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On the structure of higher order quantum maps

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We study higher order quantum maps in the context of a *-autonomous category of affine subspaces. We show that types of higher order maps can be identified with certain Boolean functions that we call type functions. By an extension of this identification, the algebraic structure of Boolean functions is inherited by some sets of quantum objects including higher order maps. Using the Möbius transform, we assign to each type function a poset whose elements are labelled by subsets of indices of the involved spaces. We then show that the type function corresponds to a comb type if and only if the poset is a chain. We also devise a procedure for decomposition of the poset to a set of basic chains from which the type function is constructed by taking maxima and minima of concatenations of the basic chains in different orders. On the level of higher order maps, maxima and minima correspond to affine mixtures and intersections, respectively.

1 Introduction

Higher order quantum maps encompass all quantum objects and admissible transformations between them. Starting from the set of quantum states, the whole hierarchy of maps can be built, including at each step all transformations between objects built at the previous steps. The basic notion here is that of a quantum channel, defined as a completely positive trace preserving map. Quantum channels represent physical transformations between quantum state spaces, but all the higher order quantum maps can be described as quantum channels with a special structure, see [1, Prop. 2].

Transformations between quantum channels are given by quantum supermaps, consisting of a pair of channels connected by an ancilla [2]. We may then consider transformations from superchannels to channels and so on, building a hierarchy of quantum combs [3, 4], also called quantum games [5], which are given by quantum circuits with some free slots. The free input and output spaces of such circuits have a fixed ordering, and therefore have a definite causal structure. Causally ordered transformations include quantum testers [6] or PPOVMs [7], describing tests and measurements on quantum channels and combs, and other types of maps, e.g. no-signalling channels.

Besides the causally ordered maps, the hierarchy of higher order maps contains transformations that have indefinite causal structure, [8–10]. Such maps include, for example, transformations between quantum combs or transformations of no-signalling channels, which can be obtained as affine mixtures of quantum combs having different causal orders [8, 10]. An example is a quantum switch [9], which transforms a pair of channels Φ and Ψ into a coherent combination of their compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$.

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In recent years, several theoretical frameworks for description of higher order maps were proposed, see also the recent review [11]. In [10, 12], a formalism of types is introduced, built from elementary types (corresponding to state spaces) and transformations. A categorical approach is taken in [13–15], where a *-autonomous category Caus[$\mathcal{C}$] is constructed over a compact closed symmetric monoidal category satisfying certain properties. The work [16] builds on these results using the description of higher order objects by superoperator projections, which extends a similar description of process matrices in [17]. In a similar approach, the projections corresponding to transformations between given quantum objects were characterized in [18].

The formalism of types provides a very general framework for description of higher order maps. The inductive construction does not refer to any specific properties of quantum systems and can therefore be transferred to more general contexts. The connection of types to *-autonomous categories enables us to recognize equivalence of certain types and to apply the underlying algebraic (Boolean algebras, [16]) and logical (BV-logics, [14]) structures to study causal relations inherent in higher order maps of a given type. The combinatorial description of types proved useful in the study of signalling relations and the restrictions they pose on possible compositions of the maps, [1]. On the other hand, the axiomatic approach of type theory in itself seems insufficient for the study of constructions of higher order maps (such as QC-CC or QC-QC maps [19], routed quantum circuits [20]) or their possible physical implementations.

The aim of the present work is to gain more understanding about the structure of types of higher order quantum maps, in an approach that combines the works [13, 14], [1, 12], [16] and [18]. A crucial tool in these works is the representation of the transformations via the Choi isomorphism, which allows to identify transformations of a given type with elements of the intersection of a certain subspace in the space of multipartite hermitian operators with the cone of positive operators and a hyperplane determined by a positive multiple of the identity. Since the cone of positive operators remains the same whichever type of higher order maps we are dealing with and the hyperplanes differ only by a positive multiple, it follows that types of higher order maps can be represented in terms of subspaces of multipartite hermitian operators, or equivalently, by projections onto them. This is the idea behind the projective characterization of process matrices in [17] and higher order quantum maps in [16, 18].

In the first part (Section 2), we combine the above idea with the categorical approach of [13, 14]. We investigate the category Af, whose objects are finite dimensional vector spaces with a distinguished affine subspace. We prove that with the structures inherited from the compact closed structure of FinVect, Af becomes a *-autonomous category. The internal hom $X \multimap Y = (X \otimes Y^*)^*$ in Af directly corresponds to the set of all linear transformations mapping elements of the affine subspace related to X into the affine subspace related to Y. Restricting to spaces of hermitian matrices, we show that the intersection with the positive cone gives exactly the Choi matrices of completely positive maps preserving given affine subspaces, in particular, the higher order quantum maps can be characterized by analysing the internal homs in Af. Using the special structure of hermitian matrices, we can also obtain a projective characterization as in [18].

We define first order objects in Af as those where the distinguished affine subspace is a hyperplane. The higher order objects are then defined by applying the internal homs, or equivalently, tensor products and duality, on a finite set $\{X_1, \ldots, X_n\}$ of first order objects. Along the lines of [12], we then find an independent decomposition of the tensor product $V = V_{X_1} \otimes \cdots \otimes V_{X_n}$ of the underlying vector spaces into subspaces labelled by binary strings. We use this decomposition to define an injective map from a certain

subalgebra $\mathcal{F}_n$ of Boolean functions into objects of Af. This map depends on the choice of the set $\{X_1, \ldots, X_n\}$. Its range consists of objects that have the tensor product V as the underlying subspace and inherits the boolean lattice structure of $\mathcal{F}_n$, with $\wedge$ and $\vee$ given by corresponding operations on the affine subspaces. In categorical terms, $\wedge$ and $\vee$ are given by pullbacks and pushouts, respectively, moreover, the above map transforms tensor products of functions into tensor products of objects. Any higher order object built over n first order objects can be found in the range of such a map, the corresponding function $f \in \mathcal{F}_n$ being uniquely determined by its type. These elements in $\mathcal{F}_n$ are called the type functions. Let us note that in the case of quantum objects, the structure inherited from $\mathcal{F}_n$ corresponds to the algebraic structure of the superoperator projections studied in [16].

The second part of the paper (Section 4) is devoted to investigation of the structure of type functions. We first notice that there are special type functions p_S, $S \subseteq [n]$, whose conjugates are mapped to objects representing channels. We then use the Möbius transform to express any function in $\mathcal{F}_n$ as a combination of these functions. This leads to definition of a subposet $\mathcal{P}_f$ in 2^n of subsets $S \subseteq [n]$ such that p_S has a nonzero coefficient. We show that if f is a type function, then $\mathcal{P}_f$ is a graded poset with even rank and f is fully determined by $\mathcal{P}_f$. It is also shown that f is a type function corresponding to combs if and only if $\mathcal{P}_f$ is a chain (of even length), and the subsets in the chain determine the input and output spaces of the combs. We then introduce a causal product on $\mathcal{F}_n$, which in case of chain types corresponds to connections of the chains. This operation can be seen as a version of the 'prec' connector of [16] and the sequential product $<$ of [14, 15]. The structure theorem of type functions then proves that any type function is obtained from several basic chain types by taking maxima and minima (or minima and maxima) over their concatenations in different orders. This can be seen as a version of the normal form of [16].

In Section 4.6 we then show that instead of the poset $\mathcal{P}_f$, we may use a smaller poset $\mathcal{P}_f^0$, whose vertices are labelled by subsets of $[n]$. We show that any type function is fully determined by $\mathcal{P}_f^0$, by presenting a procedure for decomposing the poset into a set of chains, from which the type function is constructed via the structure theorem.

Let us stress that the introduced framework is only suitable for the study of features of maps that can be expressed by affine constraints. There are many important properties that are not affine, such as causal nonseparability, LOSR (local operations and shared randomness) or LOSE (local operations and shared entanglement) bipartite channels, etc. For these one should include other restrictions e.g. on the positive cone, but such considerations are outside the scope of this work.

Let us also remark that to obtain our results for higher order quantum maps, we could have restricted our investigation to the *-autonomous subcategory of quantum objects (see Section 2.2.1), which would make our constructions somewhat easier. Specifically, in the general case there is no natural identification of a linear space V with its dual, and we also have to specify some fixed element in the hyperplane for all involved first order objects. It is not yet clear whether such a more general approach can be really useful, on the other hand, the complications involved are not that difficult to tackle. The more general category of affine subspaces was used with the idea that similar considerations might be applicable for investigation of higher order maps in the framework of general probabilistic theories, see e.g. [21] for an overview of GPTs and [22] for some suggestions for higher order constructions in this framework.

2 Affine subspaces and higher order maps

2.1 The category FinVect

The category FinVect is a basic example of a compact closed category, whose structure underlies all the results in this paper. For a standard reference on category theory and symmetric monoidal categories, see [23]. For more on compact closed categories, see [24]. For the use of categories in quantum theory, see [25].

Let FinVect be the category of finite dimensional real vector spaces with linear maps. We will denote the usual tensor product by $\otimes$, then $(\mathrm{FinVect}, \otimes, I = \mathbb{R})$ is a symmetric monoidal category, with the associators, unitors and symmetries given by the obvious isomorphisms

$$\begin{aligned} \alpha_{U,V,W} &: (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \\ \lambda_V &: I \otimes V \simeq V, \qquad \rho_V : V \otimes I \simeq V, \\ \sigma_{U,V} &: U \otimes V \simeq V \otimes U. \end{aligned}$$

For any permutation $\pi \in \mathscr{S}_n$, we will denote the isomorphism $V_1 \otimes \cdots \otimes V_n \to V_{\pi^{-1}(1)} \otimes \cdots \otimes V_{\pi^{-1}(n)}$ again by π.

Let $(-)^* : V \mapsto V^*$ be the usual vector space dual, with duality denoted by $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$. We will use the canonical identification $V^{**} = V$ and $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$. With this duality, FinVect is compact closed. This means that for each object V, there are morphisms $\eta_V : I \to V^* \otimes V$ (the 'cup') and $\epsilon_V : V \otimes V^* \to I$ (the 'cap') such that the following 'snake' identities hold:

$$(\epsilon_V \otimes V) \circ (V \otimes \eta_V) = V, \qquad (V^* \otimes \epsilon_V) \circ (\eta_V \otimes V^*) = V^*, \tag{1}$$

here we denote the identity map on the object V by V. We will see in Example 2.2 below that in the case related to quantum theory, that is when V is the space of complex hermitian matrices, the two morphisms are given by (a multiple of) the maximally entangled state. In fact, as it is now well established, the identities in (1) are exactly what lies behind fundamental protocols such as quantum teleportation [25]. These identities also enable us to identify linear maps with their Choi matrices, which is an important tool in the higher order quantum theory.

Let us identify the 'cup' and 'cap' morphisms for an object V in FinVect. First, η_V is a linear map $\mathbb{R} \to V^* \otimes V$, which can be identified with the element $\eta_V(1) \in V^* \otimes V$ and $\epsilon_V \in (V \otimes V^*)^* = V^* \otimes V$ is again an element of the same space. Choose a basis $\{e_i\}$ of V, let $\{e_i^*\}$ be the dual basis of V^*, that is, $\langle e_i^*, e_j \rangle = \delta_{i,j}$. Let us define

$$\eta_V(1) = \epsilon_V := \sum_i e_i^* \otimes e_i.$$

It is easy to see that this definition does not depend on the choice of the basis, indeed, ϵ_V is the linear functional on $V \otimes V^*$ defined by

$$\langle \epsilon_V, x \otimes x^* \rangle = \langle x^*, x \rangle, \qquad x \in V,\ x^* \in V^*.$$

It is also easily checked that the snake identities (1) hold.

For two objects V and W in FinVect, we will denote the set of all morphisms (i.e. linear maps) $V \to W$ by $\mathrm{FinVect}(V, W)$. An object $V \multimap W$ is called an internal hom if for every object U there exists an isomorphism (called currying)

$$\mathrm{FinVect}(U \otimes V, W) \simeq \mathrm{FinVect}(U, V \multimap W)$$

natural in U, V and W. In a compact closed category an internal hom always exists and can be given as $V \multimap W = V^* \otimes W$.

The set $\mathrm{FinVect}(V, W)$ itself has a structure of a real vector space and it is well known that we have the identification $\mathrm{FinVect}(V, W) \simeq V^* \otimes W = V \multimap W$. This can be given as follows: for each $f \in \mathrm{FinVect}(V, W)$, we have $C_f := (V^* \otimes f)(\epsilon_V) = \sum_i e_i^* \otimes f(e_i) \in V^* \otimes W$. Conversely, since $\{e_i^*\}$ is a basis of V^*, any element $w \in V^* \otimes W$ can be uniquely written as $w = \sum_i e_i^* \otimes w_i$ for $w_i \in W$, and since $\{e_i\}$ is a basis of V, the assignment $f(e_i) := w_i$ determines a unique map $f : V \to W$. The relations between $f \in \mathrm{FinVect}(V, W)$ and $C_f \in V^* \otimes W$ can be also written as

$$\langle\, C_f, x \otimes y^* \,\rangle = \langle\, \epsilon_V, x \otimes f^*(y^*) \,\rangle = \langle\, f^*(y^*), x \,\rangle = \langle\, y^*, f(x) \,\rangle, \qquad x \in V,\ y^* \in W^*,$$

here $f^* : W^* \to V^*$ is the adjoint of f.

The following two examples are most important for us.

Example 2.1. Let $V = \mathbb{R}^N$. In this case, we fix the canonical basis $\{|i\rangle,\ i = 1, \ldots, N\}$. We will identify $(\mathbb{R}^N)^* = \mathbb{R}^N$, with duality $\langle\, x, y \,\rangle = \sum_i x_i y_i$, in particular, we identify $I = I^*$. We then have $\epsilon_V = \sum_i |i\,\rangle \otimes |i\,\rangle$ and if $f : \mathbb{R}^N \to \mathbb{R}^M$ is given by the matrix A in the two canonical bases, then $C_f = \sum_i |i\,\rangle \otimes A|i\,\rangle$ is the vectorization of A.

Example 2.2. Let $V = M_n^h$ be the space of $n \times n$ complex hermitian matrices. We again identify $(M_n^h)^* = M_n^h$, with duality $\langle\, A, B \,\rangle = \mathrm{Tr}\, AB$. Let us choose the basis in M_n^h, given as

$$\left\{ |j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|,\ j \le k,\ i\Big(|j\,\rangle\langle\, k| - |k\,\rangle\langle\, j|\Big),\ j < k \right\}.$$

Then one can check that

$$\left\{ \frac{1}{2}\Big(|j\,\rangle\langle\, k| + |k\,\rangle\langle\, j|\Big),\ j \le k,\ \frac{i}{2}\Big(|k\,\rangle\langle\, j| - |j\,\rangle\langle\, k|\Big),\ j < k \right\}$$

is the dual basis and we have

$$\epsilon_V = \sum_{j,k} |j\,\rangle\langle\, k| \otimes |j\,\rangle\langle\, k|,$$

note that up to normalization, this is a maximally entangled state in $\mathbb{C}^n \otimes \mathbb{C}^n$. For any $f : M_n^h \to M_m^h$,

$$C_f = \sum_{j,k} |j\,\rangle\langle\, k| \otimes f(|j\,\rangle\langle\, k|)$$

is the Choi matrix of f.

In the rest of this work, we will use the following notations. For subspaces in a vector space, we will use $\vee$ and $\wedge$ for the usual lattice operations on subspaces, that is, for $V_1, V_2 \subseteq V$, we put $V_1 \wedge V_2 = V_1 \cap V_2$ and $V_1 \vee V_2 = \mathrm{Span}(V_1 \cup V_2)$. The direct sum will be denoted by $+$, that is, $V_1 + V_2 = V_1 \vee V_2$ with $V_1 \wedge V_2 = \{0\}$. For $V_1, \ldots, V_n \subseteq V$, we denote the direct sum by $\sum$, that is, $\sum_i V_i = \bigvee V_i$ with $V_i \wedge V_j = \{0\}$. The notation $\oplus$ is reserved for the orthogonal direct sum of subspaces in M_n^h with inner product $\langle\, \cdot, \cdot \,\rangle$.

2.2 The category Af

We now introduce the category Af, whose objects are of the form $X = (V_X, A_X)$, where V_X is an object in FinVect and $A_X \subseteq V_X$ is a proper affine subspace, see Appendix B for definitions and basic properties. Morphisms $X \xrightarrow{f} Y$ in Af are linear maps $f : V_X \to V_Y$

such that $f(A_X) \subseteq A_Y$. For two objects X and Y and a linear map $f : V_X \to V_Y$, we write $X \xrightarrow{f} Y$ with the meaning that f is a morphism in Af. In particular, if $V_X = V_Y = V$, then $X \xrightarrow{id_V} Y$ means that $A_X \subseteq A_Y$. The set of all morphisms $X \xrightarrow{f} Y$ in Af will be denoted by $\mathrm{Af}(X, Y)$.

