hence no $w_2 < v$ s.t. $w_2 \in \text{imp}(p(v))$, thus Condition (1) is also satisfied for $w_1 = v$.

(ii) Or, the state after the measurement of w_2 is of the form $\lambda_u(\alpha_{w_2})|\phi\rangle$ where $|\phi\rangle$ does not depend on α_{w_2} . Thus, in $\mathcal{P}$, the measurement of v after corrections produces either $\lambda_u(\alpha_{w_2})\langle +_{\alpha_v}^{\lambda_v}|_v|\phi\rangle$ or $R\sigma_u\lambda_u(\alpha_{w_2})\langle -_{\alpha_v}^{\lambda_v}|_v|\phi\rangle = e^{i\theta_0}\lambda_u(-\alpha_{w_2})R\sigma_u\langle -_{\alpha_v}^{\lambda_v}|_v|\phi\rangle$ that are equal up to a global phase. When $\alpha_{w_2} = 0$, we get $\langle +_{\alpha_v}^{\lambda_v}|_v|\phi\rangle = e^{i\theta_1}\lambda_u(0)R\sigma_u\langle -_{\alpha_v}^{\lambda_v}|_v|\phi\rangle$, thus $\lambda_u(\alpha_{w_2})\lambda_u(0)\langle +_{\alpha_v}^{\lambda_v}|_v|\phi\rangle = e^{i\theta}\lambda_u(-\alpha_{w_2})\langle +_{\alpha_v}^{\lambda_v}|_v|\phi\rangle$ so $\langle +_{\alpha_v}^{\lambda_v}|_v|\phi\rangle$ is eigenvector of $\lambda_u(2\alpha_{w_2})$. Therefore u is in the state $|+_{\frac{\lambda_u}{k\pi}}\rangle$. Hence the original pattern is not robust deterministic as u is measured with Pauli λ_u .

- Otherwise, if $RC = CR$ xor $\sigma = P$, then there is a single $\mathcal{P}'$ s.t. $\mathcal{P} \triangleright_u \mathcal{P}'$. Let C' (resp. R') be the corrector of u (resp. v) in $\mathcal{P}'$.

We show that $(p', <)$ is a Shadow Pauli Flow that satisfies Condition (1) and that $(p', <)$ induces $\mathcal{P}$. Similarly to the previous case, C is consistent with $p'(u)$ and Condition (1) is satisfied for $w_1 = u$. Moreover, $p'(v)$ also induces R and Condition (1) is also satisfied for $w_1 = v$.

- Finally, for the remaining case, there are several $\mathcal{P}'_i$ s.t. $\mathcal{P} \triangleright_u \mathcal{P}'_i$. Let $R'_0 = R'_1 = R$, $R'_2 = R'_3 = \emptyset$, $C'_0 = C'_2 = \widetilde{Q}_v C$ and $C'_1 = C'_3 = \widetilde{Q}'_v C$ be respectively the correctors of v and u in the corresponding $\mathcal{P}'_i$.

We show that $(p'_0, <)$ is a Shadow Pauli Flow that satisfies Condition (1) and that $(p'_0, <)$ induces $\mathcal{P}$. First notice that according to Definition 10 the corrector $C'_0 = \widetilde{Q}_v C$, consistent with $p'_0(u)$ in $\mathcal{P}'_0$, acts on v so $v \in p'_0(u) \cup \text{Odd}(p'_0(u))$ so $v \in \text{imp}(p'_0(u))$.

To show that $(p'_0, <)$ is a Shadow Pauli Flow, it is enough to show that $D_v := p'_2(v)$ satisfies the Shadow Pauli Flow condition. Indeed in $\mathcal{P}'_2$, there is no Pauli correction for v , thus all the other vertices in $D_v \cup \text{Odd}(D_v)$ are smaller than v and cannot be anachronistically impacted by other corrections as they are Pauli-measured.

To show that Condition (1) is satisfied, w.l.o.g., assume $Q_v = X$, $Q'_v = Y$ and $P = Z$. So before measurement of v in $\mathcal{P}'_2$, v is in a state in $\{|0\rangle, |1\rangle\}$.

Let $|\varphi\rangle$ be the state of the system before the measurement of u in $\mathcal{P}'_2$. There exist four (not necessarily normalized) vectors $|\phi_{i,j}\rangle$ s.t.

$$|\varphi\rangle = |0_v 0_u\rangle |\phi_{00}\rangle + |0_v 1_u\rangle |\phi_{01}\rangle + |1_v 0_u\rangle |\phi_{10}\rangle + |1_v 1_u\rangle |\phi_{11}\rangle$$

Let $|\Psi_{s_u}\rangle$ be the state of the system after the application of the measurements of u , and appropriate corrections.

