Hidden time-nonlocal Floquet symmetries

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We investigate the Floquet spectrum of a detuned, driven two-level system and show that it exhibits exact quasienergy crossings when the detuning is an integer multiple of the energy quantum of the driving field. This behavior can be explained by a hidden time-nonlocal parity, which allows the Floquet modes to be classified as even or odd. Then a generic feature is the emergence of exact crossings between quasienergies of different parity. A constructive proof of the existence of the symmetry is based on a scalar recurrence relation. Moreover, we present a general scheme for its numerical computation, which can be applied to models beyond the two-level system. Analytical results are illustrated with numerical data.

1 Introduction

Quasienergies reflect the spectral properties of ac-driven quantum systems and are a cornerstone of Floquet theory [1, 2, 3]. They determine the phase factors of the long-time dynamics and, thus, their splittings set the corresponding time scales. Of particular interest are quasienergy crossings at which these time scales diverge so that the quantum dynamics may become frozen. A prominent effect that relies on this is coherent destruction of tunneling (CDT) [4], which can already be understood within a two-level approximation [5]. This appealing prediction has spurred a wealth of experiments with double quantum dots [6, 7], superconducting qubits [8, 9], atoms in optical lattices [10], and optical waveguides [11]. Lately, exact quasienergy crossings attracted attention, because in their vicinity, the dissipative behavior is rather sensitive to small parameter variation [12, 13].

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Quasienergies are eigenvalues of the Floquet Hamiltonian, which is a Hermitian operator in Sambe space, i.e., in the product space of the underlying Hilbert space and that of time-periodic functions [1, 2]. As such, they exhibit generic features of quantum-mechanical spectra, in particular level repulsion [14]. Therefore, as a function of any system parameter, quasienergies are expected to form avoided crossings (also called anti-crossings) unless a symmetry or integral of motion is present. For the example of CDT in an undetuned two-level system, the emergence of exact crossings is enabled by a spatio-temporal symmetry known as generalized parity [15].

For large driving frequencies, the CDT Hamiltonian can be approximated by a time-independent effective Hamiltonian in which the tunnel matrix element is renormalized by the zeroth-order Bessel function of the first kind. At its roots, tunneling is suppressed [5]. This approximation scheme can be generalized to the presence of an "integer detuning," i.e., a detuning that matches n energy quanta of the driving field. The resulting renormalization by the nth-order Bessel function has been verified numerically [16, 17, 18] and observed experimentally [6, 7]. Since all Bessel functions of the first kind possess roots, within a high-frequency approximation one finds exact crossings. This raises the question of whether, beyond the approximation, these crossings are indeed exact or just narrowly avoided. And if they are exact, to which symmetry or integral of motion can they be attributed? Related questions have been addressed recently also for the (time-independent) Rabi Hamiltonian [19, 20], for which exact level crossings are enabled by a hidden symmetry [21].

Here, we develop an approach for finding hidden time-nonlocal symmetries of Floquet systems and use it to explain the emergence of exact crossings in the Floquet spectrum of the driven two-level system. The work is organized as follows. In Sec. 2, we introduce our model and, from numerical findings, conjecture the condition for the emergence of exact quasienergy crossings, and consequently the existence of a hidden symmetry. The strategy for finding such symmetry and its application to our model is presented in Sec. 3. In Sec. 4, we develop a scheme for its numerical computation, while conclusions are drawn in Sec. 5. An intuitive derivation for the particular case $n = 1$ and details of the numerical approach are given in the Appendix.

2 The driven two-level system

We consider the driven two-level system described by the pseudospin Hamiltonian

$$H(t) = \frac{\epsilon}{2}\sigma_z + \beta\sigma_x + \alpha\sigma_z \cos(\Omega t), \qquad (1)$$

with detuning ϵ, tunneling matrix element β, driving amplitude α, and frequency Ω. For convenience and without loss of generality, we assume α, β, ϵ, $\Omega \geq 0$ and choose units with $\hbar = 1$.

For such a time-periodic Hamiltonian, the Floquet theorem states that a complete set of solutions of the Schrödinger equation is of the form $|\psi(t)\rangle = e^{-iqt}|\phi(t)\rangle$, with quasienergy q. The Floquet mode $|\phi(t)\rangle = |\phi(t+T)\rangle$, with $T = 2\pi/\Omega$, shares the time-periodicity of the Hamiltonian. Therefore, it can be considered an element of Sambe space, i.e., Hilbert space extended by the space of T-periodic functions [1, 2]. By inserting this ansatz into the Schrödinger equation one readily finds that the Floquet modes obey the eigenvalue equation

$$\left(H(t) - i\frac{\partial}{\partial t}\right)|\phi(t)\rangle = q\,|\phi(t)\rangle\,, \qquad (2)$$

with the Floquet Hamiltonian, or quasienergy operator, $H(t) - i\partial_t$. It is straightforward to show that when $|\phi(t)\rangle$ is a Floquet mode with quasienergy q, then, for any integer k, $e^{ik\Omega t}|\phi(t)\rangle$ is a Floquet mode with quasienergy $q + k\Omega$. Both modes are equivalent as they correspond to the same solution of the Schrödinger equation. This constitutes the Brillouin-zone structure of the Floquet spectrum. Irrespective of the choice of the Brillouin zone, a complete set of non-equivalent Floquet modes at equal times forms an orthonormal basis of the Hilbert space [2].