For any object X, we put

$$L_X := \mathrm{Lin}(A_X), \quad S_X := \mathrm{Span}(A_X), \quad D_X = \dim(V_X), \quad d_X = \dim(L_X).$$

By (B.1) and (B.2), we have

$$A_X = a + L_X = S_X \cap \{\tilde{a}\}^*, \tag{2}$$

for any choice of elements $a \in A_X$ and $\tilde{a} \in A_X^*$.

We will next introduce a monoidal structure $\otimes$ as follows. For two objects X and Y, we put $V_{X\otimes Y} = V_X \otimes V_Y$ and construct the affine subspace $A_{X\otimes Y}$ as the affine span of

$$A_X \otimes A_Y = \{a \otimes b,\ a \in A_X,\ b \in A_Y\}.$$

Fix any $a_X \in A_X$, $\tilde{a}_X \in A_X^*$ and $a_Y \in A_Y$, $\tilde{a}_Y \in A_Y^*$. Since $a_X \otimes a_Y \in A_X \otimes A_Y \subseteq \{\tilde{a}_X \otimes \tilde{a}_Y\}^*$, the affine span of $A_X \otimes A_Y$ is a proper affine subspace and we have by Lemma B.1

$$A_{X\otimes Y} := \mathrm{Aff}(A_X \otimes A_Y) = \{A_X \otimes A_Y\}^{**}. \tag{3}$$

Lemma 2.1. *For any $a_X \in A_X$, $a_Y \in A_Y$, we have*

$$L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y) = \mathrm{Span}(\{x \otimes y - a_X \otimes a_Y,\ x \in A_X,\ y \in A_Y\}) \tag{4}$$
$$= (a_X \otimes L_Y) + (L_X \otimes a_Y) + (L_X \otimes L_Y) \tag{5}$$

(here $+$ denotes the direct sum of subspaces). We also have

$$S_{X\otimes Y} = S_X \otimes S_Y.$$

Proof. The equality (4) follows from Lemma B.1. For any $x \in A_X$, $y \in A_Y$ we have

$$x \otimes y - a_X \otimes a_Y = a_X \otimes (y - a_Y) + (x - a_X) \otimes a_Y + (x - a_X) \otimes (y - a_Y),$$

so that $L_{X\otimes Y} = \mathrm{Lin}(A_X \otimes A_Y)$ is contained in the subspace on the RHS of (5). Let d be the dimension of this subspace, then clearly

$$d_{X\otimes Y} \le d \le d_X + d_Y + d_X d_Y.$$

On the other hand, any element of S_X has the form tx for some $t \in \mathbb{R}$ and $x \in A_X$, so that it is easily seen that $S_X \otimes S_Y = S_{X\otimes Y}$. Hence

$$d_{X\otimes Y} = \dim(L_{X\otimes Y}) = \dim(S_{X\otimes Y}) - 1 = \dim(S_X)\dim(S_Y) - 1 = (d_X + 1)(d_Y + 1) - 1$$
$$= d_X + d_Y + d_X d_Y.$$

This completes the proof.

□

Lemma 2.2. *Let $I = (\mathbb{R}, \{1\})$. Then $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category.*

Proof. Note that this structure is inherited from the symmetric monoidal structure in FinVect. To show that $\otimes$ is a functor, we have to check that for $X_1 \xrightarrow{f} Y_1$ and $X_2 \xrightarrow{g} Y_2$ in Af, we have $X_1 \otimes Y_1 \xrightarrow{f\otimes g} X_2 \otimes Y_2$ which amounts to showing that

$$(f \otimes g)(A_{X_1\otimes Y_1}) \subseteq A_{X_2\otimes Y_2}.$$

Let $x \in A_{X_1}$, $y \in A_{Y_1}$, then $f(x)\otimes g(y) \in A_{X_2} \otimes A_{Y_2} \subseteq A_{X_2\otimes Y_2}$. Since $A_{X_1\otimes Y_1}$ is the affine subspace generated by $A_{X_1} \otimes A_{Y_1}$, the above inclusion follows by linearity of $f \otimes g$.

It only remains to prove that the associators, unitors and symmetries from FinVect are morphisms in Af. We will prove this for the associators $\alpha_{X,Y,Z} : V_X \otimes (V_Y \otimes V_Z) \to (V_X \otimes V_Y) \otimes V_Z$, the other proofs are similar. We need to check that $\alpha_{X,Y,Z}(A_{X\otimes(Y\otimes Z)}) \subseteq A_{(X\otimes Y)\otimes Z}$. It is easily checked that $A_{X\otimes(Y\otimes Z)}$ is the affine span of elements of the form $x \otimes (y \otimes z)$, $x \in A_X$, $y \in A_Y$ and $z \in A_Z$, and we have

$$\alpha_{X,Y,Z}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z \in A_{(X\otimes Y)\otimes Z}$$

for all such elements. The desired inclusion follows by linearity.

□

By Corollary B.2, the dual affine subspace A^*_X is a proper affine subspace in V^*_X, so that $X^* := (V^*_X, A^*_X)$ is an object in Af. We have $X^{**} = X$ and the corresponding subspaces are related as

$$L_{X^*} = S^\perp_X, \qquad S_{X^*} = L^\perp_X. \tag{6}$$

It is easily seen that for any $X \xrightarrow{f} Y$, the adjoint map satisfies $f^*(A^*_Y) \subseteq A^*_X$, so that $Y^* \xrightarrow{f^*} X^*$ and the duality $(-)^*$ is a full and faithful functor $\mathrm{Af}^{op} \to \mathrm{Af}$. As we will see below, Af with this monoidal structure and duality is not compact closed. Nevertheless, we next show that it is *-autonomous, which is a weaker property meaning that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category and the duality $(-)^*$ is a full and faithful contravariant functor such that the internal hom is given by $X \multimap Y = (X \otimes Y^*)^*$, see [26] for details. This property will be crucial for the structure of higher order objects studied further.

Theorem 2.3. *$(\mathrm{Af}, \otimes, I)$ is a *-autonomous category, with duality $(-)^*$, such that $I^* = I$.*

Proof. By Lemma 2.2, we have that $(\mathrm{Af}, \otimes, I)$ is a symmetric monoidal category. We have also seen that the duality $(-)^*$ is a full and faithful contravariant functor. We only need to check the natural isomorphisms

$$\mathrm{Af}(X \otimes Y, Z^*) \simeq \mathrm{Af}(X, (Y \otimes Z)^*).$$

Since FinVect is compact, we have the natural isomorphisms

$$\mathrm{FinVect}(V_X \otimes V_Y, V^*_Z) \simeq \mathrm{FinVect}(V_X, V^*_Y \otimes V^*_Z),$$

determined by the equalities

$$\langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle, \qquad x \in V_X,\ y \in V_Y,\ z \in V_Z,$$

for $f \in \mathrm{FinVect}(V_X \otimes V_Y, V^*_Z)$ and the corresponding morphism $\hat{f} \in \mathrm{FinVect}(V_Z, V^*_Y \otimes V^*_Z)$. Since $A_{X\otimes Y}$ is an affine span of $A_X \otimes A_Y$, we see that f is in $\mathrm{Af}(X \otimes Y, Z^*)$ if and only if $f(x \otimes y) \in A^*_Z$ for all $x \in A_X$, $y \in A_Y$, that is,

$$1 = \langle\, f(x \otimes y), z \,\rangle = \langle\, \hat{f}(x), y \otimes z \,\rangle \qquad \forall x \in A_X,\ \forall y \in A_Y,\ \forall z \in A_Z.$$

But this is equivalent to

$$\hat{f}(x) \in (A_Y \otimes A_Z)^* = A^*_{Y\otimes Z}, \qquad \forall x \in A_X,$$

which means that $\hat{f} \in \mathrm{Af}(X, (Y \otimes Z)^*)$.

□

A *-autonomous category is compact closed if it satisfies $(X \otimes Y)^* = X^* \otimes Y^*$. In general, $X \parr Y = (X^* \otimes Y^*)^*$ defines a dual symmetric monoidal structure that is different from $\otimes$. We next show that Af is not compact closed.

Proposition 2.4. *For objects in* Af*, we have* $(X \otimes Y)^* = X^* \otimes Y^*$ *exactly in one of the following situations:*

(i) $X \simeq I$ *or* $Y \simeq I$*,*

(ii) $d_X = d_Y = 0$*,*

(iii) $d_{X^*} = d_{Y^*} = 0$*.*

Proof. Since FinVect is compact, we have $V_{(X\otimes Y)^*} = (V_X \otimes V_Y)^* = V_X^* \otimes V_Y^* = V_{X^*\otimes Y^*}$. It is also easily seen by definition that $A_{X^*} \otimes A_{Y^*} = A_X^* \otimes A_Y^* \subseteq A^*_{X\otimes Y} = A_{(X\otimes Y)^*}$, so that we always have $A_{X^*\otimes Y^*} \subseteq A_{(X\otimes Y)^*}$. Hence the equality holds if and only if $d_{X^*\otimes Y^*} = d_{(X\otimes Y)^*}$. From Lemma 2.1, we see that

$$d_{X^*\otimes Y^*} = d_{X^*} + d_{Y^*} + d_{X^*}d_{Y^*}.$$

On the other hand, we have using (6) that $L_{(X\otimes Y)^*} = S^\perp_{X\otimes Y} = (S_X \otimes S_Y)^\perp$, so that

$$d_{(X\otimes Y)^*} = D_X D_Y - \dim(S_X)\dim(S_Y) = D_X D_Y - (d_X + 1)(d_Y + 1).$$

Taking into account that by (6) we have $d_{X^*} = D_X - d_X - 1$, similarly for d_{Y^*}, we obtain

$$d_{(X\otimes Y)^*} - d_{X^*\otimes Y^*} = d_X d_{Y^*} + d_{X^*}d_Y.$$

This is equal to 0 iff $d_X d_{Y^*} = d_Y d_{X^*} = 0$, which amounts to the conditions in the lemma.

□

In a *-autonomous category, the internal hom can be identified as $X \multimap Y = (X \otimes Y^*)^*$. The underlying vector space is $V_{X\multimap Y} = (V_X \otimes V_Y^*)^* = V_X^* \otimes V_Y = V_X \multimap V_Y$ and we have seen in Section 2.1 that we may identify this space with $\mathrm{FinVect}(V_X, V_Y)$, by $f \leftrightarrow C_f$. This property is extended to Af, in the following sense.

Proposition 2.5. *For any objects* X, Y *in* Af*, the map* $f \mapsto C_f$ *is a bijection of* $\mathrm{Af}(X, Y)$ *onto* $A_{X\multimap Y}$*. In particular,* A_X^* *can be identified with* $\mathrm{Af}(X, I)$*.*

Proof. Let $f \in \mathrm{FinVect}(V_X, V_Y)$. Since by definition $A_{X\multimap Y} = A^*_{X\otimes Y^*} = (A_X \otimes A_Y^*)^*$, we see that $C_f \in A_{X\multimap Y}$ if and only if for all $x \in A_X$ and $\tilde{y} \in A_Y^*$, we have

$$1 = \langle\, C_f, x \otimes \tilde{y} \,\rangle = \langle\, \tilde{y}, f(x) \,\rangle.$$

This latter statement is clearly equivalent to $f(A_X) \subseteq A_Y$, so that $f \in \mathrm{Af}(X, Y)$.

□

2.2.1 Quantum and classical objects

We now restrict to objects such that the underlying vector spaces are spaces of hermitian matrices, as in Example 2.2. We may also restrict morphisms between such spaces to completely positive maps. We show that this restriction amounts to taking an intersection of $A_{X\multimap Y}$ with the cone of positive semidefinite matrices. This, and subsequent examples, shows that for characterization of sets of higher order quantum maps it is enough to work with the category Af.

An object X of Af will be called quantum if $V_X = M_n^h$ for some n and A_X is an affine subspace such that both A_X and A_X^* contain a positive multiple of the identity matrix E_n[1]. (recall that we identify $(M_n^h)^* = M_n^h$).

Proposition 2.6. *Let X, Y be quantum objects in* Af. *Then*

(i) X^* and $X \otimes Y$ are quantum objects as well.

(ii) Let $V_X = M_n^h$, $V_Y = M_m^h$. Then for any $f \in \mathrm{FinVect}(M_n^h, M_m^h)$, we have $C_f \in A_{X\multimap Y} \cap M_{mn}^+$ if and only if f is completely positive and

$$f(A_X \cap M_n^+) \subseteq A_Y \cap M_m^+.$$

Proof. The statement (i) is easily seen from $A_X \otimes A_Y \subseteq A_{X\otimes Y}$ and $A_X^* \otimes A_Y^* \subseteq A_{X\otimes Y}^*$. To show (ii), let $C_f \in A_{X\multimap Y} \cap M_{mn}^+$. By the properties of the Choi isomorphism f is completely positive and by Proposition 2.5, $f(A_X) \subseteq A_Y$, this proves one implication. For the converse, note that we only need to prove that under the given assumptions, $f(A_X) \subseteq A_Y$, for which it is enough to show that $A_X \subseteq \mathrm{Aff}(A_X \cap M_n^+)$. To see this, let $t_X E_n \in A_X$ for some $t_X > 0$. Any element in A_X can be written in the form $t_X E_n + v$ for some $v \in L_X$. Since $t_X E_n \in int(M_n^+)$, there is some $s > 0$ such that $a_\pm := t_X E_n \pm sv \in M_n^+$, and since $\pm sv \in L_X$, we see that $a_\pm \in A_X \cap M_n^+$. It is now easily checked that

$$t_X E_n + v = \frac{1+s}{2s} a_+ + \frac{s-1}{2s} a_- \in \mathrm{Aff}(A_X \cap M_n^+).$$

□

Remark 2.1. Note that the same results can be obtained if we only assume that both A_X and A_X^* contain an interior point of the positive cone M_n^+.

Remark 2.2. The above results show that the quantum objects with the morphisms restricted by complete positivity form a *-autonomous subcategory in Af. Observe also that this subcategory is equivalent to the category Caus[**CPM**], constructed in [13], which is referred to as higher-order causal category for quantum theory.

Let $X = (M_n^h, A_X)$ be a quantum object. Since both A_X and A_X^* contain a positive multiple of the identity E_n and $\langle E_n, a \rangle = \mathrm{Tr}\,[a]$ for any $a \in M_n^h$, we see that the object X is uniquely identified by the dimension $n_X = n$, the subspace $S_X \in M_n^h$ containing E_n and a positive constant $c_X > 0$ such that

$$A_X = \{a \in S_X,\ \mathrm{Tr}\,[a] = c_X\}. \tag{7}$$

We then have

$$L_X = \{v \in S_X,\ \mathrm{Tr}\,[v] = 0\}, \qquad S_X = L_X \oplus \mathbb{R}\{E_n\},$$

[1]We use the notation E_n, and not I_n, to avoid the slight chance that it might be confused with the monoidal unit.

where $\oplus$ denotes the orthogonal direct sum with respect to the inner product $\langle a, b \rangle = \mathrm{Tr}\,[ab]$ in M_n^h.

Since we identify M_n^h and $(M_n^h)^*$ using $\langle \cdot, \cdot \rangle$, we also identify the annihilator $S_X^\perp$ with the orthogonal complement of S_X in M_n^h. We obtain

$$S_{X^*} = L_{X^*} \oplus \mathbb{R}\{E_n\} = S_X^\perp \oplus \mathbb{R}\{E_n\}. \tag{8}$$

Let $a \in A_X$ and $\tilde{a} \in A_{X^*}$. Then $\tilde{a} = \tilde{v} + tE_n$ for $\tilde{v} \in S_X^\perp$ and $t \in \mathbb{R}$ and from

$$1 = \langle a, \tilde{a} \rangle = t\mathrm{Tr}\,[a] = tc_X$$

we get $t = c_X^{-1}$, so that $c_{X^*} = \mathrm{Tr}\,[\tilde{a}] = c_X^{-1} n$. It is also clear from (3) that if Y is another quantum object, then we have $c_{X \otimes Y} = c_X c_Y$.

The description of the affine subspace for a quantum object in (7) is, up to the positivity constraint, basically the same as the quantum object set in [18, Def. 1], where the linear subspace S_X is characterized as the range of an orthogonal projection (with respect to the Hilbert-Schmidt inner product). As we have seen in Proposition 2.5, the affine subspace $A_{X \multimap Y}$ is precisely the set of Choi matrices of maps between affine sets A_X and A_Y. Our next result then corresponds to [18, Theorem 2], where this set is described in terms of the corresponding projections and trace constraints.