$$|\Psi_0\rangle = |0_v\rangle |\phi_{00}\rangle + |1_v\rangle |\phi_{10}\rangle.$$

$$|\Psi_1\rangle = \tilde{Q}_v |0_v\rangle C |\phi_{01}\rangle + \tilde{Q}_v |1_v\rangle C |\phi_{11}\rangle = |1_v\rangle C |\phi_{01}\rangle + |0_v\rangle C |\phi_{11}\rangle.$$

$$\exists \beta \text{ s.t. } e^{i\beta} C |\phi_{01}\rangle = |\phi_{10}\rangle \text{ and } e^{i\beta} C |\phi_{11}\rangle = |\phi_{00}\rangle.$$

Therefore,

$$|\varphi\rangle = |0_v 0_u\rangle |\phi_{00}\rangle + e^{-i\beta} |1_v 1_u\rangle C |\phi_{00}\rangle + |1_v 0_u\rangle |\phi_{10}\rangle + e^{-i\beta} |0_v 1_u\rangle C |\phi_{10}\rangle$$

with $|\phi_{00}\rangle = 0$ or $|\phi_{10}\rangle = 0$. w.l.o.g. we can assume that $|\phi_{10}\rangle = 0$.

$$|\varphi\rangle = |0_v 0_u\rangle |\phi_{00}\rangle + e^{-i\beta} |1_v 1_u\rangle C |\phi_{00}\rangle.$$

We have $|\Psi_0\rangle = |0_v\rangle |\phi_{00}\rangle$. As $\mathcal{P}'_0$ is robustly deterministic:

- If $s_v = 0$: $\langle +_{\alpha_v}^{X,Y} | |\Psi_0\rangle = \frac{1}{\sqrt{2}} |\phi_{00}\rangle$ and,
- If $s_v = 1$: $R \langle -_{\alpha_v}^{X,Y} | |\Psi_0\rangle = \frac{1}{\sqrt{2}} e^{-i\alpha_v} R |\phi_{00}\rangle$.

Therefore robust determinism of $\mathcal{P}'_0$ implies that $R |\phi_{00}\rangle = e^{i\gamma} |\phi_{00}\rangle$. Notice that γ does not depend on the particular angle of measurement α_v .

- If $\sigma = P$ and $RC = CR$, C and R have same eigenvectors and $C |\phi_{00}\rangle = e^{i\delta} |\phi_{00}\rangle$. Therefore $|\varphi\rangle = (|0_v 0_u\rangle + e^{-i(\beta+\delta)} |1_v 1_u\rangle) \otimes |\phi_{00}\rangle$. After the measurement of v in $\mathcal{P}$, we get
 - * If $s_v = 0$: $\langle +_{\alpha_v}^{X,Y} | |\varphi\rangle = (\langle +_{\alpha_v}^{X,Y} | 0\rangle |0_u\rangle + e^{-i(\beta+\delta)} \langle +_{\alpha_v}^{X,Y} | 1\rangle |1_u\rangle) \otimes |\phi_{00}\rangle = \frac{1}{\sqrt{2}} (|0_u\rangle + e^{-i(\beta+\delta+\alpha_v)} |1_u\rangle) \otimes |\phi_{00}\rangle$.
 - * If $s_v = 1$: $\sigma_u R \langle -_{\alpha_v}^{X,Y} | |\varphi\rangle = e^{i\gamma} (\langle -_{\alpha_v}^{X,Y} | 0\rangle |0_u\rangle - e^{-i(\beta+\delta)} \langle -_{\alpha_v}^{X,Y} | 1\rangle |1_u\rangle) \otimes |\phi_{00}\rangle = \frac{e^{i\gamma}}{\sqrt{2}} (|0_u\rangle + e^{-i(\beta+\delta+\alpha_v)} |1_u\rangle) \otimes |\phi_{00}\rangle$.

So u is separable and is in the state $|+_{\delta'-\alpha_v}^{X,Y}\rangle$.

- If $\sigma = I$ and $RC = -CR$,

in $\mathcal{P}$ measuring v first we get:

- * If $s_v = 0$: $\langle +_{\alpha_v}^{X,Y} | |\varphi\rangle = \frac{1}{\sqrt{2}} (|0_u\rangle |\phi_{00}\rangle + e^{-i(\beta+\alpha_v)} |1_u\rangle C |\phi_{00}\rangle) = Z_u(\alpha_v) (\frac{1}{\sqrt{2}} (|0_u\rangle |\phi_{00}\rangle + e^{-i\beta} |1_u\rangle C |\phi_{00}\rangle))$ which satisfies the condition.
- * If $s_v = 1$: $R \langle -_{\alpha_v}^{X,Y} | |\varphi\rangle = \frac{1}{\sqrt{2}} R (|0_u\rangle |\phi_{00}\rangle - e^{-i(\beta+\alpha_v)} |1_u\rangle C |\phi_{00}\rangle) = \frac{e^{i\gamma}}{\sqrt{2}} (|0_u\rangle |\phi_{00}\rangle + e^{-i(\beta+\alpha_v)} |1_u\rangle C |\phi_{00}\rangle) = e^{i\gamma} Z_u(\alpha_v) (\frac{1}{\sqrt{2}} (|0_u\rangle |\phi_{00}\rangle + e^{-i\beta} |1_u\rangle C |\phi_{00}\rangle))$ which satisfies the condition.

□