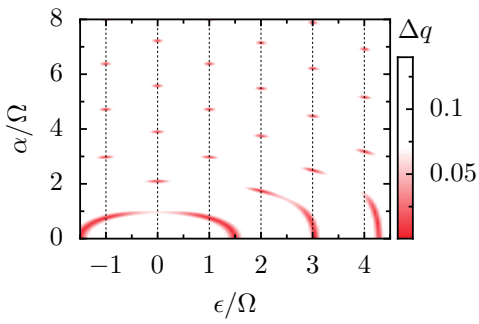

Figure 1: Minimal quasienergy splitting as a function of the detuning and the driving amplitude for tunnel coupling $\beta = 1.3\,\Omega$. The color scale is chosen such that regions with particularly small values are highlighted. The vertical dashed lines mark detunings that are integer multiples of the driving frequency, $\epsilon = n\Omega$.

Figure 1 shows the minimal quasienergy splitting of Hamiltonian (1) as a function of detuning and driving amplitude. Unless the driving amplitude is rather small, the regions with tiny splittings are centered at integer values of ϵ/Ω. Below, we demonstrate that for such integer detuning the splittings are indeed zero. This implies that then the Floquet spectra as a function of the driving amplitude exhibit exact crossings. In Fig. 2, this is illustrated for the special case $\epsilon = \Omega$. Therefore, in accordance with the generic properties of quantum-mechanical spectra [14], we can conclude that the Hamiltonian (1) must possess a symmetry or an integral of motion that characterizes the Floquet modes.

In the absence of detuning ($\epsilon = 0$), the relevant symmetry is the well-known generalized parity $G = \sigma_x P$, where $P = \exp[(T/2)\partial_t]$ shifts time by half a driving period [5, 15]. The Floquet modes are eigenstates of G, i.e., $G\,|\phi_\nu(t)\rangle = \sigma_x\,|\phi_\nu(t+T/2)\rangle = j_\nu\,|\phi_\nu(t)\rangle$. Owing to the T-periodicity of the Floquet modes, $G^2\,|\phi_\nu(t)\rangle = |\phi_\nu(t)\rangle$, which implies that the eigenvalues j_ν are ± 1.

3 Time-nonlocal symmetries

For nonzero detuning $\epsilon \neq 0$, Hamiltonian (1) lacks an obvious symmetry. Therefore, we search for a hidden spatio-temporal symmetry with the structure of the generalized parity [15]. Specifically, we look for a generally time-dependent

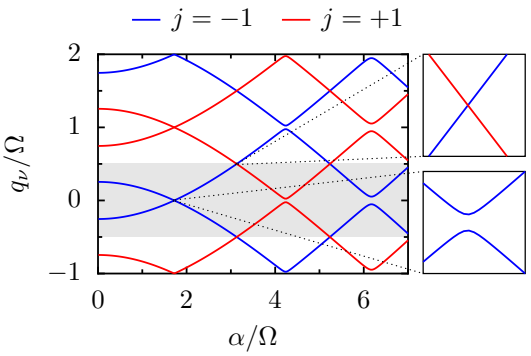

Figure 2: Floquet spectrum of the driven two-level system as a function of the driving amplitude α for tunneling matrix element $\beta = 2.7\,\Omega$ and detuning $\epsilon = \Omega$ (i.e., $n = 1$). The gray box marks the first Brillouin zone. The parity $j = \pm 1$, derived below, is indicated by the color—red for $+1$ and blue for -1. Equivalent modes in neighboring Brillouin zones have opposite parity. The zooms indicate that only quasienergies with the same parity form avoided crossings, while those with different parity cross exactly.

operator $Q(t)$ such that $J(t) = Q(t)P$ acts as a parity-like symmetry operator on the Floquet modes $|\phi_\nu(t)\rangle$, i.e.,

$$J(t)\,|\phi_\nu(t)\rangle = j_\nu\,|\phi_\nu(t)\rangle\,, \tag{3}$$

with $j_\nu = \pm 1$. The corresponding relation for $Q(t)$ reads

$$Q(t)\,|\phi_\nu(t+T/2)\rangle = j_\nu\,|\phi_\nu(t)\rangle\,. \tag{4}$$

Since two equivalent Floquet modes from neighboring Brillouin zones differ by a phase factor $e^{i\Omega t}$, they will have opposite parity. This is already the case for the generalized parity in the absence of detuning [15]. It underlines that while equivalent modes correspond to the same solution of the Schrödinger equation, they are different solutions of the eigenvalue equation (2).

From the T-periodicity of the Floquet modes follows $Q(t+T/2)\,|\phi_\nu(t)\rangle = j_\nu\,|\phi_\nu(t+T/2)\rangle$. Therefore, $Q(t)Q(t+T/2)\,|\phi_\nu(t)\rangle = |\phi_\nu(t)\rangle$ for all Floquet modes and hence $Q(t)Q(t+T/2) = \mathbb{I}$. Moreover, since $|j_\nu| = 1$, Eq. (4) implies that $Q(t)$ maps an orthonormal basis into another orthonormal basis, and therefore it must be unitary, which leads to

$$Q^\dagger(t) = Q(t+T/2). \tag{5}$$

Using this relation twice yields that $Q(t)$ must be a T-periodic function of time.