Lemma 2.7. *Let $X = (M_n^h, A_X)$, $A_X = \{a \in S_X,\ \mathrm{Tr}\,[a] = c_X\}$, $Y = (M_m^h, A_Y)$, $A_Y = \{a \in S_Y,\ \mathrm{Tr}\,[a] = c_Y\}$ be quantum objects. Then $X \multimap Y = (M_{mn}^h, A_{X \multimap Y})$, where $A_{X \multimap Y}$ is determined by*

$$S_{X \multimap Y} = (S_X^\perp \otimes M_m^h) \oplus (S_X \otimes L_Y) \oplus \mathbb{R}\{E_{mn}\}, \qquad c_{X \multimap Y} = nc_X^{-1} c_Y.$$

Proof. From $X \multimap Y = (X \otimes Y^*)^*$, we have by Lemma 2.1 and (6), together with the above considerations, that

$$S_{X \multimap Y} = L_{X \multimap Y} \oplus \mathbb{R}\{E_{mn}\}, \quad c_{X \multimap Y} = mn(c_X c_{Y^*})^{-1} = (c_X c_Y^{-1} m)^{-1} mn = nc_X^{-1} c_Y$$

and

$$L_{X \multimap Y} = (S_X \otimes S_{Y^*})^\perp = (S_X^\perp \otimes M_m^h) \vee (M_n^h \otimes L_Y) = (S_X^\perp \otimes M_m^h) \oplus (S_X \otimes L_Y).$$

□

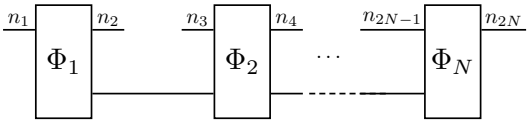
Example 2.3 (States, channels and combs). A basic example of a quantum object is $X = (M_n^h, A_X)$, determined by $S_X = V_X = M_n^h$ and $c_X = 1$. Let us denote this object by $\mathcal{S}_n$. The set $A_{\mathcal{S}_n} \cap M_n^+$ is the set of n by n density matrices. By Proposition 2.5, $\mathcal{C}_{m,n} := \mathcal{S}_m \multimap \mathcal{S}_n$ is a quantum object as well, with underlying vector space M_{mn}^h, and Proposition 2.6 shows that $A_{\mathcal{C}_{m,n}} \cap M_{mn}^+$ is the set of Choi matrices of completely positive maps $f : M_m \to M_n$ mapping states to states, i. e. quantum channels.

Note that the dual object $\mathcal{C}_{m,n}^* = \mathcal{S}_m \otimes \mathcal{S}_n^*$ represents the set of Choi matrices of replacement channels $M_n \to M_m$, that is, channels that map any state in M_n to a fixed state in M_m. We also have $\mathcal{S}_n^* = \mathcal{S}_n \multimap I = \mathcal{C}_{n,1}$ and $\mathcal{S}_n = I \multimap \mathcal{S}_n = \mathcal{C}_{1,n}$. We can proceed inductively as follows. Put

$$\mathcal{C}_{m,n,k,l}^2 = \mathcal{C}_{m,n} \multimap \mathcal{C}_{k,l} = (\mathcal{S}_m \multimap \mathcal{S}_n) \multimap (\mathcal{S}_k \multimap \mathcal{S}_l),$$

then $A_{\mathcal{C}_{m,n,k,l}^2} \cap M_{mnkl}^+$ is the set of Choi matrices of completely positive maps $M_{mn} \to M_{kl}$, mapping Choi matrices of channels to Choi matrices of channels, or quantum 2-combs,

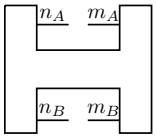
[2, 4]. The 1-combs coincide with quantum channels. The quantum $(N+1)$-combs for any N is the set of Choi matrices of completely positive maps mapping N-combs to 1-combs. The corresponding objects in Af are then given as $\mathcal{C}^N_{n_1,\ldots,n_{2N}} = \mathcal{C}^{N-1}_{n_2,\ldots,n_{2N-2}} \multimap \mathcal{C}_{n_1,n_{2N}}$. By Proposition 2.6, these are all quantum objects. It was proved in [4] that the completely positive maps corresponding to N-combs have the form



for some channels $\Phi_1, \ldots, \Phi_N$.

Example 2.4 (No-signalling channels and process matrices). No-signalling channels describe the situation when no signalling is allowed between two parties A and B. As it was shown in [9, 27], this corresponds to bipartite channels that can be obtained by affine combinations of channels acting locally on A and B. In our setting, the bipartite channels are represented by the object $\mathcal{C}_{n_An_B,m_Am_B}$ and the no-signalling channels by $\mathcal{C}_{n_A,m_A} \otimes \mathcal{C}_{n_B,m_B}$, see also [13].

Process matrices were introduced in [8] as positive linear functionals that map tensor products of pairs (or more generally tuples) of Choi matrices of channels to 1:



By linearity, this property extends to all no-signalling channels. In our setting, the process matrices correspond to $(\mathcal{C}_{n_A,m_A} \otimes \mathcal{C}_{n_B,m_B})^*$.

We can define classical objects in Af in a similar way, replacing M_n^h by $\mathbb{R}^N$ and the positive cone by $\mathbb{R}^N_+$, and we require that both A_X and A_X^* contains a positive multiple of the unit vector $e_N = (1, \ldots, 1) \in \mathbb{R}^N$. A statement similar to Proposition 2.6 holds in this case, with complete positivity replaced by positivity. Again, it can be seen that the *-autonomous subcategory of classical objects and positive morphisms is equivalent to the category Caus[$\mathbf{Mat}(\mathbb{R}_+)$] of [13].

We can similarly treat classical-to-quantum and quantum-to-classical maps as morphisms between these types of objects, satisfying appropriate positivity assumptions.

Example 2.5 (Partially classical maps). We may similarly define the basic classical object as

$$\mathcal{P}_k := (\mathbb{R}^k, \{x \in \mathbb{R}^k,\ \sum_i x_i = 1\}).$$

In this case, $\mathcal{A}_{\mathcal{P}_k} \cap \mathbb{R}^k_+$ is the probability simplex. We then obtain further useful objects by combining $\mathcal{P}_k$ with quantum objects. For example, it can be easily seen that $\mathcal{S}_n \multimap \mathcal{P}_k$ intersected with the cone $M_n^+ \otimes \mathbb{R}^n_+$ corresponds to k-outcome measurements. Similarly, we obtain k-outcome quantum instruments from $\mathcal{S}_m \multimap (\mathcal{S}_n \otimes \mathcal{P}_k)$, quantum multimeters from $(\mathcal{S}_m \otimes \mathcal{P}_k) \multimap \mathcal{P}_l$, quantum testers from $\mathcal{C}^N_{n_1,\ldots,n_{2N}} \multimap \mathcal{P}_k$, etc. In general, maps of the form $X^* \multimap \mathcal{P}_k$ can be used to define probabilistic objects related to X.

2.3 First order and higher order objects

We say that an object X in Af is first order if $d_X = D_X - 1$, equivalently, $S_X = V_X$. Another equivalent condition is $d_{X^*} = 0$, which means that A_X is determined by a single element $\tilde{a}_X \in V_X^*$ as

$$A_X = \{\tilde{a}_X\}^*, \qquad A_X^* = \{\tilde{a}_X\}.$$

Note that first order objects, resp. their duals, are exactly those satisfying condition (iii), resp. condition (ii), in Proposition 2.4, in particular, $(X \otimes Y)^* = X^* \otimes Y^*$ for first order objects X and Y.

Higher order objects in Af are objects obtained from a finite set $\{X_1, \ldots, X_n\}$ of first order objects by taking tensor products and duals. The above is indeed a set, so that all the objects are different (though they may be isomorphic) and the ordering is not essential. We will also assume that the monoidal unit I is not contained in this set. By definition of $X \multimap Y$, and since we may identify $X \multimap I$ with X^*, we see that higher order objects are also generated by applying the internal hom inductively on $\{X_1, \ldots, X_n\}$ if we allow $X_i = I$ for some i. It follows that the objects introduced in Examples 2.3, 2.4 and 2.5 are indeed higher order objects in Af according to the above definition.

Of course, any first order object is also higher order with $n = 1$. Note that we cannot say that a higher order object generated from $\{X_1, \ldots, X_n\}$ is automatically "of order n", as the following lemma shows.

Lemma 2.8. *Let X, Y be first order, then $X \otimes Y$ is first order as well.*

Proof. We have $S_{X\otimes Y} = S_X \otimes S_Y = V_X \otimes V_Y = V_{X\otimes Y}$.

□

As we have seen, higher order objects are obtained by applying the internal hom iteratively. The next lemma shows some properties of such iterations.

Lemma 2.9. *Let X, Y, Z be any objects in* Af *and let F_0, F_1, F_2 be first order objects. Then we have*

(i) $Z \multimap (X \multimap Y) \simeq (Z \otimes X) \multimap Y \simeq X \multimap (Z \multimap Y)$.

(ii) $(Z \multimap F_1) \multimap (F_0 \multimap F_2) \simeq ((F_0 \multimap Z) \multimap F_1) \multimap F_2$.

(iii) If $F = (V_F, \{\tilde{a}_F\}^*)$ is first order, then $Z \multimap F$ is determined as

$$A_{Z\multimap F} = \{w \in V_Z^* \otimes V_F, (id \otimes \tilde{a}_F)(w) \in A_Z^*\}.$$

Proof. The statement (i) is a well known property (currying) of closed symmetric monoidal categories and (iii) is easily seen from the definition and properties of $\multimap$. The proof of (ii) is given in Appendix B.1.

□

Note also that since we identify $X^{**} = X$ for any object X, the isomorphisms in (i) above are given by the symmetries in FinVect, that is, by permutations of the components in the tensor products of the underlying vector spaces. To save some parentheses, we also assume that the internal homs associate to the right, so we write $X \multimap Y \multimap Z$ instead of $X \multimap (Y \multimap Z)$.

Example 2.6 (Channels and Combs). Let X and Y be first order objects in Af. As we have seen, $C_1(X,Y) := X \multimap Y$ is then a higher order object, called a channel or 1-comb (We slightly abuse the terminology here). We will inductively construct higher order objects in Af, similarly as in Example 2.3. An N-comb over first order objects $X_1, \dots, X_{2N}$ is an object

$$\begin{aligned} C_N(X_1,\dots,X_{2N}) &:= C_{N-1}(X_2,\dots,X_{2N-1}) \multimap (X_1 \multimap X_{2N}) \\ &\simeq X_1 \multimap C_{N-1}(X_2,\dots,X_{2N-1}) \multimap X_{2N} \\ &\simeq X_1 \multimap (X_2 \multimap \cdots \multimap (X_N \multimap X_{N+1}) \multimap \cdots \multimap X_{2N-1}) \multimap X_{2N} \\ &\simeq (\dots(X_1 \multimap X_2) \multimap X_3) \multimap \dots) \multimap X_{2N-1}) \multimap X_{2N} \end{aligned}$$

where the isomorphisms follow by repeated use of the statements (i) and (ii) of Lemma 2.9. Note that the first two isomorphism are obtained using only the part (i) of the Lemma, so that they hold even if the involved object are not first order, but we need to assume first order objects for the isomorphism in the last line. The subspace A_{C_N} for an N-comb C_N can be found inductively, using Lemma 2.9 (iii). If $X_1, \dots, X_{2N}$ are quantum objects, then $\tilde{a}_{X_i}$ is always a multiple of the identity, so we obtain the characterization of quantum combs in [4].

3 Combinatorial description of higher order objects

Types of higher order quantum maps were introduced in [10, 12], where a combinatorial description of the corresponding subspaces is given. More precisely, it is shown that the subspace for the corresponding type of higher order objects can be combined from some independent system of linear subspaces labelled by binary strings.

In this section we show that we can have a similar description of higher order objects in Af, though the construction is slightly more complicated, see also Remark 3.1 below. We will use boolean functions to characterize the subsets of binary strings corresponding to the type, which will turn out useful for further description of types. We will need the definitions, notations and results given in Appendix B for affine subspaces, Appendix A.4 for boolean functions and the notations given at the end of Section 2.1 for operations on subspaces of a vector space. The notation $\boxplus$ will always mean the disjoint union of subsets of natural numbers and the operadic composition of permutations, as in Appendix A.

For a first order object $X = (V_X, \{\tilde{a}_X\}^*)$, let us pick an element $a_X \in A_X$. We have a direct sum decomposition

$$V_X = L_{X,0} + L_{X,1}, \tag{9}$$

where $L_{X,0} := \mathbb{R}\{a_X\}$, $L_{X,1} := \{\tilde{a}_X\}^\perp = L_X$. We define the conjugate object as $\tilde{X} = (V_X^*, \{a_X\}^*)$. Note that we always have $\tilde{a}_X \in A_{\tilde{X}}$ and with the choice $a_{\tilde{X}} = \tilde{a}_X$, we obtain $\tilde{\tilde{X}} = X$ and

$$L_{\tilde{X},u} = L_{X,1-u}^\perp, \qquad u \in \{0,1\}. \tag{10}$$

These definitions depend on the choice of a_X, but we will assume below that this choice is fixed and that we choose $a_{\tilde{X}} = \tilde{a}_X$. Since we will always work with a finite set of objects at a time, this will not create any problems.

The situation is simplified in the case of quantum or classical objects, where we have an identification of V_X with its dual V_X^* by a standard inner product, and $\tilde{a}_X$ is a (positive) multiple of the identity by definition. The hyperplane $\{\tilde{a}_X\}^*$ contains a (necessarily unique) positive multiple of identity as well, which we then choose to be the distinguished element a_X. Any first order quantum object X can be also determined by the dimension n_X and a

positive constant c_X such that $X = (M^h_{n_X}, A_X = \{a,\ \mathrm{Tr}\,[a] = c_X\})$. In this case, we have $\tilde{a}_X = c_X^{-1} E_{n_X}$, $a_X \in n_X^{-1} c_X E_{n_X}$ and the conjugate object is the first order quantum object determined by $n_{\tilde{X}} = n_X$ and $c_{\tilde{X}} = n_X c_X^{-1}$. This gives us the quite familiar decomposition of any hermitian operator as a multiple of the identity and a traceless part:

$$L_{X,0} = L_{\tilde{X},0} = \mathbb{R}\{E_{n_X}\}, \qquad L_{X,1} = L_{\tilde{X},1} = \{E_n\}^\perp = \{a \in M^h_{n_X},\ \mathrm{Tr}\,[a] = 0\} \tag{11}$$

Similarly for classical objects.

Remark 3.1. It is also possible to use an identification $V_X \simeq V_X^*$ by fixing some inner product in V_X. The element $\tilde{a}_X$ then becomes identified with an element of V_X, and $A_X = \{\tilde{a}_X\}^*$ contains a (unique) multiple of $\tilde{a}_X$. Then both V_X and V_X^* are decomposed into $\mathrm{span}(\tilde{a}_X)$ and its orthogonal complement. But note that the choice of the inner product is quite arbitrary, and the constructions below would still depend on this choice. We could also choose another inner product in M^h_n and obtain a different decomposition. Nevertheless, the subspaces obtained for higher order objects would still remain the same, regardless of the chosen approach.

Let $X_1, \dots, X_n$ be first order objects in Af. Let $a_{X_i} \in A_{X_i}$ be fixed and let $\tilde{X}_i$ be the conjugate first order objects. Let us denote $V_i = V_{X_i}$ and

$$L_{i,u} := L_{X_i,u}, \qquad \tilde{L}_{i,u} := L_{\tilde{X}_i,u} \qquad u \in \{0,1\},\ i \in [n].$$

For a string $s \in \{0,1\}^n$, we define

$$L_s := L_{1,s_1} \otimes \cdots \otimes L_{n,s_n}, \qquad \tilde{L}_s := \tilde{L}_{1,s_1} \otimes \cdots \otimes \tilde{L}_{n,s_n},$$

then by (9) we have the direct sum decompositions

$$V := V_1 \otimes \cdots \otimes V_n = \sum_{s \in \{0,1\}^n} L_s, \qquad V^* = V_1^* \otimes \cdots \otimes V_n^* = \sum_{s \in \{0,1\}^n} \tilde{L}_s.$$

Lemma 3.1. *For any $s \in \{0,1\}^n$, we have*

$$L_s^\perp = \sum_{t \in \{0,1\}^n} (1 - \chi_s(t)) \tilde{L}_t, \qquad \tilde{L}_s^\perp = \sum_{t \in \{0,1\}^n} (1 - \chi_s(t)) L_t.$$

Here $\chi_s : \{0,1\}^n \to \{0,1\}$ is the characteristic function of s, that is, $\chi_s(t) = 1$ iff $t = s$.

Proof. Using (10) and the direct sum decomposition of V_i^*, we get

$$\begin{aligned}(L_{1,s_1} \otimes \cdots \otimes L_{n,s_n})^\perp &= \bigvee_j \left(V_1^* \otimes \cdots \otimes V_{j-1}^* \otimes \tilde{L}_{j,1-s_j} \otimes V_{j+1}^* \otimes \cdots \otimes V_n^* \right) \\ &= \bigvee_j \left(\sum_{\substack{t \in \{0,1\}^n \\ t_j \neq s_j}} \tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right) = \sum_{\substack{t \in \{0,1\}^n \\ t \neq s}} \left(\tilde{L}_{1,t_1} \otimes \cdots \otimes \tilde{L}_{n,t_n} \right).\end{aligned}$$

The proof of the other equality is the same. □

For the fixed first order objects $X_1, \dots, X_n$ and their conjugate objects $\tilde{X}_1, \dots, \tilde{X}_n$, we introduce the following definitions. Put $a := a_{X_1} \otimes \cdots \otimes a_{X_n}$, $\tilde{a} := \tilde{a}_{X_1} \otimes \cdots \otimes \tilde{a}_{X_n}$. For $f \in \mathcal{F}_n$ (see Appendix A.5) define

$$S_f = S_f(X_1, \dots, X_n) := \sum_{s \in \{0,1\}^n} f(s) L_s, \qquad A_f = A_f(X_1, \dots, X_n) := S_f \cap \{\tilde{a}\}^*. \tag{12}$$

It is clear from definition that A_f is an affine subspace. Since $f(\theta_n) = 1$, the space S_f always contains the subspace $L_0 = L_{1,0} \otimes \cdots \otimes L_{n,0} = \mathbb{R}\{a\}$ and it is clear that $L_s \subseteq \{\tilde{a}\}^\perp$ for any $s \neq \theta_n$. It follows that $a \in A_f$, so that $A_f \neq \emptyset$, and since $A_f \subseteq \{\tilde{a}\}^*$, we see that A_f is proper and $\tilde{a} \in A_f^*$. It is easy to see that we have

$$\mathrm{Lin}(A_f) = \sum_{s \in \{0,1\}^n \setminus \{\theta_n\}} f(s) L_s, \qquad \mathrm{Span}(A_f) = S_f.$$

We may now define the objects

$$X_f = X_f(X_1, \ldots, X_n) := (V, A_f(X_1, \ldots, X_n))$$

in Af.

Proposition 3.2. *Let $X_1, \ldots, X_n$ be first order objects and $\tilde{X}_1, \ldots, \tilde{X}_n$ the conjugate objects. The map $\mathcal{F}_n \ni f \mapsto X_f(X_1, \ldots, X_n) \in \mathrm{Af}$ is injective and we have the following properties.*

(i) For the least and the largest element in $\mathcal{F}_n$,

$$X_{p_n} = \tilde{X}_1^* \otimes \cdots \otimes \tilde{X}_n^* = (\tilde{X}_1 \otimes \cdots \otimes \tilde{X}_n)^*, \qquad X_{1_n} = X_1 \otimes \cdots \otimes X_n.$$

(ii) The dual object satisfies

$$X_f^*(X_1, \ldots, X_n) = X_{f^*}(\tilde{X}_1, \ldots, \tilde{X}_n).$$

(iii) Let $f_1 \in \mathcal{F}_{n_1}$, $f_2 \in \mathcal{F}_{n_2}$. For the decomposition $[n] = [n_1] \boxplus [n_2]$,

$$X_{f_1 \otimes f_2}(X_1, \ldots, X_n) = X_{f_1}(X_1, \ldots, X_{n_1}) \otimes X_{f_2}(X_{n_1+1}, \ldots, X_n).$$

(iv) For any $\sigma \in \mathscr{S}_n$ we have an isomorphism

$$X_{f \circ \sigma}(X_1, \ldots, X_n) \xrightarrow{\sigma} X_f(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)}).$$

Proof. Since L_s, $s \in \{0,1\}$ is an independent decomposition of V, the subspace S_f has a unique decomposition in terms of L_s. It follows that the map $f \mapsto A_f$, and hence also $f \mapsto X_f$ is injective. We have

$$S_{p_n} = L_{\theta_n} = \mathbb{R}\{a\} \qquad S_{1_n} = \sum_{s \in \{0,1\}^n} L_s = V,$$

Since $X_1 \otimes \cdots \otimes X_n = (V, \{\tilde{a}\}^*)$ and $\tilde{X}_1 \otimes \cdots \otimes \tilde{X}_n = (V^*, \{a\}^*)$, this proves (i). For (ii), it is enough to prove that $A_f^*(X_1, \ldots, X_n) = A_{f^*}(\tilde{X}_1, \ldots, \tilde{X}_n)$. To see this, we compute using Lemma 3.1 and the fact that the subspaces form an independent decomposition,

$$\mathrm{Span}(A_f^*) = \mathrm{Lin}(A_f)^\perp = \left(\sum_{s \in \{0,1\}^n \setminus \{0\}} f(s) L_s \right)^\perp = \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} L_s^\perp$$

$$= \bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} \left(\sum_{t \in \{0,1\}^n} (1 - \chi_s(t)) \tilde{L}_t \right)$$

$$= \sum_{t \in \{0,1\}^n} \left(\bigwedge_{\substack{s \in \{0,1\}^n \\ s \neq 0, f(s)=1}} (1 - \chi_s(t)) \tilde{L}_t \right) = \sum_{t \in \{0,1\}^n} f^*(t) \tilde{L}_t.$$

To see the last equality, note that

$$\bigwedge_{\substack{s\in\{0,1\}^n\\ s\neq 0, f(s)=1}} (1-\chi_s(t)) = \begin{cases} 1 & \text{if } t=\theta_n \\ 1-f(t) & \text{if } t\neq\theta_n \end{cases} = f^*(t).$$

The statement (iii) is easily seen from the definitions. To show (iv), compute

$$\begin{aligned}\sigma^{-1}(S_f(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)})) &= \sigma^{-1}(\sum_s f(s)L_{\sigma^{-1}(1),s_1}\otimes\cdots\otimes L_{\sigma^{-1}(n),s_n}) \\ &= \sum_s f(s)L_{1,s_{\sigma(1)}}\otimes\cdots\otimes L_{n,s_{\sigma(n)}} = S_{f\circ\sigma}(X_1,\dots,X_n).\end{aligned}$$

It follows that

$$A_f(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)}) = S_f(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)})\cap\{\sigma(\tilde{a})\}^* = \sigma(A_{f\circ\sigma}(X_1,\dots,X_n)).$$

□

With the decomposition $[n]=[n_1]\boxplus[n_2]$ and $f\in\mathcal{F}_{n_1}$, $g\in\mathcal{F}_{n_2}$, we will use the notations

$$f\to g := (f\otimes g^*)^*, \qquad f\parr g := (f^*\otimes g^*)^*. \tag{13}$$

Corollary 3.3. *With the above notations, we have*

$$X_{f\to g}(X_1,\dots,X_{n_1+n_2}) = X_f(\tilde{X}_1,\dots,\tilde{X}_{n_1})\multimap X_g(X_{n_1+1},\dots,X_n)$$

and

$$X_{f\parr g} = X_f(X_1,\dots,X_{n_1})\parr X_g(X_{n_1+1},\dots,X_n).$$

Proof. Immediate from Proposition 3.2. □

Since $\{L_s,\ s\in\{0,1\}^n\}$ is an independent decomposition of V, the subspaces S_f form a distributive sublattice in the lattice of subspaces of V and we clearly have $f\leq g$ if and only if $S_f\subseteq S_g$, and $S_{f\wedge g}=S_f\cap S_g$, $S_{f\vee g}=S_f\vee S_g$. With these operations and complementation given as $S_f\mapsto S_{f^*}$, each sequence $X_1,\dots,X_n$ of first order objects defines a representation of the boolean algebra $\mathcal{F}_n$ as a boolean lattice (i. e. a complemented distributive lattice) of subspaces of a vector space $V=V_{X_1}\otimes\cdots\otimes V_{X_n}$, with a one dimensional subspace $\mathbb{R}a$ as the bottom and the whole space as the top element.

If all the first order objects are quantum, we have an identification $V=V^*=M_N^h$ and both a and $\tilde{a}$ are positive multiples of the identity E_N. It follows that all $X_f(X_1,\dots,X_n)$ are quantum objects as well. Further, by (11) the subspaces $S_f(X_1,\dots,X_n)$ remain unchanged if some X_i are replaced with $\tilde{X}_i$, in particular, we get by (8)

$$S_{f^*}(X_1,\dots,X_n) = S_{f^*}(\tilde{X}_1,\dots,\tilde{X}_n) = S_{X_f^*(X_1,\dots,X_n)} = S_f(X_1,\dots,X_n)^\perp + \mathbb{R}\{E_N\}.$$

This shows that the complementation in $\mathcal{F}_n$ is reflected as the orthocomplementation in the sublattice of subspaces in M_N^h containing $\mathbb{R}\{E_N\}$.

Let us also note that in terms of the objects in the category Af, the lattice structure in $\{X_f(X_1\dots,X_n),\ f\in\mathcal{F}_n\}$ is obtained as follows. We have $f\leq g$ if and only if $X_f\xrightarrow{id_V}X_g$

(which means that $A_f \subseteq A_g$). It can be shown that if $k \leq f, g \leq h$, then the following diagrams are a pullback resp. pushout:

$$\begin{array}{ccc} X_{f\wedge g} & \xrightarrow{id_V} & X_f \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle id_V} \\ X_g & \xrightarrow[id_V]{} & X_h \end{array} \qquad \begin{array}{ccc} X_k & \xrightarrow{id_V} & X_f \\ {\scriptstyle id_V}\downarrow & & \downarrow{\scriptstyle id_V} \\ X_g & \xrightarrow[id_V]{} & X_{f\vee g} \end{array}$$

This holds in particular for the bottom and top elements $k = p_n$ and $h = 1_n$. In the situation that the order of the first order objects $X_1, \ldots, X_n$ is not fixed, we may replace the identity arrows above by appropriate permutations, as in Proposition 3.2 (iv).

Our main goal in this paragraph is to show that the higher order objects are precisely those of the form $Y = X_f(X_1, \ldots, X_n)$ for some choice of the first order objects $X_1, \ldots, X_n$ and a function f that belongs to a special subclass $\mathcal{T}_n \subseteq \mathcal{F}_n$. The elements of this subclass will be called the type functions, or types, and are defined as those functions in $\mathcal{F}_n$ that can be obtained by taking the constant function 1_1 in each coordinate and then repeatedly applying complementation and tensor products of such functions in any order. The set of indices for which the corresponding coordinate was subjected to taking the dual an odd number of times will be called the inputs (of f) and denoted by $I = I_f$, indices in $O = O_f := [n] \setminus I_f$ will be called outputs. The reason for this terminology will become clear later. It is easy to observe that if $f \in \mathcal{T}_n$, then $O_{f^*} = I_f$ and $I_{f^*} = O_f$. Further, for $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$, we have $O_{f_1 \otimes f_2} = O_{f_1} \boxplus O_{f_2}$ and $I_{f_1 \otimes f_2} = I_{f_1} \boxplus I_{f_2}$, see (A.3) for the definition of $\boxplus$.

We have the following description of the sets of type functions.

Proposition 3.4. *The system $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ is the smallest system such that*

1. *$\mathcal{T}_1 = \mathcal{F}_1$, $\mathcal{T}_n \subseteq \mathcal{F}_n$ for all n,*
2. *For $[n] = [n_1] \boxplus [n_2]$, $\mathcal{T}_{n_1} \otimes \mathcal{T}_{n_2} \subseteq \mathcal{T}_n$,*
3. *$\mathcal{T}_n$ is invariant under permutations: if $f \in \mathcal{T}_n$, then $f \circ \sigma \in \mathcal{T}_n$ for any $\sigma \in \mathscr{S}_n$,*
4. *$\mathcal{T}_n$ is invariant under complementation: if $f \in \mathcal{T}_n$ then $f^* \in \mathcal{T}_n$.*

Proof. It is clear by construction that any system of subsets $\{\mathcal{S}_n\}_n$ with these properties must contain the type functions and that $\{\mathcal{T}_n\}_n$ itself has these properties. □

To simplify some of the statements in the next sections, we also introduce the set $\mathcal{T}_0$ of trivial type functions on the empty string ε, containing a unique element $1_\varepsilon : \varepsilon \mapsto 1$. It is easy to see that for any $f \in \mathcal{T}_n$, we have $1_\varepsilon^* = 1_\varepsilon$ and $1_\varepsilon \otimes f = f$.

Assume that Y is a higher order object constructed from a set of distinct first order objects $Y_1, \ldots, Y_n$, $Y_i = (V_{Y_i}, \{\tilde{a}_{Y_i}\}^*)$. Let us fix elements $a_{Y_i} \in A_{Y_i}$ and construct the conjugate objects $\tilde{Y}_i$. By compactness of FinVect, we may assume (relabeling the objects if necessary) that the vector space of Y has the form

$$V_Y = V := V_1 \otimes \cdots \otimes V_n,$$

where V_i is either V_{Y_i} or $V_{Y_i}^*$, according to whether Y_i was subjected to taking duals an even or odd number of times. Similarly as for the type functions, the indices such that

the first case is true will be called the outputs and the subset of outputs in $[n]$ will be denoted by O, or O_Y, when we need to specify the object. The set $I = I_Y := [n] \setminus O_Y$ is the set of inputs. Note that although we cannot yet exclude that Y was constructed from Y_i in several different ways, the input and output spaces are always the same, fixed in the structure of V.

Theorem 3.5. *Let Y be a higher order object, constructed from first order objects $Y_1, \dots, Y_n$. For $i \in [n]$, let $X_i = Y_i$ if $i \in O_Y$ and $X_i = \tilde{Y}_i$ for $i \in I_Y$. There is a unique function $f \in \mathcal{T}_n$, with $O_f = O_Y$, such that*

$$Y = X_f = (V, A_f(X_1, \dots, X_n)).$$

Conversely, let $X_1, \dots, X_n$ be first order objects and let $f \in \mathcal{T}_n$. Then $Y = X_f$ is a higher order object with $O_Y = O_f$, with underlying first order objects $Y_1, \dots, Y_n$, where $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$.

Proof. Since the map $f \mapsto X_f$ is injective, uniqueness is clear. To show existence of this function, we will proceed by induction on n. For $n = 1$, the assertion is easily seen to be true, since in this case, we we have either $Y = Y_1$ or $Y = Y_1^*$. In the first case, $O = \{1\}$, $I = \emptyset$, $X_1 = Y_1$ and

$$S_Y = V_Y = V_1 = L_{1,0} + L_{1,1} = 1(0)L_{1,0} + 1(1)L_{1,1} = S_1(X_1),$$

so in this case $f \in \mathcal{T}_1$ is the constant 1. If $Y = Y_1^*$, we have $O = \emptyset$, $I = \{1\}$, $X_1 = \tilde{Y}_1$, and then

$$S_Y = \mathbb{R}\{\tilde{a}_{Y_1}\} = L_{1,0} = p_1(0)L_{1,0} + p_1(1)L_{1,1} = S_{p_1}(X_1),$$

so that $f = 1^* = p_1 \in \mathcal{T}_1$. It is clear that $O_f = O_Y$ in both cases.

Assume now that the assertion is true for all $m < n$. By construction, Y is either the tensor product $Y = Z_1 \otimes Z_2$, with Z_1 constructed from $Y_1, \dots, Y_m$ and Z_2 from $Y_{m+1}, \dots, Y_n$, or Y is the dual of such a product. Let us assume the first case. It is clear that $O_{Z_1} \boxplus O_{Z_2} = O_Y$, and similarly for I, so that the corresponding objects $X_1, \dots, X_m$ and $X_{m+1}, \dots, X_n$ remain the same. By the induction assumption, there are functions $f_1 \in \mathcal{T}_m$ and $f_2 \in \mathcal{T}_{n-m}$ such that $O_{f_1} = O_{Z_1}$, $O_{f_2} = O_{Z_2}$ and, by Proposition 3.2(iii),

$$Y = Z_1 \otimes Z_2 = X_{f_1}(X_1, \dots, X_m) \otimes X_{f_2}(X_{m+1}, \dots, X_n) = X_{f_1 \otimes f_2}(X_1, \dots, X_n).$$

This implies the assertion, with $f = f_1 \otimes f_2 \in \mathcal{T}_n$ and $O_f = O_{f_1} \boxplus O_{f_2} = O_Y$. To finish the proof, it is now enough to observe that if the assertion holds for Y then it also holds for Y^*. So assume that $Y = X_f(X_1, \dots, X_n)$ for some $f \in \mathcal{T}_n$, then by Proposition 3.2(ii), $Y^* = X_f^* = X_{f^*}(\tilde{X}_1, \dots, \tilde{X}_n)$. By the construction of conjugate objects, we have $\tilde{X}_i = \tilde{\tilde{Y}}_i = Y_i$ if $i \in I_Y$ and $\tilde{X}_i = \tilde{Y}_i$ if $i \in O_Y$. Since by definition and the assumption, $O_{Y^*} = I_Y = I_f = O_{f^*}$, this proves the statement.

The converse is proved by a similar induction argument, using Proposition 3.2.

□

Let us stress that in general, the objects X_f depend on the choice of the elements a_{X_i}. From the above proof, it is clear that the construction in Theorem 3.5 does not depend on the choice of the elements $a_{Y_i} \in A_{Y_i}$. Furthermore, if all the first order objects are quantum, we have $S_Y = S_f(Y_1, \dots, Y_n)$, since the space S_f is unchanged if some of the objects are replaced by conjugates. If Y_i are determined by the dimension m_i and positive constants c_i, then $\tilde{a}_i = c_i^{-1} E_{m_i}$ and $a_i = n_i^{-1} c_i E_{m_i}$. It follows that $Y = (M_N^h, A_Y)$ is a quantum object determined by the subspace $S_f(Y_1, \dots, Y_n)$ and $c = \frac{c_O}{c_I} m_I$, with $N = \Pi_i m_i$, $m_I = \Pi_{i \in I} m_i$, $c_O = \Pi_{i \in O} c_i$ and $c_I = \Pi_{ß \in I} c_i$.