To compute $Q(t)$ for a given Hamiltonian, we derive its equation of motion by applying $i\partial_t$ to Eq. (4). The resulting time derivatives of the Floquet modes can be replaced using the Floquet equation (2), yielding $i\partial_t Q(t)\,|\phi_\nu(t+T/2)\rangle = [H(t)Q(t) - Q(t)H(t+T/2)]\,|\phi_\nu(t+T/2)\rangle$. Since this equation holds for all Floquet modes, it follows that

$$i\frac{\partial}{\partial t}Q(t) = [H_+(t), Q(t)] + \{H_-(t), Q(t)\}, \tag{6}$$

where $2H_\pm(t) = H(t) \pm H(t+T/2)$. In the particular case of Hamiltonian (1), $H_+ = \epsilon\sigma_z/2 + \beta\sigma_x$ is the time-independent part and $H_-(t) = \alpha\sigma_z\cos(\Omega t)$, which appears in the anticommutator, is the driving.

Our goal is to find an operator $Q(t)$ that complies with the generic properties derived so far as well as with the specific symmetries of the model Hamiltonian (1) considered below. In our practical calculations, the starting point will be a solution $\tilde{Q}(t)$ of Eq. (6), which *a priori* may not be unitary. From the Hermitian adjoint of Eq. (6), we find that it will obey

$$i\frac{\partial}{\partial t}\tilde{Q}(t)\tilde{Q}^\dagger(t) = [H(t), \tilde{Q}(t)\tilde{Q}^\dagger(t)]. \tag{7}$$

This implies that if the operator $\tilde{Q}(t)\tilde{Q}^\dagger(t)$ commutes with $H(t)$, it will be time-independent. Below we use this relation to normalize a solution $\tilde{Q}(t)$ with a time-independent factor such that it becomes unitary and still obeys Eq. (6).

Here, a caveat is in order. Obvious solutions of Eq. (4) read

$$Q(t) = \sum_\nu j_\nu \Pi_\nu(t), \tag{8}$$

with $\Pi_\nu(t) \equiv |\phi_\nu(t)\rangle\langle\phi_\nu(t+T/2)|$. The sum encompasses a complete set of non-equivalent Floquet modes, for instance those belonging to a given Brillouin zone, and the coefficients $j_\nu = \pm 1$ are chosen arbitrarily. Unitarity of $Q(t)$ is ensured by the fact that such set of modes forms a basis of the Hilbert space. However, without further specification, such solutions are useless for the present purpose, because they would assign a parity j_ν to each Floquet mode by hand. Moreover, the resulting "symmetry" would depend on how the Floquet modes are labeled and, hence, on the (arbitrary) choice of the Brillouin zone. Therefore, for a meaningful parity operator, the signs j_ν must be specified in a unique manner such that $Q(t)$ becomes a continuous function of all parameters.

3.1 Anti-unitary symmetries

Before attempting to solve Eq. (6), we examine the anti-unitary symmetries of the Floquet Hamiltonian $H(t)-i\partial_t$ [22]. In particular, we consider time-reversal symmetry and particle-hole symmetry.[1] The simultaneous presence of both symmetries implies a chirality, which may substitute either one of the other two. While these symmetries cannot explain the emergence of exact crossings, they do facilitate the solution of Eq. (6).

Time-reversal symmetry. Both the Hamiltonian (1) and $-i\partial_t$ possess time-reversal symmetry as they are invariant under the transformation $\Theta = (K, t \to -t)$, where K denotes complex conjugation. Applying Θ to Eq. (4) and using that for all Floquet modes, $\Theta\,|\phi_\nu(t)\rangle$ equals $|\phi_\nu(t)\rangle$ up to a phase factor, leads to the operator identity

$$Q(t) = Q^*(-t). \tag{9}$$

Therefore, the Fourier coefficients of $Q(t)$ must be real matrices. Note that here the Pauli matrices are introduced solely for a compact pseudospin notation. They do not represent angular momenta and need not change sign under time reversal.

Particle-hole symmetry. Transformation with σ_y inverts the sign of Hamiltonian (1), while complex conjugation does the same with $-i\partial_t$. Hence, the combined operation $C = \sigma_y K$ inverts the sign of the Floquet Hamiltonian, i.e., $C[H(t)-i\partial_t]C^{-1} = -[H(t)-i\partial_t]$. Therefore, the quasienergies come in pairs with opposite sign, $q_{\bar\nu} = -q_\nu$, and C maps the corresponding Floquet modes to each other, i.e., $C\,|\phi_\nu(t)\rangle$ and $|\phi_{\bar\nu}(t)\rangle$ differ at most by a phase factor. Consequently, $Q(t)C\,|\phi_\nu(t+T/2)\rangle = j_{\bar\nu} C\,|\phi_\nu(t)\rangle$. Acting with C on Eq. (4) and using the relation just derived, yields

$$\begin{aligned} CQ(t)\,|\phi_\nu(t+T/2)\rangle &= j_\nu C\,|\phi_\nu(t)\rangle \\ &= j_\nu j_{\bar\nu} Q(t) C\,|\phi_\nu(t+T/2)\rangle\,. \end{aligned} \tag{10}$$

Since this holds for all Floquet modes, the product $j_\nu j_{\bar\nu}$ must be the same for all particle-hole related pairs of modes. Moreover, $Q(t)$ must obey the operator identity

$$CQ(t)C^{-1} = \sigma_y Q^*(t)\sigma_y = \mp Q(t), \tag{11}$$

where the upper sign holds for $j_\nu j_{\bar\nu} = -1$.