4 The type functions

The aim of this section is to gain some understanding into the structure and properties of the set of type functions. We start by an important example.

Example 4.1. Let $T \subseteq [n]$. It is easily seen that the function p_T (see Example A.1 in Appendix A.4) is a type function, since we have

$$p_T(s) = \Pi_{j\in T}(1 - s_j) = \Pi_{j\in T}1^*(s_j) = (\otimes_{j\in T}1^*)(s).$$

By definition, T is the set of inputs for p_T. Let $Y_1, \ldots, Y_n$ be first order objects, $Y_i = (V_i, \{\tilde{a}_{Y_i}\}^*)$ and let $X_1, \ldots, X_n$ be first order objects as in Theorem 3.5, so that $X_i = \tilde{Y}_i$ if $i \in T$ and $X_i = Y_i$ otherwise. Let $k = |T|$ and let $\sigma \in \mathscr{S}_n$ be such that $\sigma^{-1}(T) = [k]$. Then $p_T \circ \sigma = p_k \otimes 1_{n-k}$. By Proposition 3.2, it follows that we have the isomorphism

$$X_{p_T}(X_1, \ldots, X_n) \overset{\sigma}{\simeq} X_{p_k\otimes 1_{n-k}}(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)}) = Y_T^* \otimes Y_{[n]\setminus T},$$

here $Y_T := \otimes_{j\in T}\tilde{X}_j = \otimes_{j\in T}Y_j$ and $Y_{[n]\setminus T} := \otimes_{j\in[n]\setminus T}X_j = \otimes_{j\in[n]\setminus T}Y_j$ are first order object by Lemma 2.8. In particular, $A_{Y_T^*} = \{\tilde{a}_T\}$, where $\tilde{a}_T := \otimes_{i\in T}\tilde{a}_{Y_i}$. It follows that any element in

$$A_f(X_1, \ldots, X_n) \simeq A_{Y_T^*\otimes Y_{[n]\setminus T}}$$

is of the form $\tilde{a}_T \otimes \omega$ for some state $\omega \in A_{Y_{[n]\setminus T}}$. This corresponds to a channel in $Y_T \multimap Y_{[n]\setminus T}$ that maps any state in A_{Y_T} to the fixed state ω. In this way, we can see that the type function p_T describes the type of replacement channels with set of input indices in T.

By duality, the elements of T are output indices for the conjugate function p_T^*, so this time we put $X_j = Y_j$ for $j \in T$ and $X_j = \tilde{Y}_j$ otherwise. Using Proposition 3.2 (ii) and the result for p_T, we obtain the isomorphisms

$$X_{p_T^*}(X_1, \ldots, X_n) = X_{p_T}^*(\tilde{X}_1, \ldots, \tilde{X}_n) \overset{\sigma}{\simeq} (Y_T^* \otimes Y_{[n]\setminus T})^* \overset{\rho}{\simeq} Y_{[n]\setminus T} \multimap Y_T,$$

where ρ is the transposition in $\mathscr{S}_2$. It follows that the type function $p_T^* = 1 - p_T + p_n$ corresponds to the type of channels with output indices in T.

Lemma 4.1. *Let $f \in \mathcal{T}_n$ and let $O_f = O$, $I = I_f$. Then*

$$p_I \leq f \leq p_O^*.$$

Proof. This is obviously true for $n = 1$. Indeed, in this case, $\mathcal{T}_1 = \mathcal{F}_1 = \{1_1 = p_\emptyset, 1_1^* = p_1\}$. If $f = 1_1$, then $O = [1]$, $I = \emptyset$ and

$$p_I = p_\emptyset = 1_1 = p_O^*,$$

the case $f = p_1$ is obtained by taking complements. Assume that the assertion holds for $m < n$. Let $f \in \mathcal{T}_n$ and assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$, $h \in \mathcal{T}_{n-m}$. By the assumption,

$$p_{I_g} \otimes p_{I_h} \leq g \otimes h \leq p_{O_g}^* \otimes p_{O_h}^* \leq (p_{O_g} \otimes p_{O_h})^*,$$

the last inequality follows from Lemma A.2. With the decomposition $[n] = [m][m+1, n]$, we have $O_f = O_g \boxplus O_h$, $I_f = I_g \boxplus I_h$, so that by Lemma A.3, $p_{O_f} = p_{O_g} \otimes p_{O_h}$ and similarly for p_{I_f}. Now notice that for any $f \in \mathcal{T}_n$ we have either $f \approx g \otimes h$ or $f \approx (g \otimes h)^*$ (see Appendix A.5 for the definition of $\approx$). Since the inequality is easily seen to be preserved by permutations, and reversed by duality which also switches the input and output sets, the assertion is proved.

□

Combining this with Proposition 3.2, we get the following result (cf. [1, Proposition 2]).

Corollary 4.2. *Let Y be a higher order objects constructed from first order objects $Y_1, \dots, Y_n$, $O_Y = O$, $I_Y = I$. Then there are $\sigma_1, \sigma_2 \in \mathscr{S}_n$ such that we have the morphisms*

$$(Y_I^* \otimes Y_O) \xrightarrow{\sigma_1} Y \xrightarrow{\sigma_2} (Y_I \multimap Y_O).$$

We also obtain a simple way to identify the output indices of a type function.

Proposition 4.3. *For $f \in \mathcal{T}_n$, $j \in O_f$ if and only if $f(e^j) = 1$, here $e^j = \delta_{1,j} \dots \delta_{n,j}$.*

Proof. Let $j \in O_f$, then by Lemma 4.1, $p_{I_f}(e^j) = 1 \leq f(e^j)$, so that $f(e^j) = 1$. Conversely, if $f(e^j) = 1$, then by the other inequality in lemma 4.1, $p_{O_f}(e^j) = 0$, whence $j \in O_f$. □

Example 4.2. The type functions for $n = 2$ are given as (writing $\bar{u} = 1 - u$ for $u \in \{0, 1\}$, and $s = s_1 s_2$):

$$1_2(s) = 1, \quad p_2(s) = \bar{s}_1 \bar{s}_2, \quad p_{\{1\}}(s) = \bar{s}_1, \quad p^*_{\{1\}}(s) = 1 - \bar{s}_1 + \bar{s}_1 \bar{s}_2,$$

and functions obtained from these by permutation, which gives 6 different elements. We have seen in Appendix A.5 that $\mathcal{F}_n$ has $2^{2^n - 1}$ elements, so that $\mathcal{F}_2$ has 8 elements in total. The two remaining functions are

$$g(s) = 1 - \bar{s}_1 - \bar{s}_2 + 2\bar{s}_1 \bar{s}_2, \qquad g^*(s) = \bar{s}_1 + \bar{s}_2 - \bar{s}_1 \bar{s}_2.$$

It can be checked directly from Lemma 4.1 and Proposition 4.3 that g is not a type function. Indeed, if $g \in \mathcal{T}_2$, we would have $O_g = \emptyset$, so that $p_2 \leq g \leq p^*_\emptyset = p_2$, which is obviously not the case. Clearly, also the complement $g^* \notin \mathcal{T}_2$. Notice also that $g^* = p_{\{1\}} \vee p_{\{2\}}$, so that $\mathcal{T}_2$ is not a lattice.

Since $\mathcal{F}_2$ can be identified as a sublattice in $\mathcal{F}_n$ for all $n \geq 2$ as $\mathcal{F}_2 \ni f \mapsto f \otimes 1_{n-2} \in \mathcal{F}_n$, the above example shows that $\mathcal{T}_n$, $n \geq 2$ is a subposet in the distributive lattice $\mathcal{F}_n$ but not a sublattice, so that for $f_1, f_2 \in \mathcal{T}_n$, none of $f_1 \wedge f_2$ or $f_1 \vee f_2$ has to be a type function. Nevertheless, we have by the above results that all type functions with the same output indices are contained in the interval $p_I \leq f \leq p^*_O$, which is a distributive lattice. Elements of such an interval will be called subtypes. It is easily seen that for $n = 2$ all subtypes are type functions, but it is not difficult to find a subtype for $n = 3$ which is not in $\mathcal{T}_3$ (see also Example 4.7 below). The objects corresponding to subtypes are not necessarily higher order objects, but are embedded in $Y_I \multimap Y_O$ and contain the replacement channels. If f_1 and f_2 have the same output set, then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are subtypes. By the remarks below Proposition 3.2, the corresponding objects can be obtained by pushouts resp. pullbacks of the higher order objects corresponding to f_1 and f_2. A relevant example of a subtype is shown in Section 4.2.1 (Example 4.9). An interesting non-example is given below.

Example 4.3 (Bistochastic channels). As we have seen in Example 4.2, the function $g = p^*_{\{1\}} \wedge p^*_{\{2\}}$ is not a subtype, but nevertheless it corresponds to an interesting set of channels. Indeed, $p^*_{\{1\}}$ describes the type of channels with output space labeled by 1 and input space by 2, while $p^*_{\{2\}}$ has the input and output space reversed. The function g then corresponds to their intersection, which is precisely the set of bistochastic (or unital) channels. This is not a (sub)type since the input and output spaces are not fixed. Nevertheless, using the structure of $\mathcal{F}_n$ and the construction in Theorem 3.5, one can characterize higher order

maps over bistochastic channels. It was shown in [28] that the bistochastic channels can be interpreted as bidirectional with respect to the time direction, and higher order maps over them contain operations with indefinite time direction, such as the quantum time flip.

4.1 The poset $\mathcal{P}_f$

By Theorem A.1, any boolean function has a unique expression of the form

$$f = \sum_{T\subseteq[n]} \hat{f}_T p_T,$$

where $\hat{f}$ is the Möbius transform of f. Using this, we introduce a poset related to f, which will be useful for description of the structure of f. We will need the definitions and basic results in Appendix A.2.

Let $\mathcal{P}_f$ be the subposet in the distributive lattice 2^n, of elements such that $\hat{f}_T \neq 0$. The main result of this paragraph is that any type function $f \in \mathcal{T}_n$ is fully determined by $\mathcal{P}_f$. We first need to show how some of the operations on type functions are reflected on $\mathcal{P}_f$.

Lemma 4.4. *Let $f \in \mathcal{T}_n$.*

(i) If $\sigma \in \mathscr{S}_n$, then $S \mapsto \sigma^{-1}(S)$ is an isomorphism of $\mathcal{P}_f$ onto $\mathcal{P}_{f\circ\sigma}$.

(ii) For $g \in \mathcal{T}_m$ and the decomposition $[n+m] = [n] \boxplus [m]$, we have $\mathcal{P}_{f\otimes g} \simeq \mathcal{P}_f \times \mathcal{P}_g$, with the isomorphism given by $(S,T) \mapsto S \boxplus T$ and

$$\widehat{(f\otimes g)}_{(S,T)} = \hat{f}_S \hat{g}_T.$$

Proof. The statement is proved using Lemma A.3. We have

$$f\circ\sigma = \sum_{S\subseteq[n]} \hat{f}_S p_S \circ \sigma = \sum_{S\subseteq[n]} \hat{f}_S p_{\sigma^{-1}(S)} = \sum_{S\subseteq[n]} \hat{f}_{\sigma(S)} p_S.$$

The statement (i) now follows by uniqueness of the Möbius transform. Similarly, for $s = s^1 s^2$,

$$\begin{aligned} f\otimes g(s) = f(s^1)g(s^2) &= \sum_{S\subseteq[n]}\sum_{T\subseteq[m]} \hat{f}_S\hat{g}_T p_S(s^1)p_T(s^2) = \sum_{S\subseteq[n]}\sum_{T\subseteq[m]} \hat{f}_S\hat{g}_T (p_S\otimes p_T)(s) \\ &= \sum_{S\subseteq[n]}\sum_{T\subseteq[m]} \hat{f}_S\hat{g}_T p_{S\boxplus T}(s). \end{aligned}$$

This proves (ii).

□

Theorem 4.5. *Let $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank $r(f) := r(\mathcal{P}_f) \leq n$. Moreover, we have*

$$f = \sum_{S\in\mathcal{P}_f} (-1)^{\rho_f(S)} p_S,$$

where ρ_f is the rank function of $\mathcal{P}_f$.

Proof. We will proceed by induction on n. Assume that $n = 1$. Then $2^n = \{\emptyset, [1]\}$ and $\mathcal{T}_1 = \{1_1, p_1\}$. For both type functions, $\mathcal{P}_f$ is a singleton, which is clearly a graded poset, with rank $k = 0$ and trivial rank function $\rho_f \equiv 0$. We have

$$1_1 = p_\emptyset = (-1)^{\rho(\emptyset)} p_\emptyset.$$

The statement for $f = p_1$ follows by duality.

Assume next that the statement holds for all $m < n$ and let $f \in \mathcal{T}_n$. By construction, it is enough to show that the property is invariant under permutations and complement and that it holds for any f of the form $f = f_1 \otimes f_2$ for type functions $f_1 \in \mathcal{T}_{n_1}$, $f_2 \in \mathcal{T}_{n_2}$. So assume f has the desired property and let $\sigma \in \mathscr{S}_n$. It is clear by the isomorphism in Lemma 4.4 (i) that $\mathcal{P}_{f\circ\sigma}$ is a graded poset as well, with the same even rank as f and rank function $\rho_{f\circ\sigma} = \rho_f \circ \sigma$. Then

$$f \circ \sigma = \sum_{S\subseteq[n]} (-1)^{\rho_f(S)} p_S \circ \sigma = \sum_{S\subseteq[n]} (-1)^{\rho_f\circ\sigma(S)} p_S.$$

Further, assume that we have

$$\begin{aligned} f^* = 1 - f + p_n &= (1 - \hat{f}_\emptyset) p_\emptyset - \sum_{\emptyset,[n]\neq S\subseteq[n]} \hat{f}_S p_S + (1 - \hat{f}_{[n]}) p_n \\ &= (1 - \hat{f}_\emptyset) 1 - \sum_{\substack{S\in\mathcal{P}_f \\ \emptyset\neq S,[n]\neq S}} (-1)^{\rho_f(S)} p_S + (1 - \hat{f}_{[n]}) p_n. \end{aligned} \tag{14}$$

If $\emptyset \in \mathcal{P}_f$, then $\emptyset$ is the least element of $\mathcal{P}_f$, so that $\rho_f(\emptyset) = 0$ and therefore $\hat{f}_\emptyset = (-1)^0 = 1$. Similarly, if $[n] \in \mathcal{P}_f$, then $[n]$ is the largest element in $\mathcal{P}_f$, hence it is the last element in any maximal chain. It follows that $\rho_f([n]) = r(f)$ and hence $\hat{f}_{[n]} = (-1)^{r(f)} = 1$ (since the rank $r(f)$ is even). Therefore the equality (14) implies that $\mathcal{P}_{f^*}$ differs from $\mathcal{P}_f$ only in the bottom and top elements: $\emptyset \in \mathcal{P}_f$ iff $\emptyset \notin \mathcal{P}_{f^*}$ and $[n] \in \mathcal{P}_f$ iff $[n] \notin \mathcal{P}_{f^*}$. It follows that $\mathcal{P}_{f^*}$ is graded as well, with rank equal to $r(f) - 2$, $r(f)$ or $r(f) + 2$, which in any case is even. Furthermore, this also implies that for all $S \in \mathcal{P}_f$, $S \notin \{\emptyset, [n]\}$, we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$, according to whether $\emptyset$ was added or removed. The statement now follows from (14).

Now assume that $f = f_1 \otimes f_2$. By the induction assumption, both $\mathcal{P}_{f_1}$ and $\mathcal{P}_{f_2}$ are graded posets. By Lemma 4.4, $\mathcal{P}_f \simeq \mathcal{P}_{f_1} \times \mathcal{P}_{f_2}$, so that $\mathcal{P}_f$ is a graded poset as well, with rank function $\rho_f(S,T) = \rho_{f_1}(S) + \rho_{f_2}(T)$ and rank $r(f) = r(f_1) + r(f_2)$. By Lemma 4.4 (ii), we get

$$f = \sum_{S\subseteq[n_1],T\subseteq[n_2]} (\widehat{f_1})_S (\widehat{f_2})_T p_S \otimes p_T = \sum_{S\subseteq[n_1],T\subseteq[n_2]} (-1)^{\rho_{f_1}(S)+\rho_{f_2}(T)} p_{S\boxplus T}.$$

This finishes the proof.

□

Remark 4.1. Notice that we need to assume n to be known. Indeed, for any m and f, $\mathcal{P}_f$ and $\mathcal{P}_{f\otimes 1_m}$ are the same, but the two type functions are different. In particular, the corresponding constructions of higher order objects are different. See also the examples in Section 4.2.1.

In the course of the above proof, we have also shown the following.

Corollary 4.6. *Let $f \in \mathcal{T}_n$. Then*

$$\mathcal{P}_{f^*} = \mathcal{P}_f \triangle \{\emptyset, [n]\},$$

here $\triangle$ denotes the symmetric difference of two sets: $A \triangle B = (A \cup B) \setminus (A \cap B)$.

4.2 Labelled Hasse diagrams

We introduce labels for the elements of $\mathcal{P}_f$ in the following way. For $S \in \mathcal{P}_f$, put

$$L_S := \{i \in [n] \ : \ i \in S, \ \forall S' \subsetneq S, i \notin S'\}.$$

In other words, i is a label for S if S is a minimal element in the subposet of elements containing i in $\mathcal{P}_f$. We will use the notation $L_{S,f}$ if the function f has to be specified. It is easily seen that for $S \in \mathrm{Min}(\mathcal{P}_f)$, $L_S = S$ and for any $S \in \mathcal{P}_f$, $S = \cup_{S' \subseteq S} L_{S'}$. It follows that $f \in \mathcal{T}_n$ (with known n) is fully determined by the order relation on $\mathcal{P}_f$ and the label sets. All the information about f can be therefore obtained from the labelled Hasse diagram of $\mathcal{P}_f$. Some examples of type functions and their labelled Hasse diagrams will be given in Section 4.2.1.

Let us denote

$$I_f^F := \cap_{S \in \mathrm{Min}(\mathcal{P}_f)} L_S, \qquad O_f^F := [n] \setminus \cup_{S \in \mathcal{P}_f} L_S.$$

It is easily checked by Proposition 4.3 that any $i \in O_f^F$ is an output index, since in this case we have $f(e^i) = f(\theta_n) = 1$. Such elements will be called the free outputs of f. If f has some free outputs, then necessarily $[n] \notin \mathcal{P}_f$. Similarly, any $j \in I_f^F$ is an input of f, since j must be contained in any $T \in \mathcal{P}_f$, so that $p_T(e^j) = 0$ for all $T \in \mathcal{P}_f$ and consequently $f(e^j) = 0$. Such elements will be called free inputs of f. The elements of $I_f^F \cup O_f^F$ will be called free indices of f.

It is clear that we have

$$f \approx p_k \otimes g \otimes 1_l \tag{15}$$

where $k = |I_f^F|$, $l = |O_f^F|$ and $g \in \mathcal{T}_{n-k-l}$ has no free indices. Note that we may have $k = 0$, in which case f has no free inputs and $p_k = 1_\varepsilon$ is trivial, similarly for l (see the paragraph below Proposition 3.4). As posets, $\mathcal{P}_f \simeq \mathcal{P}_g$, with labels

$$L_{S,f} = \begin{cases} \sigma(L_{S,g}), & \text{if } S \notin \mathrm{Min}(\mathcal{P}_f) \\ \sigma(L_{S,g}) \cup I_f^F, & \text{otherwise,} \end{cases}$$

for some $\sigma \in \mathscr{S}_n$. Clearly, $n - k - l$ has to be specified for g.

The two distinguished elements $\emptyset$ and $[n]$, if present in $\mathcal{P}_f$, can be easily recognized from its structure as a labelled poset. Indeed, $\emptyset \in \mathcal{P}_f$ if and only if $\mathcal{P}_f$ has the smallest element and it has an empty label. Similarly, $[n] \in \mathcal{P}_f$ if and only if $\mathcal{P}_f$ has the largest element and $\cup_{S \in \mathcal{P}_f} L_S = [n]$. The basic operations on type functions are obtained as follows.

Corollary 4.7. *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$. Then*

(i) For $\sigma \in \mathscr{S}_n$, $\mathcal{P}_{f \circ \sigma} \simeq \mathcal{P}_f$, with the labels changed as $L_S \mapsto \sigma^{-1}(L_S)$.

(ii) If $[n] \in \mathcal{P}_{f^*}$, then $L_{[n],f^*} = O_f^F$. All other elements and labels remain the same.

(iii) Assume the decomposition $[n+m] = [n] \boxplus [m]$. Then $\mathcal{P}_{f \otimes g} \simeq \mathcal{P}_f \times \mathcal{P}_g$, with label sets

$$L_{(S,T)} = \begin{cases} L_S \cup (n + L_T), & \text{if } S \in \mathrm{Min}(\mathcal{P}_f), \ T \in \mathrm{Min}(\mathcal{P}_g) \\ L_S, & \text{if } S \notin \mathrm{Min}(\mathcal{P}_f), \ T \in \mathrm{Min}(\mathcal{P}_g) \\ n + L_T, & \text{if } S \in \mathrm{Min}(\mathcal{P}_f), \ T \notin \mathrm{Min}(\mathcal{P}_g) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. We only need to prove the statements on the label sets. This is quite clear in (i). In (ii), if $[n] \in \mathcal{P}_{f^*}$, then the only new indices not appearing below $[n]$ can be the free outputs of f. In (iii), assume that $i \in L_{(S,T)}$, then $S \boxplus T$ must be a minimal element in $\mathcal{P}_{f\otimes g}$ containing i. Hence, either $i \in S$ or $i \in n+T$. In the first case, $i \in (S' \boxplus T') \leq (S \boxplus T)$ whenever $i \in S' \leq S$ and $T' \leq T$, so we must have $i \in L_S$ and $T \in \mathrm{Min}(\mathcal{P}_g)$. Similarly, for $i \in n+T$, we get $i \in n + L_T$ and $S \in \mathrm{Min}(\mathcal{P}_f)$.

□

We next show that the input and output sets of $f \in \mathcal{T}_n$ can be easily recognized from the labels in $\mathcal{P}_f$.

Proposition 4.8. *Let $f \in \mathcal{T}_n$ and $i \in [n]$. Then*

(i) All $S \in \mathcal{P}_f$ such that $i \in L_S$ have the same rank, which will be denoted by $r_f(i)$. If $i \in O_f^F$, we put $r_f(i) := r(f) + 1$.

(ii) $i \in O_f$ if and only if $r_f(i)$ is odd.

Proof. As before, we proceed by induction on n. Both assertions are quite trivial for $n = 1$, so assume the statements hold for $m < n$. It is easily seen that the properties are invariant under permutations. Assume (i) and (ii) hold for $f \in \mathcal{T}_n$ and consider f^*. If $i \in L_{[n],f^*}$, then i cannot be contained in the label set of any other element, so (i) is true. Also, by Corollary 4.7, $L_{[n],f^*} = O_f^F$, so that i is an input of f^*. Since $[n]$ is the largest element of $\mathcal{P}_f$, $\rho_f([n]) = r(f)$ is even, so that (ii) holds as well. By duality, both statements hold if $i \in O_{f^*}^F$. In all other cases, $i \in L_{S,f}$ if and only if $i \in L_{S,f^*}$, so (i) is true for f^*. By the proof of Proposition 4.5 we have $\rho_{f^*}(S) = \rho_f(S) \pm 1$ for any S, depending only on the fact whether $\emptyset \in \mathcal{P}_f$. This implies that (i) and (ii) are preserved by complementation.

It is now enough to assume that $f = g \otimes h$ for some $g \in \mathcal{T}_m$ and $h \in \mathcal{T}_{n-m}$. Suppose without loss of generality that $i \in [m]$, then $i \in L_{S\boxplus T,f}$ if and only if $i \in L_{S,g}$ and $T \in \mathrm{Min}(h)$. Since then $\rho_h(T) = 0$, we have by the induction assumption

$$\rho_f(S \boxplus T) = \rho_g(S) + \rho_h(T) = \rho_g(S) = r_g(i).$$

The statement (ii) follows from the fact that $i \in O_f$ if and only if $i \in O_g$.

□

4.2.1 Examples

In this paragraph, we will give some examples of simple type functions, their corresponding Hasse diagrams and types of higher order maps.

Example 4.4 (Free indices). Let $f \in \mathcal{T}_n$ and let I^F/O^F be the free inputs/outputs of f. Up to a permutation, we may assume that $I^F = [k]$ and $O^F = [n-l, n]$ and then $f = p_k \otimes g \otimes 1_l$ for some type function $g \in \mathcal{T}_{n-k-l}$. It follows that

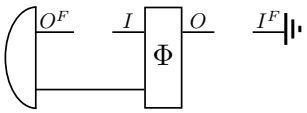
$$S_f = (\otimes_{i=1}^k \tilde{a}_i) \otimes S_g \otimes (\otimes_{i=n-l+1}^n V_l).$$

We clearly have $O_f = O^F \boxplus O$ and $I_f = I^F \boxplus I$, where $O = O_g$ and $I = I_g$.

Assume that all the basic first order objects are the usual quantum state spaces (elementary systems), so that $Y_i = (M_{n_i}^h, \{\mathrm{Tr}\,[a] = 1\})$ and $\tilde{a}_i = E_{n_i}$. Let $K = \Pi_{i=1}^k n_i$, $L = \Pi_{i=n-l+1}^n n_i$. Then we have $A_f = E_K \otimes A_g \otimes A_{\mathcal{S}_L}$, (see Example 2.3 for the notation). Any channel of type f has the Choi matrix of the form

$$C = E_K \otimes \sum_j \alpha_j C_j \otimes \rho_j = E_K \otimes D,$$

where $D = \sum_j \alpha_j C_j \otimes \rho_j$ for some channels C_j of type g and states ρ_j on M_L^h, with $\sum_j \alpha_j = 1$, so that $\mathrm{Tr}_O D = E_I \otimes \rho$, where $\mathrm{Tr}[\rho] = 1$. By the characterization of quantum combs in [4], we see that C has the form



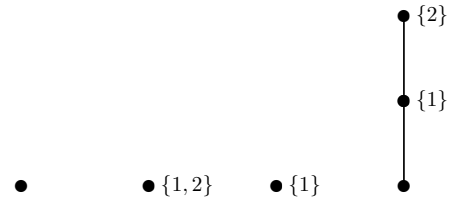
If g is the type of all channels between elementary systems $I \to O$, then clearly Φ is some channel $I \otimes R \to O$, for some ancillary system R. In general, we may always suppose that $g = (h \to k) = (h \otimes k^*)^*$ for some types h and k, and it can be proved by induction that Φ is a map of type $h \otimes 1 \to k$, where the function 1 represents the ancillary input of Φ. Using the currying equality in Lemma 2.9 (i), we obtain that $(h \otimes 1 \to k) \approx (1 \to g)$.

To describe further examples, we introduce the following notations: for $n \in \mathbb{N}$, put

$$\gamma_n := \begin{cases} \sum_{j=0}^{n} (-1)^j p_{[j]}, & \text{if } n \text{ is even} \\ \sum_{j=1}^{n} (-1)^{j-1} p_{[j]} & \text{if } n \text{ is odd} \end{cases} \tag{16}$$

(here we put $[0] = \emptyset$). Note that $\gamma_n \in \mathcal{F}_n$ and $\gamma_n^* = \gamma_{n-1} \otimes 1_1$ for $n \in \mathbb{N}$, where we put $\gamma_0 = 1_\varepsilon$. For each n, the poset $\mathcal{P}_{\gamma_n}$ is a chain of even length and we will see in Section 4.3 below that $\gamma_n \in \mathcal{T}_n$.

Example 4.5 ($\mathcal{T}_2$). The labelled Hasse diagrams of elements in $\mathcal{T}_2$ described in Example 4.2 are up to permutations as follows:



The diagram (a) corresponds to the function $1 = p_\emptyset$, so the diagram has only one element with no label. This describes first order objects $Y_1 \otimes Y_2$ (states). This type has two free outputs and no inputs. Diagram (b) corresponds to $p_2 = p_{\{1,2\}}$, which is $Y_1^* \otimes Y_2^*$ (the unit functional). Here we have two free inputs and no outputs. The function in (c) is $p_{\{1\}} = p_1 \otimes 1_1$, which describes replacement channels $Y_1 \to Y_2$. In this case we have one free input and one free output. The diagram in (d) corresponds to the function $\gamma_2 = p_{\{1\}}^*$. As we have seen in Example 4.1, this function is related to channels $Y_2 \multimap Y_1$. This type has no free indices, note that the input/output indices follow the results of Proposition 4.8.

We now depict maps of the above types in a form more familiar in quantum theory:

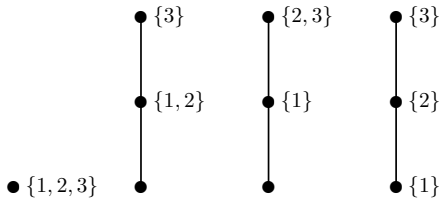


Example 4.6 ($\mathcal{T}_3$). Note that by the above example, all the posets corresponding to elements in $\mathcal{T}_2$ are chains. This is also true for $n = 3$. Indeed, up to a permutation that does not change the chain structure, any $f \in \mathcal{T}_3$ is either a product of two elements $g \in \mathcal{T}_2$ and $h \in \mathcal{T}_1$, or the dual of such a product. Since g must be a chain and $|\mathcal{P}_h| = 1$, their product must be a chain as well. Taking the dual of a chain only adds/removes the least/largest elements, so the dual of a chain must be a chain as well. All the diagrams depicted in (a)-(d) above describe functions contained in $\mathcal{T}_3$, with an additional free output 3. The corresponding maps have the form



(a) (b) (c) (d)

The only other elements of $\mathcal{T}_3$ are (up to permutations):



(a*) (b*) (c*) (d*)

Notice that in $\mathcal{T}_3$, the function with diagram in (a*) is the conjugate of the function with diagram in (a), etc. The diagram in (d*) corresponds to γ_3, which is a channel with an additional free input 1:



(a*) (b*) (c*) (d*)

Example 4.7 ($\mathcal{T}_3$ subtypes). We will next find all subtypes for $n = 3$, given by the input and output sets $I = \{1, 2\}$ and $O = \{3\}$, all the other nontrivial subtypes can be found by permutations and duals. The subtypes are given by the condition

$$p_{\{1,2\}} \leq f \leq p^*_{\{3\}}.$$

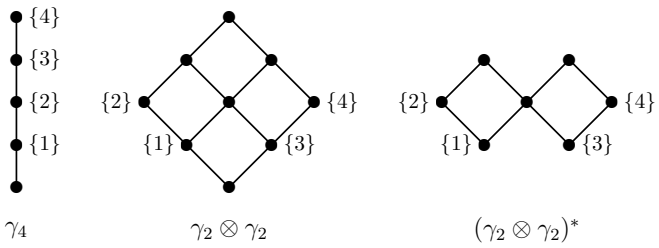
The function of the left hand side is equal to 1 on all strings s with $s_1 = s_2 = 0$; the functions on the right are equal to 0 whenever $s \neq \theta_3$ and $s_3 = 0$. This leaves us with only 3 strings where the values of f are not fixed: 101, 011 and 111, and we have $p_{\{1,2\}}(s) = 0$, $p^*_{\{3\}}(s) = 1$ for all these strings. This means that there are 8 subtypes in this case. All types in this interval are described by the chains in Examples 4.2 and 4.6. The two extremal cases correspond to the diagram (b) in Example 4.2, resp. diagram (c*) in Example 4.6 (with $\{3\}$ in the middle and $\{1, 2\}$ on the top). The only remaining types with the given input/output sets are two chains of the form (d*), with $\{3\}$ in the middle, let us denote these functions by f_1 and f_2. Including infima and suprema, we obtain the first 5 subtypes in following table (note that $f_1 \wedge f_2 = p_{\{1,2\}}$):

s	$p_{\{1,2\}}$	$p^*_{\{3\}}$	f_1	f_2	$f_1 \vee f_2$	g_1	g_2	g_3
101	0	1	1	0	1	0	0	1
011	0	1	0	1	1	0	1	0
111	0	1	0	0	0	1	1	1

Hence there are only four types in this interval, and only one additional subtype that is obtained by taking joins and meets of chains. The remaining subtypes g_1, g_2, g_3 have no clear interpretation and the corresponding subspaces S_{g_i} depend on the choice of the element $a_{Y_i} \in A_{Y_i}$, $i = 1, 2$, e.g.

$$S_{g_1} = \tilde{a}_{Y_1} \otimes \tilde{a}_{Y_2} \otimes V_{Y_3} + \{a_{Y_1}\}^\perp \otimes \{a_{Y_2}\}^\perp \otimes \{\tilde{a}_{Y_3}\}^\perp.$$

Example 4.8 ($\mathcal{T}_4$). The only elements in $\mathcal{T}_4$ such that the posets are not isomorphic to any of those given above have the diagrams:



γ_4 $\qquad$ $\gamma_2 \otimes \gamma_2$ $\qquad$ $(\gamma_2 \otimes \gamma_2)^*$

The corresponding type functions and related higher order objects are:

1. $\gamma_4(s) = 1 - \bar{s}_1 + \bar{s}_1\bar{s}_2 - \bar{s}_1\bar{s}_2\bar{s}_3 + \bar{s}_1\bar{s}_2\bar{s}_3\bar{s}_4$, related to 2-combs (superchannels), see Proposition 4.10 below,

2. $(\gamma_2 \otimes \gamma_2)(s) = 1 - \bar{s}_1 - \bar{s}_3 + \bar{s}_1\bar{s}_2 + \bar{s}_1\bar{s}_3 + \bar{s}_3\bar{s}_4 - \bar{s}_1\bar{s}_2\bar{s}_3 - \bar{s}_1\bar{s}_3\bar{s}_4 + \bar{s}_1\bar{s}_2\bar{s}_3\bar{s}_4$ related to no-signalling bipartite channels,

3. $(\gamma_2 \otimes \gamma_2)^*(s) = \bar{s}_1 + \bar{s}_3 - \bar{s}_1\bar{s}_2 - \bar{s}_1\bar{s}_3 - \bar{s}_3\bar{s}_4 + \bar{s}_1\bar{s}_2\bar{s}_3 + \bar{s}_1\bar{s}_3\bar{s}_4$ related to process matrices (Example 2.4).