To obtain the most general form of $Q(t)$ that respects these anti-unitary symmetries, we start with a general 2×2 matrix. Particle-hole symmetry requires that $Q(t)$ obeys Eq. (11) which relates its first and second row such that it must be of the form

$$\tilde{Q}(t) = \begin{pmatrix} \lambda(t) & \mu(t) \\ \pm\mu^*(t) & \mp\lambda^*(t) \end{pmatrix} \tag{12}$$

$$\equiv \sum_k e^{-ik\Omega t} \begin{pmatrix} \lambda_k & \mu_k \\ \pm\mu_{-k} & \mp\lambda_{-k} \end{pmatrix}. \tag{13}$$

Due to time-reversal symmetry, the Fourier coefficients of the matrix elements, λ_k and μ_k, must be real. The tilde indicates that so far unitarity is not ensured.

Finally, we employ Eq. (5) to relate matrix elements at times t and $t+T/2$ as

$$\lambda^*(t) = \lambda(t+T/2), \tag{14}$$

$$\mu(t) = \mp\,\mu(t+T/2), \tag{15}$$

which for the Fourier coefficients means

$$\lambda_{-k} = (-)^k \lambda_k, \tag{16}$$

$$\mu_k = \mp\,(-)^k \mu_k. \tag{17}$$

While the first relation links different coefficients, the second one yields that for the upper sign, $\mu_k = 0$ when k is even, while for the lower sign, μ_k vanishes for odd k.

3.2 Recurrence relations

Inserting the Fourier series (13) into the equation of motion (6) yields a set of four coupled recurrence equations for the matrix elements of the coefficients $\tilde{Q}_k$, where only two equations are independent. From the diagonal matrix elements, one finds

$$-k\Omega\lambda_k - \beta(\mu_k \mp \mu_{-k}) + \alpha(\lambda_{k-1} + \lambda_{k+1}) = 0. \tag{18}$$

The off-diagonal matrix elements provide the relation

$$\begin{aligned} (\epsilon - k\Omega)\mu_k &= \beta(\lambda_k \pm \lambda_{-k}) \\ &= \beta[1 \pm (-)^k]\lambda_k, \end{aligned} \tag{19}$$

[1] In accordance with Ref. [22], we use the terminology established for random matrices in the context of many-body Hamiltonians [23].

Table 1: Fourier coefficients of the functions $\lambda(t)$ and $\mu(t)$ for the integer detunings $\epsilon = n\Omega$ with $n = 0, \ldots, 4$. To achieve a compact notation, we have defined the abbreviations $D_0 = 2\Omega^2 - 2\alpha^2 + \beta^2$ and $D_1 = 6\Omega^2 - \alpha^2 - 2\beta^2$. All coefficients with index $|k| > n$ vanish owing to the break condition. For $n = 0$, $\lambda(t) = 0$ such that $Q(t) = \sigma_x$ and J becomes the generalized parity.

	k	-3	-2	-1	0	1	2	3	4
$n=0$	λ_k				-				
	μ_k				1				
$n=1$	λ_k			-	β	-			
	μ_k			-	-	α			
$n=2$	λ_k		-	$-\alpha\beta$	$\beta\Omega$	$\alpha\beta$	-		
	μ_k		-	-	β^2	-	α^2		
$n=3$	λ_k	-	$\alpha^2\beta$	$-2\alpha\beta\Omega$	$\beta(2\Omega^2 - \alpha^2 + \beta^2)$	$2\alpha\beta\Omega$	$\alpha^2\beta$	-	
	μ_k	-	-	$-\alpha\beta^2$	-	$2\alpha\beta^2$	-	α^3	
$n=4$	λ_k	$-\alpha^3\beta$	$3\alpha^2\beta\Omega$	$-\alpha\beta D_1$	$2\beta\Omega D_0$	$\alpha\beta D_1$	$3\alpha^2\beta\Omega$	$\alpha^3\beta$	-
	μ_k	-	$\alpha^2\beta^2$	-	$\beta^2 D_0$	-	$3\alpha^2\beta^2$	-	α^4

where the second equality follows from Eq. (16). This relation contains only Fourier coefficients with equal index and can be used to eliminate in Eq. (18) the dependence on $\mu_{\pm k}$.

Once more, we make use of knowledge of the CDT case $\epsilon = 0$ in which $Q(t) = \sigma_x$ consists of only the Fourier component with $k = 0$. While this appears impossible in the detuned case, we employ a slightly weaker condition, namely that the symmetry operator should have a finite number of Fourier components. Let us therefore assume that there exists an integer $n \geq 0$ such that $\tilde{Q}_k = 0$ and, thus, $\lambda_k = 0 = \mu_k$ for all $|k| > n$. In particular, $\mu_{n+1} = \lambda_{n+1} = \lambda_{n+2} = 0$, such that Eq. (18) for $k = n + 1$ reads

$$\lambda_n = 0, \tag{20}$$

while Eq. (19) simplifies to

$$(\epsilon - n\Omega)\mu_n = 0. \tag{21}$$

A non-trivial solution requires that the prefactor $\epsilon - n\Omega$ in the latter equation vanishes, i.e., the detuning must match an integer multiple of the driving frequency, $\epsilon = n\Omega$. Such integer detuning has been identified as condition for the existence of a hidden symmetry also for the Rabi model [21]. Owing to the assumption $\epsilon > 0$, Eq. (21) also implies $\mu_{-n} = 0$, such that μ_n remains the only non-vanishing matrix element of $\tilde{Q}_n$. This finally allows us to set the sign of $Q(t)$ in a unique manner and independently of the parameter values by choosing μ_n real and positive. For convenience, we set $\mu_n = \alpha^n$ and take care for a proper normalization later. Together with Eq. (17), this determines a further sign, namely the one in Eq. (11), which must read $\mp = (-)^{n+1}$.

Summarizing the relations obtained so far, we find that for $|k| < n$, the recurrence relation in Eq. (18) can be written as

$$\alpha\lambda_{k-1} - (k\Omega + b_k)\lambda_k + \alpha\lambda_{k+1} = 0, \tag{22}$$

where the second term has been expressed in terms of λ_k by making use of Eq. (19) and introducing the shorthand notation

$$b_k = \begin{cases} \frac{4k\beta^2}{(n^2-k^2)\Omega} & \text{if } n+k \text{ even,} \\ 0 & \text{else.} \end{cases} \tag{23}$$

With the boundary condition $\lambda_n = \lambda_{n+1} = 0$, its evaluation is straightforward. Since $b_{-k} = -b_k$, the solution will be consistent with Eq. (16) and with the condition that all λ_k must vanish for $k \leq -n$.

In a last step, we have to normalize $Q(t)$ such that it becomes unitary and still obeys the equation of motion (6). This would be impossible if the required normalization factor were time-dependent. Here, however, the form of $\tilde{Q}(t)$ in Eq. (12) ensures that $\tilde{Q}(t)\tilde{Q}^\dagger(t)$ is proportional to a unit matrix. Then the right-hand side of Eq. (7) vanishes, and we can conclude that $\tilde{Q}(t)\tilde{Q}^\dagger(t)$ is time-independent. Therefore, any solution of Eq. (6) that complies with particle-hole symmetry can be normalized by a time-independent

factor such that the corresponding unitary $Q(t)$ obeys the same equation of motion.

The resulting Fourier coefficients λ_k and μ_k for integer detuning up to $\epsilon = 4\Omega$ are compiled in Table 1. In the time domain, for $n = 1$,

$$Q(t) \propto \begin{pmatrix} \beta & \alpha e^{-i\Omega t} \\ -\alpha e^{i\Omega t} & \beta \end{pmatrix} \tag{24}$$

while for $n = 2$,

$$Q(t) \propto \begin{pmatrix} \beta\Omega - 2i\alpha\beta\sin(\Omega t) & \beta^2 + \alpha^2 e^{-2i\Omega t} \\ \beta^2 + \alpha^2 e^{2i\Omega t} & -\beta\Omega - 2i\alpha\beta\sin(\Omega t) \end{pmatrix}. \tag{25}$$

These operators become unitary upon division by the square roots of $\alpha^2+\beta^2$ and $(\alpha^2+\beta^2)^2+(\beta\Omega)^2$, respectively. For $n = 1$, an intuitive alternative derivation of $Q(t)$ is provided in Appendix A.

3.3 Consequences for the Floquet spectrum

Since $J(t)$ is T-periodic, it can be considered as an operator in Sambe space, just like the Floquet Hamiltonian $H(t) - i\partial_t$. Then, from Eq. (5), it follows that it is Hermitian. Equations (2) and (3) identify the Floquet modes as their common eigenstates. Hence, each mode $|\phi_\nu(t)\rangle$ can be characterized by the corresponding eigenvalues, namely a quasienergy q_ν and a parity j_ν. The latter is given by the expectation value

$$j_\nu = \frac{1}{T}\int_0^T dt\, \langle\phi_\nu(t)|\, J(t)\, |\phi_\nu(t)\rangle = \pm 1, \tag{26}$$

with the time integration stemming from the inner product in Sambe space [2]. The practical computation is simplified by the fact that Eq. (4) holds already in Hilbert space, such that

$$j_\nu = \langle\phi_\nu(t)|\, Q(t)\, |\phi_\nu(t + T/2)\rangle\,, \tag{27}$$

which turns out to be time independent. In our numerical calculations, we evaluate it at $t = 0$.

Generally the eigenvalues of Hermitian operators, as a function of any parameter, exhibit level repulsion, unless they belong to modes from different symmetry classes [14]. In the present case, such symmetry is given by the hidden parity $J(t)$, which, for integer detuning, allows the emergence of exact crossings. To visualize this behavior, we have diagonalized numerically for $\epsilon = \Omega$ the Floquet Hamiltonian to obtain the quasienergies and the Floquet modes, as well as the expectation value of $J(t)$ determined by Eq. (27) with the (normalized) operator $Q(t)$ in Eq. (24). Figure 2 shows the resulting Floquet spectrum extended over three Brillouin zones. The color of the lines reflects the value of j_ν. The avoided and exact crossings verify the consequences of the hidden parity.

This observation is consistent with the von Neumann-Wigner theorem [14] which states that in the presence of time-reversal symmetry, one needs to adjust two independent parameters to obtain a degeneracy. Here, this is done in the following way. For arbitrary values of Ω and β, one has to (i) adjust the detuning to an integer multiple of the driving frequency and (ii) choose a particular amplitude α. Alternatively, one may start with an arbitrary value of α and will find degeneracies for particular values of β. It is elucidating to consider non-integer detunings in the context of the von Neumann-Wigner theorem. For arbitrary fixed detuning ϵ, one may search for values of α and β such that the quasienergies are degenerate. However, there seems to exist only the trivial solution $\beta = 0$. For this value, however, the nature of the problem is entirely different, because the system possesses the time-local symmetry σ_z.