Example 4.9 (Adapters). An adapter is a map that transforms process matrices to process matrices. Let $g = (\gamma_2 \otimes \gamma_2)^* \in \mathcal{T}_4$ be the type functions of process matrices, then the type function for adapters is $a = g \to g = (g \otimes g^*)^* \in \mathcal{T}_8$. The poset $\mathcal{P}_a$ is already rather big and complicated, we will show the diagram of the corresponding reduced poset in Example 4.13 (Section 4.6). In the quantum case, this type was described in [29] as the set of operations in the resource theory of causal connection. As a possible choice of free operations, the set of admissible free preserving (AFP) adapters was suggested. If we denote the set of inputs of the source/target process matrix by I/I', the free preserving adapters are those of the type $f = p_I \to p_{I'} = (p_I \otimes p^*_{I'})^*$. The AFP adapters then correspond to the subtype $a \wedge f$.

4.3 Chains and combs

We have seen that for some type functions the poset $\mathcal{P}_f$ is a chain, which is also a basic example of a graded poset. A chain in 2^n has the form $\mathcal{C} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, $S_i \subseteq [n]$. Note that the length of the chain $\mathcal{C}$ is $N - 1$. It is clear that $\mathcal{C}$ is graded with rank $N - 1$ and rank function $\rho(S_i) = i - 1$.

Proposition 4.9. *For a chain $\mathcal{C} = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_N\}$, the function*

$$f = f_{\mathcal{C}} := \sum_{i=1}^{N} (-1)^{i-1} p_{S_i}$$

is a type function if and only if N odd. In this case, we say that f is a chain type.

Proof. By Proposition 4.5, if $f \in \mathcal{T}_n$, then the rank of f must be even, so that N must be odd. We will show that the converse is true. We proceed by induction on N. For $N = 1$, we have $f = p_{S_1} \in \mathcal{T}_n$. Assume that the statement holds for all odd numbers $M < N$ and let $\mathcal{C}$ be a chain as above. It is easily checked that

$$f \approx p_{n_1} \otimes g \otimes 1_{n-n_N},$$

where $n_i := |S_i|$ and $g \in \mathcal{F}_{n_N - n_1}$ is the function for a chain $\mathcal{C}'$ in $2^{n_N - n_1}$ of the form $\mathcal{C}' := \{\emptyset \subsetneq S_2' \subsetneq \cdots \subsetneq [n_N - n_1]\}$. Since f is a type function if g is, this shows that we may assume that the chain $\mathcal{C}$ contains $\emptyset$ and $[n]$. But then

$$f = 1 + \sum_{j=2}^{N-1} (-1)^{j-1} p_{S_j} + p_n$$

and

$$f^* = 1 - f + p_n = \sum_{j=1}^{N-2} (-1)^{j-1} p_{S_{j+1}},$$

By the induction assumption $f^* \in \mathcal{T}_n$, hence also $f = f^{**} \in \mathcal{T}_n$. □

Let $f \in \mathcal{T}_n$ be a chain type and let $\mathcal{P}_f = \{S_1 \subsetneq \cdots \subsetneq S_N\}$ be the corresponding chain. There is a decomposition of $[n]$ given as

$$T_0 := S_1, \quad T_j := S_{j+1} \setminus S_j, \ j = 1, \ldots, N-1, \quad T_N := [n] \setminus S_N. \tag{17}$$

It is clear that the label sets are given as $L_{S_j} = T_{j-1}$, $j = 1, \ldots, N$ and it can be easily seen from Proposition 4.8 that

$$I_f = \bigcup_{j=0}^{(N-1)/2} T_{2j}, \qquad O_f = \bigcup_{j=0}^{(N-1)/2} T_{2j+1} \cup O_f^F, \qquad I_f^F = T_0, \qquad O_f^F = T_N \tag{18}$$

(note that N must be odd). As we have seen, $f \approx p_{n_1} \otimes g \otimes 1_{n-n_N}$ and g is a chain type with no free indices. By Proposition 3.2, we have for any collection of first order objects

$$X_f(X_1, \ldots, X_n) \overset{\sigma}{\simeq} \tilde{X}^*_{I_f^F} \otimes X_g(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n_N - n_1)}) \otimes X_{O_f^F},$$

for some $\sigma \in \mathscr{S}_n$. We will show below that chain types correspond to an important kind of higher order objects.

Proposition 4.10. *Let $f \in \mathcal{T}_n$ be a chain type with $\mathcal{P}_f = \{\emptyset = S_1 \subsetneq \cdots \subsetneq S_N = [n]\}$, with label sets $T_i = L_{S_{i+1}}$, $i = 1, \ldots, n$. Let $Y = X_f(X_1, \ldots, X_n)$ for some first order objects $X_1, \ldots, X_n$. Then for $N \geq 3$, Y is an $(N-1)/2$-comb. More precisely, let $Y_1, \ldots, Y_n$ be such that $Y_i = X_i$ for $i \in O_f$ and $Y_i = \tilde{X}_i$ for $i \in I_f$. Then*

$$Y \overset{\sigma}{\simeq} Y_{T_{N-1}} \multimap (Y_{T_{N-2}} \multimap \cdots \multimap (Y_{T_{\frac{N+1}{2}}} \multimap Y_{T_{\frac{N-1}{2}}}) \multimap \cdots \multimap Y_{T_2}) \multimap Y_{T_1}$$

where we put $Y_T = \otimes_{j \in T} Y_j$.

Proof. Let $Y_1, \ldots, Y_n$ be as assumed, then by (18),

$$Y_{T_i} = \begin{cases} \otimes_{j \in T_i} X_j, & \text{if } i \text{ is odd,} \\ \otimes_{j \in T_i} \tilde{X}_i, & \text{if } i \text{ is even.} \end{cases}$$

We will proceed by induction on N. Let $N = 3$, then $f = 1 - p_{S_2} + p_n$, and we see from Example 4.1 that $Y \overset{\sigma}{\simeq} Y_{T_2} \multimap Y_{T_1}$. Assume the assertion is true for $N-2$. As in the proof of Proposition 4.9, we see that

$$f^* = \sum_{i=1}^{N-2} (-1)^{i-1} p_{S_{i+1}} \approx p_{n_2} \otimes g \otimes 1_{n-n_{N-1}}$$

where $g \in \mathcal{T}_{n_{N-1}-n_2}$ is the chain type for a chain $\{\emptyset \subsetneq \sigma(S_3 \setminus S_2) \subsetneq \cdots \subsetneq \sigma(S_{N-1} \setminus S_2) = [n_{N-1} - n_2]\}$, for some $\sigma \in \mathscr{S}_n$ such that $\sigma(S_2) = [n_2]$ and $\sigma(T_{N-1}) = \sigma([n] \setminus S_{N-1}) = [n - n_{N-1}]$. By Proposition 3.2, we see that

$$X_f(X_1, \ldots, X_n) = X^*_{f^*}(\tilde{X}_1, \ldots, \tilde{X}_n) \overset{\sigma}{\simeq} (Y_{T_{N-1}} \otimes \tilde{X}_g \otimes Y^*_{T_1})^* = Y_{T_{N-1}} \multimap \tilde{X}_g \multimap Y_{T_1}$$

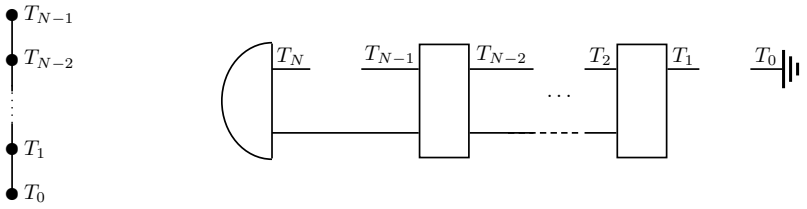
where $\tilde{X}_g = X_g(\tilde{X}_{\sigma^{-1}(1)}, \ldots, \tilde{X}_{\sigma^{-1}(n_{N-1})})$. Since g satisfies the induction assumption, and $\tilde{\tilde{X}}_i = X_i$, we obtain

$$\tilde{X}_g \overset{\sigma'}{\simeq} Y_{T_{N-2}} \multimap \cdots \multimap (Y_{T_{\frac{N+1}{2}}} \multimap Y_{T_{\frac{N-1}{2}}}) \multimap \cdots \multimap Y_{T_2},$$

for some permutation σ'. Since Y_{T_j} are first order objects, this proves the result (see Example 2.6).

□

Using the above result and Example 4.4, we find the quantum comb corresponding to a general chain of odd length N with possibly nontrivial free inputs ($T_0 = S_1 \neq \emptyset$) and free outputs ($T_N = [n] \setminus S_N \neq \emptyset$). The chain and the comb are depicted below. Note that the causal ordering of the spaces in the comb goes down the chain, so it is opposite to the order of the chain.



4.4 Connecting chains: the causal product

It is easy to see that two chains can be appended to create a single chain using the ordinal sum, and any chain of more than one elements can be decomposed as an ordinal sum of chains (see Appendix A.2 for the definitions). Such operations are trickier for chain types, since the chains have to be of even length.

To motivate our next definition, let us consider the chain type γ_6 of length 6. It can be easily seen that it can be composed of two chain types of length 2 and 4 as follows

$$\gamma_6(s) = \gamma_2(s_1 s_2) - \bar{s}_1 \bar{s}_2 + \bar{s}_1 \bar{s}_2 \gamma_4(s_3 s_4 s_5 s_6) = ((\gamma_2 - p_2) \otimes 1_2 + p_2 \otimes \gamma_4)(s^1 s^2), \tag{19}$$

where $s^1 = s_1 s_2$ and $s^2 = s_3 \dots s_6$. As we have noticed above, the causal ordering of the spaces in the corresponding comb goes down the chain, so that the part of the chain corresponding to γ_2 cannot influence the outputs in the part described by γ_4. Hence we can say that γ_2 is in the causal future of γ_4. We will next extend the expression in (19) for any pair of functions in $\mathcal{F}_n$.

For a fixed decomposition $[n] = [n_1] \boxplus [n_2]$ and functions $f_1 \in \mathcal{F}_{n_1}$, $f_2 \in \mathcal{F}_{n_2}$, we define their causal product as

$$f_1 \lhd f_2 := (f_1 - p_{n_1}) \otimes 1_{n_2} + p_{n_1} \otimes f_2.$$

For $s^1 \in \{0,1\}^{n_1}$ and $s^2 \in \{0,1\}^{n_2}$, this function acts as

$$(f_1 \lhd f_2)(s^1 s^2) = f_1(s^1) - p_{n_1}(s^1) + p_{n_1}(s^1) f_2(s^2) = \begin{cases} f_1(s^1), & \text{if } s^1 \neq \theta_{n_1}, \\ f_2(s^2), & \text{if } s^1 = \theta_{n_1}. \end{cases} \tag{20}$$

Remark 4.2. As in the case of chains, we may interpret $f_1 \lhd f_2$ as 'f_1 is in the causal future of f_2'. For first order objects $X_1, \dots, X_n$, the corresponding subspace has the form

$$S_{f_1 \lhd f_2}(X_1, \dots, X_n) = L_{f_1}(X_1, \dots, x_{n_1}) \otimes V_{n_2} + a_{n_1} \otimes S_{f_2}(X_{n_1+1}, \dots, X_n),$$

with $V_{n_2} = \otimes_{i=1}^{n_2} V_{X_{n_1+i}}$ and $a_{n_1} = \otimes_{j=1}^{n_1} a_{X_j}$. Notice that in the case of quantum objects, this corresponds to the one-way signalling composition $\succ$ defined in [16] for the corresponding orthogonal projections, i.e. denoting by P_f the projection onto S_f, we have $P_{f_1 \lhd f_2} = P_{f_1} \succ P_{f_2}$.

The following properties are immediate from (20). See also [16] for the corresponding properties of $\succ$.

Lemma 4.11. *Let $f_1, g_1 \in \mathcal{F}_{n_1}$, $f_2, g_2 \in \mathcal{F}_{n_2}$. Then $f_1 \lhd f_2 \in \mathcal{F}_{n_1+n_2}$ and we have*

(i) $(f_1 \lhd f_2)^* = f_1^* \lhd f_2^*$,

(ii) $(f_1 \vee g_1) \lhd (f_2 \vee g_2) = (f_1 \lhd f_2) \vee (g_1 \lhd g_2) = (f_1 \lhd g_2) \vee (g_1 \lhd f_2)$,

(iii) $(f_1 \wedge g_1) \lhd (f_2 \wedge g_2) = (f_1 \lhd f_2) \wedge (g_1 \lhd g_2) = (f_1 \lhd g_2) \wedge (g_1 \lhd f_2)$.

Moreover, for any $f_3 \in \mathcal{F}_{n_3}$, and for the decomposition $[n] = [n_1] \boxplus [n_2] \boxplus [n_3]$, we have

$$(f_1 \lhd f_2) \lhd f_3 = f_1 \lhd (f_2 \lhd f_3).$$

We can also combine f_1 and f_2 in the opposite order:

$$f_2 \lhd f_1 := 1_{n_1} \otimes (f_2 - p_{n_2}) + f_1 \otimes p_{n_2},$$

so that

$$(f_2 \lhd f_1)(s^1 s^2) = f_2(s^2) - p_{n_2}(s^2) + p_{n_2}(s^2) f_1(s^1) = \begin{cases} f_2(s^2), & \text{if } s^2 \neq \theta_{n_2}, \\ f_1(s^1), & \text{if } s^2 = \theta_{n_2}. \end{cases} \tag{21}$$

Of course, this product has similar properties as listed in the above lemma. To avoid any confusion, we have to bear in mind the fixed decomposition $[n] = [n_1] \boxplus [n_2]$ and that for the concatenation $s = s^1 s^2$, f_i acts on s^i.

Lemma 4.12. *In the situation as above, we have*

$$f_1 \otimes f_2 = (f_1 \lhd f_2) \wedge (f_2 \lhd f_1), \qquad f_1 \parr f_2 = (f_1 \lhd f_2) \vee (f_2 \lhd f_1).$$

Proof. The first equality follows again by straightforward computation from (20) and (21): let $s^1 \in \{0,1\}^{n_1}$, $s^2 \in \{0,1\}^{n_2}$ and compute

$$\begin{aligned}(f_1 \lhd f_2) \wedge (f_2 \lhd f_1)(s^1 s^2) &= \Big(f_1(s^1) + p_{n_1}(s^1)(f_2(s^2) - 1)\Big)\Big(f_2(s^2) + p_{n_2}(s^2)(f_1(s^1) - 1)\Big) \\ &= f_1(s^1) f_2(s^2),\end{aligned}$$

here the last equality follows from the fact that $f_i(s^i)(1 - f_i(s^i)) = 0$ (since $f_i(s^i) \in \{0,1\}$) and the fact that p_{n_1} is the least element in $\mathcal{F}_{n_1}$, so that $p_{n_1}(s^1)(f_1(s^1) - 1) = p_{n_1}(s^1) - p_{n_1}(s^1) = 0$.

The second equality in the statement follows easily from the first, using the definition of $\parr$ in (13), Lemma 4.11 (i) and the de Morgan laws in $\mathcal{F}_n$. □

Using the last part of Lemma 4.11, for a decomposition $[n] = \boxplus_i [n_i]$ and $f_i \in \mathcal{F}_{n_i}$, we may define the function $f_1 \lhd \ldots \lhd f_k \in \mathcal{F}_n$. Note that we have for $s = s^1 \ldots s^k$,

$$\begin{aligned}(f_1 \lhd \ldots \lhd f_k)(s) &= (f_1 - p_{n_1})(s^1) + p_{n_1}(s^1)(f_2 - p_{n_2})(s^2) + \cdots + p_{n-n_k}(s^1 \ldots s^{k-1}) f_k(s^k) \\ &= \begin{cases} f_1(s_1) & \text{if } s^1 \neq \theta_{n_1} \\ f_2(s^2) & \text{if } s^1 = \theta_{n_1}, s^2 \neq \theta_{n_2} \\ \ldots & \\ f_k(s^k) & \text{if } s^1 = \theta_{n_1}, \ldots, s^{k-1} = \theta_{n_{k-1}}. \end{cases}\end{aligned}$$

For any permutation $\pi \in \mathscr{S}_k$, we define $f_{\pi^{-1}(1)} \lhd \ldots \lhd f_{\pi^{-1}(k)} \in \mathcal{F}_n$ in an obvious way.