4 Numerical computation of the symmetry operator

While we have already proven that for any integer detuning $\epsilon = n\Omega$, a hidden parity exists, its analytical calculation becomes increasingly tedious for larger n. For a numerical solution, one may follow the scheme derived above and numerically iterate the recurrence relation (22). This will provide the Fourier coefficients λ_k and μ_k, and eventually $Q(t)$. Here, however, we follow a different route, because it is instructive to obtain the Fourier coefficients Q_k directly from the Floquet modes. Moreover, such numerical solution is not restricted to the two-level system and, thus, hints on how to compute time-nonlocal symmetries for other systems.

To this end, we start from Eq. (8) and write the Fourier components of $Q(t)$ in the form

$$Q_k = \sum_\nu j_\nu \Pi_{\nu,k}, \tag{28}$$

where the operators $\Pi_{\nu,k}$ are the Fourier components of $\Pi_\nu(t)$ defined in Sec. 3. These can be

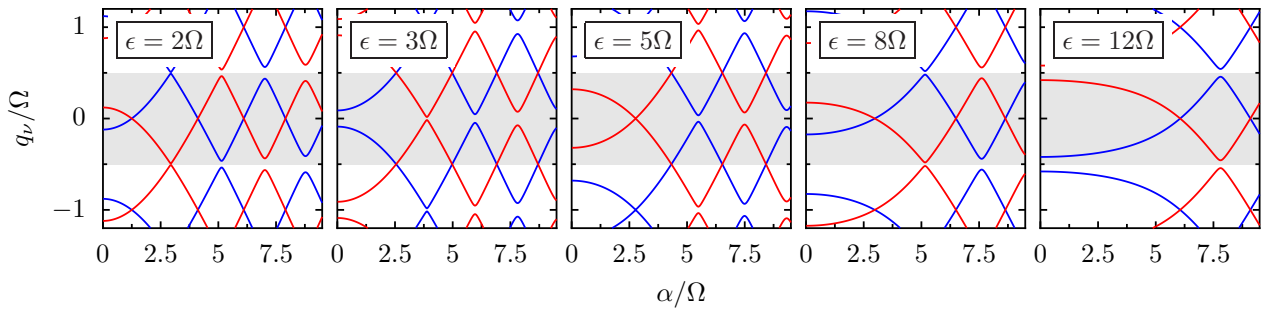

Figure 3: Floquet spectra as a function of the driving amplitude for various integer detunings $\epsilon = n\Omega$. The tunnel matrix element $\beta = 2.7\Omega$ and the color code of the parity are as in Fig. 2. As there, some crossings of levels with equal parity appear exact, but in fact are narrowly avoided.

expressed in terms of the sidebands of the Floquet modes as $\Pi_{\nu,k} = \sum_{k'}(-1)^{k'} \left|\phi_{\nu,k'+k}\right\rangle \left\langle\phi_{\nu,k'}\right|$. The sum is restricted to d non-equivalent Floquet modes, where d is the dimension of the Hilbert space. While the $\Pi_{\nu,k}$ depend on the choice of the Brillouin zone, it is crucial that the resulting operator $Q(t)$ is independent of this arbitrariness.

The analytical insight gained so far suggests the need for a break condition. We therefore assume that the Fourier coefficients Q_k in Eq. (28) vanish when $|k|$ exceeds some *a priori* unknown value. This assumption leads to an overdetermined set of homogeneous linear equations for the coefficients j_ν, which in general admits only the trivial solution $j_\nu = 0$. A non-trivial solution for the j_ν can then exist only if a time-nonlocal symmetry is present. In that case, such a solution can be obtained from any subset of d equations with sufficiently large Fourier index, while the remaining equations may be used for consistency checks. Since the system of equations is homogeneous, the solution is determined only up to an overall sign (or, more generally, a phase factor). As in the analytical construction, this residual freedom must be fixed in a unique manner. Based on the considerations above, this can be achieved by choosing the only non-vanishing matrix element of Q_k with the largest index to be real and positive. For details of the implementation of the numerical scheme, see Appendix B.

To verify the existence of a hidden time-nonlocal symmetry also for $\epsilon = n\Omega$ with $n > 1$, we have computed the Floquet modes for various integer detunings by diagonalizing the Floquet Hamiltonian. To find $Q(t)$, we have computed the $\Pi_{\nu,k}$ and have determined the parities j_ν as described in the last paragraph. Then for each Floquet mode, we have evaluated Eq. (27) at time $t = 0$ to obtain the parity of each mode, including all equivalent ones. The corresponding expression in Sambe space, Eq. (26), has been used for confirmation. Figure 3 shows the resulting spectra, where again the color of the curves refers to the value of j_ν. Within numerical precision it assumes the values ± 1. Besides verifying our conjecture for the crossings, this also demonstrates that our numerical approach is reliable even for rather large integer detunings.

Finally, let us us remark that the numerical scheme with the time-local ansatz $J(t) = \sum_\nu j_\nu |\phi_\nu(t)\rangle\langle\phi_\nu(t)|$ leads to the trivial result $J(t) = \mathbb{I}$. This underlines that time-nonlocality is an essential constituent of the present symmetry.

5 Discussion and conclusions

We have developed the concept of hidden time-nonlocal Floquet symmetries. It is based on an ansatz with a spatio-temporal transformation that resembles the classic generalized parity [15] known from CDT. The main difference is that the spatial part now is time dependent. Its properties can be determined from a conjectured automorphism of the Floquet modes. It turned out that the mapping $Q(t)$ must reflect all symmetries of the Hamiltonian, foremost the time periodicity. The practical calculation requires solving a Liouville-like equation in which the driving Hamiltonian appears within an anti-commutator.

For the driven two-level system with integer detuning, we have demonstrated the existence of such symmetry. The constructive proof makes use of the anti-unitary symmetries of the Hamiltonian, while an explicit expression can be found from a recurrence equation together with a break condition. The hidden parity partitions the

Sambe space into even and odd subspaces, allowing quasienergies from different subspaces to form exact crossings. We have verified numerically that the Floquet spectrum of our model exhibits this feature. The condition of integer detuning has been found also for the existence of a hidden integral of motion of the Rabi model [19, 20, 21]. Despite this similarity, we note several remarkable differences, such as quasienergies not being bounded from below, the possibility of exploiting anti-unitary symmetries, and the availability of a constructive existence proof.

For the numerical computation of the symmetry operator, we have developed an independent scheme not based on the recurrence relation. It turned out rather stable even for large detunings. Besides enabling a visualization in the two-level case, it provides a tool for the search for hidden time-nonlocal symmetries in other Floquet systems.

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A Intuitive solution for $n = 1$

For the smallest non-trivial integer detuning, $\epsilon = \Omega$, the invariant can be obtained in a less formal way. The idea is to find a sequence of transformations that inverts the sign of the driving, which finally is canceled by the time shift P. The mapping starts with a transformation to the interaction picture with respect to the detuning, $U_0(t) = \exp(-i\sigma_z \Omega t/2)$, which results in

$$\tilde{H}(t) = \beta\sigma_x \cos(\Omega t) - \beta\sigma_y \sin(\Omega t) + \alpha\sigma_z \cos(\Omega t). \tag{29}$$

Then transformation with $\beta\sigma_x + \alpha\sigma_z$ (we ignore the normalization) flips the sign of the second term of $\tilde{H}(t)$, while leaving the rest as is. A further transformation with $U_0(t)$ yields

$$H'(t) = -\frac{\Omega}{2}\sigma_z + \beta\sigma_x + \alpha\sigma_z \cos(\Omega t), \tag{30}$$

which is the original Hamiltonian, but with inverted detuning. Finally, transformation with σ_x moves the minus sign to the time-dependent term, such that the full transformation

$$\tilde{Q}(t) = U_0(t)(\beta\sigma_x + \alpha\sigma_z)U_0(t)\sigma_x \tag{31}$$

agrees with Eq. (24).

B Numerical solution: Implementation

It is instructive to discuss in more detail the implementation of the numerical approach outlined in Sec. 4. We seek an operator $Q(t)$ of the form (8) with a finite set of Fourier coefficients Q_k. With regard to parity symmetry, the coefficients j_ν are restricted to ± 1.

The numerical computation is based on the eigenvalue equation (2), which provides the Floquet modes and, consequently, the operators $\Pi_\nu(t) = |\phi_\nu(t)\rangle\langle\phi_\nu(t+T/2)|$. To bring Eq. (8) into a form suitable for a numerical implementation, we rearrange the Fourier coefficients of the 2×2 matrices $\Pi_\nu(t)$ and $Q(t)$ to obtain the 4-dimensional column vectors $\mathbf{\Pi}_{\nu,k}$ and $\mathbf{Q}_k$. Then the ansatz takes the form

$$\begin{pmatrix} \vdots & \vdots \\ \mathbf{\Pi}_{1,k_0} & \mathbf{\Pi}_{2,k_0} \\ \vdots & \vdots \\ \mathbf{\Pi}_{1,1} & \mathbf{\Pi}_{2,1} \\ \mathbf{\Pi}_{1,0} & \mathbf{\Pi}_{2,0} \\ \mathbf{\Pi}_{1,-1} & \mathbf{\Pi}_{2,-1} \\ \vdots & \vdots \\ \mathbf{\Pi}_{1,-k_0} & \mathbf{\Pi}_{2,-k_0} \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{0} \\ \vdots \\ \mathbf{Q}_1 \\ \mathbf{Q}_0 \\ \mathbf{Q}_{-1} \\ \vdots \\ \mathbf{0} \\ \vdots \end{pmatrix}, \tag{32}$$

which determines the coefficients Q_k up to a factor. A key non-trivial ingredient is the truncation condition $\mathbf{Q}_k = \mathbf{0}$ for $|k| \geq k_0$. For our model, we already know that for k_0, we may choose any integer $k_0 > n$. Without this knowledge, one needs a tradeoff between exceeding the index of the last non-vanishing $\mathbf{Q}_k$ and avoiding the numerical uncertainties of the Fourier components with very large index.

For the numerical implementation of Eq. (32), we choose for the Fourier components a cutoff $K > k_0$, such that the matrix on the left-hand side has dimension $4(2K+1) \times 2$. Then any two rows of the resulting matrix-vector equation with zero on the right-hand side constitute a set of two homogeneous linear equations, which generally admits only the trivial solution $j_1 = j_2 = 0$. However, if a symmetry with the desired properties exists, one finds a non-trivial solution for j_ν, which can be normalized such that $j_1 = 1$ and $j_2 = \pm 1$. This solution must be consistent with all other rows of Eq. (32) with $|k| \geq k_0$. The central rows then yield the non-vanishing coefficients Q_k and, thus, $Q(t)$. However, owing to the arbitrary normalization $j_1 = 1$, the symmetry operator $Q(t)$ is determined only up to a sign, which must be fixed uniquely in the final step. In the present case, this is straightforward, because the last non-vanishing coefficient Q_n has only one non-zero matrix element, which can be chosen real and positive.

References

[1] Jon H. Shirley. "Solution of the Schrödinger equation with a Hamiltonian periodic in time". Phys. Rev. **138**, B979 (1965).

[2] Hideo Sambe. "Steady states and quasienergies of a quantum-mechanical system in an oscillating field". Phys. Rev. A **7**, 2203 (1973).

[3] Peter Hänggi. "Driven quantum systems". In Quantum Transport and Dissipation. Chapter 5, pages 249–286. Wiley-VCH, Weinheim (1998).

[4] Frank Grossmann, Thomas Dittrich, Peter Jung, and Peter Hänggi. "Coherent destruction of tunneling". Phys. Rev. Lett. **67**, 516 (1991).

[5] Frank Großmann and Peter Hänggi. "Localization in a driven two-level dynamics". Europhys. Lett. **18**, 571 (1992).

[6] J. Stehlik, Y. Dovzhenko, J. R. Petta, J. R. Johansson, F. Nori, H. Lu, and A. C. Gossard. "Landau-Zener-Stückelberg interferometry of a single electron charge qubit". Phys. Rev. B **86**, 121303(R) (2012).

[7] F. Forster, G. Petersen, S. Manus, P. Hänggi, D. Schuh, W. Wegscheider, S. Kohler, and S. Ludwig. "Characterization of qubit dephasing by Landau-Zener-Stückelberg-Majorana interferometry". Phys. Rev. Lett. **112**, 116803 (2014).

[8] Mika Sillanpää, Teijo Lehtinen, Antti Paila, Yuriy Makhlin, and Pertti Hakonen. "Continuous-time monitoring of Landau-Zener interference in a Cooper-pair box". Phys. Rev. Lett. **96**, 187002 (2006).

[9] David M. Berns, Mark S. Rudner, Sergio O. Valenzuela, Karl K. Berggren, William D. Oliver, Leonid S. Levitov, and Terry P. Orlando. "Amplitude spectroscopy of a solid-state artificial atom". Nature (London) **455**, 51 (2008).

[10] H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo. "Dynamical control of matter-wave tunneling in periodic potentials". Phys. Rev. Lett. **99**, 220403 (2007).

[11] G. Della Valle, M. Ornigotti, E. Cianci, V. Foglietti, P. Laporta, and S. Longhi. "Visualization of coherent destruction of tunneling in an optical double well system". Phys. Rev. Lett. **98**, 263601 (2007).

[12] Georg Engelhardt, Gloria Platero, and Jianshu Cao. "Discontinuities in driven spin-boson systems due to coherent destruction of tunneling: Breakdown of the Floquet-Gibbs distribution". Phys. Rev. Lett. **123**, 120602 (2019).

[13] Sigmund Kohler. "Quantum dissipation at conical intersections of quasienergies". Phys. Rev. A **110**, 052218 (2024).

[14] Fritz Haake, Sven Gnutzmann, and Marek Kuś. "Quantum signatures of chaos". Springer Series in Synergetics. Springer. Cham (2018). 4th edition.

[15] Asher Peres. "Dynamical quasidegeneracies and quantum tunneling". Phys. Rev. Lett. **67**, 158 (1991).

[16] Michael Strass, Peter Hänggi, and Sigmund Kohler. "Nonadiabatic electron pumping: Maximal current with minimal noise". Phys. Rev. Lett. **95**, 130601 (2005).

[17] S. Ashhab, J. R. Johansson, A. M. Zagoskin, and Franco Nori. "Two-level systems driven by large-amplitude fields". Phys. Rev. A **75**, 063414 (2007).

[18] Oleh V. Ivakhnenko, Sergey N. Shevchenko, and Franco Nori. "Nonadiabatic Landau-Zener-Stückelberg-Majorana transitions, dy-

namics, and interference". Phys. Rep. **995**, 1 (2023).

[19] S. Ashhab. "Attempt to find the hidden symmetry in the asymmetric quantum Rabi model". Phys. Rev. A **101**, 023808 (2020).

[20] Vladimir V. Mangazeev, Murray T. Batchelor, and Vladimir V. Bazhanov. "The hidden symmetry of the asymmetric quantum Rabi model". J. Phys. A: Math. Theor. **54**, 12LT01 (2021).

[21] Cid Reyes-Bustos, Daniel Braak, and Masato Wakayama. "Remarks on the hidden symmetry of the asymmetric quantum Rabi model". J. Phys. A: Math. Theor. **54**, 285202 (2021).

[22] Georg Engelhardt and Jianshu Cao. "Dynamical symmetries and symmetry-protected selection rules in periodically driven quantum systems". Phys. Rev. Lett. **126**, 090601 (2021).

[23] Alexander Altland and Martin R. Zirnbauer. "Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures". Phys. Rev. B **55**, 1142 (1997).