For the smallest and the largest element in $\mathcal{F}_n$, the causal product behaves as follows.

Lemma 4.13. *Let $f \in \mathcal{F}_{n_1}$ and let $n_2 \in \mathbb{N}$. Then for the decomposition $[n_1 + n_2] = [n_1] \boxplus [n_2]$,*

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \leq 1_{n_2} \lhd f \approx (f^* \to 1_{n_2})$$

and

$$p_{n_2} \lhd f = f \otimes p_{n_2} \leq f \lhd p_{n_2} \approx (1_{n_2} \to f).$$

In particular,

$$1_{n_1} \lhd p_{n_2} \approx 1_{n_2} \to 1_{n_1} = 1 - p_{[n_1]} + p_{n_1+n_2}$$

is the chain type for $\{\emptyset \subsetneq [n_1] \subsetneq [n_1 + n_2]\}$. Similar properties hold for the decomposition $[n_1 + n_2] = [n_2] \boxplus [n_1]$.

Proof. Immediate from the definition of the causal product, Lemma 4.11 (i) and Lemma 4.12.

□

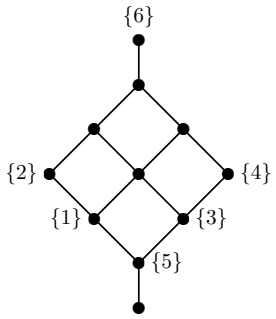
Note that if $g = 1_k \lhd f$, the elements in the set $[k]$ are output indices of g that are in the causal future of f, so that we may interpret $1_k \lhd f$ as adding a global future to the maps represented by f. Similarly, $f \lhd p_k$ may be interpreted as adding a global past. Using Lemma 4.13 and currying (Lemma 2.9 (i)), we obtain

$$1_k \lhd f \lhd p_l \approx 1_l \to (1_k \lhd f) \approx 1_l \to (f^* \to 1_k) \approx f^* \to (1_l \to 1_k).$$

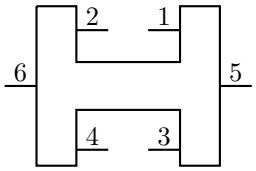
Example 4.10. Let $f \in \mathcal{T}_4$ be the process matrix type, $f = (\gamma_2 \otimes \gamma_2)^*$ (see Example 4.8, part 3.). By the above computation, we obtain

$$1_1 \lhd (\gamma_2 \otimes \gamma_2)^* \lhd p_1 \approx (\gamma_2 \otimes \gamma_2) \to \gamma_2,$$

which describes process matrices with global past and future. In the diagrams below, $(\gamma_2 \otimes \gamma_2)^*$ acts on $s_1 \ldots s_4$, 1_1 acts on s_5 and p_1 acts on s_6:



$1_1 \lhd (\gamma_2 \otimes \gamma_2)^* \lhd p_1$



Process matrix with global past and future

Remark 4.3 (Type extensions). Observe that the causal product $1_1 \lhd f$ exactly matches the inductive definition of an extension of the type f by an elementary type, introduced in [12, Def. 2]. Indeed, the elementary types of [12] correspond to the (quantum) first order objects, which are obtained from the type function 1_1. If $f = 1_1$, then $1_1 \lhd f = 1_1 \otimes 1_1 = 1_2$ is the tensor product of two elementary types, and if $f \approx g \to h$ for some type functions g and h, then we have

$$1_1 \lhd f \approx 1_1 \lhd (g \to h) \approx (g \otimes h^*) \to 1_1 \approx g \to (h^* \to 1_1) \approx g \to (1_1 \lhd h),$$

where we used Lemma 4.13 and currying. Similarly, $p_1 \lhd p_1 = p_2$ and

$$f \lhd p_1 \approx (g \to h) \lhd p_1 \approx 1_1 \to (g \to h) \approx g \to (1_1 \to h) \approx g \to (h \lhd p_1),$$

so $f \lhd p_1$ can be understood as a dual extension. We can now prove that the higher order maps are complete, in the following sense: For any $f \in \mathcal{T}_n$ and $g \in \mathcal{T}_k$, any collections of first order objects $X_0, X_1, \ldots, X_n, X_{n+1}$ and $Y_1, \ldots, Y_k$, and any morphism $X_f(X_1, \ldots, X_n) \xrightarrow{\Phi} X_g(Y_1, \ldots, Y_n)$, we have

$$X_{1_1 \lhd f \lhd p_1}(X_0, \ldots, X_{n+1}) \xrightarrow{id \otimes \Phi \otimes id} X_{1_1 \lhd g \lhd p_1}(X_0, Y_1 \ldots, Y_k, X_{n+1}) \tag{22}$$

Indeed, note that by Lemmas 4.11 (i) and 4.13, we have $(1_1 \lhd f \lhd p_1)^* = p_1 \otimes f^* \otimes 1_1$. Taking adjoints and applying Proposition 3.2, (22) is equivalent to

$$\tilde{X}_0^* \otimes X_{g^*}(\tilde{Y}_1, \ldots, \tilde{Y}_k) \otimes \tilde{X}_{n+1} \xrightarrow{id \otimes \Phi^* \otimes id} \tilde{X}_0^* \otimes X_{f^*}(\tilde{X}_1, \ldots, \tilde{X}_k) \otimes \tilde{X}_{n+1},$$

which is quite easy to see.

We will next show that the causal product is related to the ordinal sum $\star$ of the corresponding posets.

Proposition 4.14. *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$ and consider the decomposition $[n+m] = [n] \boxplus [m]$. Replace the labels of $\mathcal{P}_g$ by their translations $L_S \mapsto n + L_S = \{n + i,\ i \in L_S\}$. Then*

(a) If $[n] \in \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star (\mathcal{P}_g \setminus \{\emptyset\})$, with all labels remaining the same.

(b) If $[n] \in \mathcal{P}_f$ and $\emptyset \notin \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = (\mathcal{P} \setminus \{[n]\}) \star \mathcal{P}_g$, where the labels of $[n]$ are added to the labels of elements in $\mathrm{Min}(\mathcal{P}_g)$.

(c) If $[n] \notin \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$, then $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star (\mathcal{P}_g \setminus \{\emptyset\})$, where the free outputs of f are added to the label sets of elements in $\mathrm{Min}(\mathcal{P}_g \setminus \emptyset)$.

(d) *If* $[n] \notin \mathcal{P}_f$ *and* $\emptyset \notin \mathcal{P}_g$, *then* $\mathcal{P}_{f \lhd g} = \mathcal{P}_f \star \{\bullet\} \star \mathcal{P}_g$, *where* $\{\bullet\}$ *is a one-element poset with label* $L_\bullet = O_f^F$.

Proof. By definition of the causal product, we have

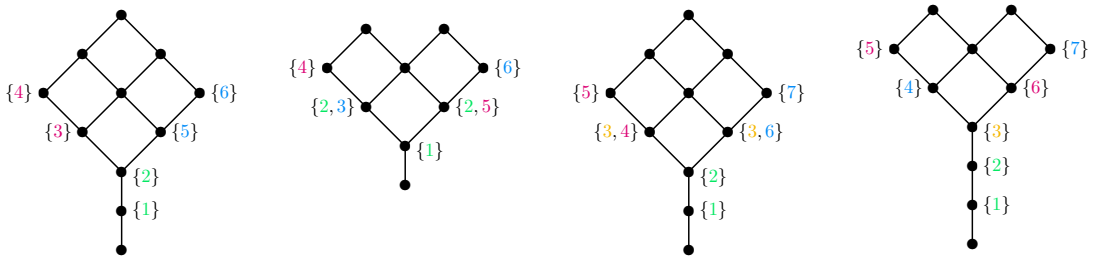
$$f \lhd g = \sum_{S \in \mathcal{P}_f \setminus \{[n]\}} \hat{f}_S p_S + (\hat{f}_{[n]} - 1 + \hat{g}_\emptyset) p_{[n]} + \sum_{T \in \mathcal{P}_g \setminus \{\emptyset\}} \hat{g}_T p_{[n] \boxplus T}.$$

The term in brackets can be equal to 1, -1, or 0, depending on whether $[n] \in \mathcal{P}_f$ and $\emptyset \in \mathcal{P}_g$. The statement is now immediate.

□

Example 4.11. The following Hasse diagrams show some examples of the causal products in cases (a)-(d) of Proposition 4.14. All type functions are of the form $\beta \lhd f$ for a chain type β. By Proposition 4.15 below, all the results are type functions. In the diagrams, the indices belonging to β are in green and yellow color. The color of the other indices indicate the signalling relations.

(a) (b) (c) (d)



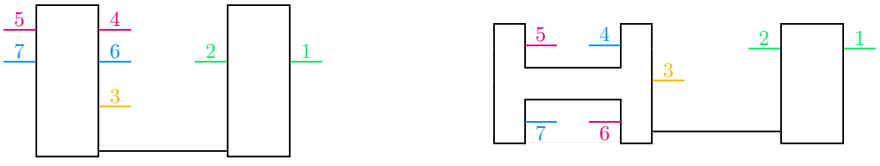
$\gamma_2 \lhd (\gamma_2 \otimes \gamma_2)$ $\gamma_2 \lhd (\gamma_2 \otimes \gamma_2)^*$ $\gamma_3^* \lhd (\gamma_2 \otimes \gamma_2)$ $\gamma_3^* \lhd (\gamma_2 \otimes \gamma_2)^*$

To identify maps of these types, note that from $f \leq p_O^*$ with $O = O_f$, we get $\beta \lhd f \leq \beta \lhd p_O^*$ for any chain β. Since $\beta \lhd p_O^*$ is again a chain type, it follows that any map of type $\beta \lhd f$ is essentially a comb, where the first tooth satisfies additional constraints given by the type f and β. In example (a), $\beta = \gamma_2$ and f is the type of bipartite no-signalling channels, hence the corresponding map is a 2-comb as depicted below, where the inputs can only signal to outputs of the same color or green. We can similarly find the maps in the case (b), here the first tooth is a process matrix, with colors indicating the signalling restrictions:



(a) (b)

In the cases (c) and (d), $\beta = \gamma_3^* = \gamma_2 \otimes 1_1 = \gamma_2 \lhd 1_1$ (Example 4.6 (d) and Lemma 4.13) and $\beta \lhd f = (\gamma_2 \lhd 1_1) \lhd f = \gamma_2 \lhd (1_1 \lhd f)$. It follows that the corresponding maps have a similar form as in (a) resp. (b), but the first tooth has an additional global output (indexed by 3). Here the pink and blue inputs can also signal to the yellow output:



(c) (d)

It can be seen from the proof of Proposition 4.14 that if both f and g are chain types with chains of N and M elements, respectively, then $f \lhd g$ is a chain type for a chain with $M+N\pm 1$ elements. Note also that that this construction can be interpreted as appending the two chains in the respective order. It is not clear that if f and g are arbitrary type functions, then $f \lhd g$ is a type function as well, but using the inequalities in Lemma 4.4 and 4.12, we have

$$p_{I_f} \otimes p_{I_g} \leq f \otimes g \leq f \lhd g \leq (f^* \otimes g^*)^* \leq (p_{O_f} \otimes p_{O_g})^*,$$

so that $f \lhd g$ is a subtype, with inputs in $I_f \boxplus I_g$ and outputs in $O_f \boxplus O_g$.

Our next result shows that if f or g is a chain type, we always obtain a type function.

Proposition 4.15. *Let $f \in \mathcal{T}_{n_1}$ and let $\beta \in \mathcal{T}_{n_2}$ be a chain type. Then both $f \lhd \beta$ and $\beta \lhd f$ are types, with outputs $O = O_f \boxplus O_\beta$ and inputs $I = I_f \boxplus I_\beta$.*

Proof. Let $\beta = \sum_{k=1}^{N} (-1)^{k-1} p_{S_k}$ for some odd N and $S_1 \subsetneq \cdots \subsetneq S_N \subseteq [n_2]$. We will proceed by induction on N. Suppose $N = 1$. If $S_1 = \emptyset$, then $\beta = 1_{n_2}$ and we have by Lemma 4.13

$$f \lhd 1_{n_2} = f \otimes 1_{n_2} \in \mathcal{T}_{n_1+n_2}$$

and

$$1_{n_2} \lhd f = (p_{n_2} \lhd f^*)^* = (f \otimes p_{n_2})^* \in \mathcal{T}_{n_1+n_2}.$$

Assume that $S_1 = [n_2]$, then $\beta = p_{n_2}$ and the assertion follows by duality. If $\emptyset \neq S_1 \subsetneq [n_2]$, then we have $\beta \approx p_{m_1} \otimes 1_{m_2} = p_{m_1} \lhd 1_{m_2}$ for $m_1 = |S_1|$, $m_1 + m_2 = n_2$. Then

$$\beta \lhd f \approx p_{m_1} \lhd (1_{m_2} \lhd f) \in \mathcal{T}_{n_1+n_2}, \quad f \lhd \beta \approx (f \lhd p_{m_1}) \lhd 1_{m_2} \in \mathcal{T}_{n_1+n_2},$$

by the first part of the proof and Lemma 4.11.

Assume next that the assertion holds for all odd numbers $M < N$. Using Proposition 4.14 (d), we see that $\beta \approx \beta_1 \lhd \beta_2$, where β_1 is an $N-2$-element chain type and β_2 is a one-element chain type. Then

$$\beta \lhd f \approx \beta_1 \lhd (\beta_2 \lhd f), \qquad f \lhd \beta \approx (f \lhd \beta_1) \lhd \beta_2$$

are type functions, by the induction assumption.

To prove the statement on the output and input indices, note that for any $i \in [n_1] \boxplus [n_2]$, we have $e^i_{n_1+n_2} = e^j_{n_1} \theta_{n_2}$ or $e^i_{n_1+n_2} = \theta_{n_1} e^k_{n_2}$ for some $j \in [n_1]$, $k \in [n_2]$. Then

$$f \lhd \beta(e^i) = f(e^j_{n_1}) \;\text{ or }\; f \lhd \beta(e^i) = \beta(e^k_{n_2}).$$

The statement on input/output indices follow from Lemma 4.1. The proof for $\beta \lhd f$ is similar.

□

4.5 The structure of type functions

The following structure theorem for type functions is one of the main results of the paper. It shows that any type function f is contained in the lattice generated by chains with the same input/output indices as f. This is quite similar to the 'normal form' of [16], which is defined in the quantum case and uses the formalism of projections and their algebraic properties. Moreover, we show that these chains are concatenations of some basic chains in different orders.

Theorem 4.16. *Let $f \in \mathcal{T}_n$. Then there is a decomposition $[n] = \boxplus_{i=1}^{k}[n_i]$, chain types $\beta_1 \in \mathcal{T}_{n_1}, \ldots, \beta_k \in \mathcal{T}_{n_k}$ such that $O_f = \boxplus_j O_{\beta_j}$, $I_f = \boxplus_j I_{\beta_j}$, finite index sets A, B and permutations $\pi_{a,b} \in \mathscr{S}_k$, $a \in A$, $b \in B$ such that*

$$f \approx \bigvee_{a\in A}\bigwedge_{b\in B}(\beta_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta_{\pi_{a,b}^{-1}(k)}) = \bigwedge_{b\in B}\bigvee_{a\in A}(\beta_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta_{\pi_{a,b}^{-1}(k)}).$$

Proof. We will once again proceed by induction on n. Since any element in $\mathcal{T}_n$ for $n \leq 3$ is a chain type, the statement clearly holds in this case. Assume the condition holds for all $m < n$. The condition is obviously invariant under permutations. Assume f can be written in the given form, then

$$f^* \approx \bigwedge_{a\in A}\bigvee_{b\in B}(\beta^*_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta^*_{\pi_{a,b}^{-1}(k)}) = \bigvee_{b\in B}\bigwedge_{a\in A}(\beta^*_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta^*_{\pi_{a,b}^{-1}(k)}).$$

Since β_j^* is a chain type for each j, this proves the statement for f^*.

It is now enough to show this form for $f = f_1 \otimes f_2$, where $f_1 \in \mathcal{T}_m$, $f_2 \in \mathcal{T}_{n-m}$ with $[n] = [m] \boxplus [n-m]$. By the induction assumption, f_1 and f_2 satisfy the conditions, so that

$$\begin{aligned} f_1 &\approx \bigvee_{a\in A}\bigwedge_{b\in B}(\beta^1_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta^1_{\pi_{a,b}^{-1}(k_1)}) = \bigwedge_{b\in B}\bigvee_{a\in A}(\beta^1_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta^1_{\pi_{a,b}^{-1}(k_1)}), \\ f_2 &\approx \bigvee_{c\in C}\bigwedge_{d\in D}(\beta^2_{\tau_{c,d}^{-1}(1)} \lhd \ldots \lhd \beta^2_{\tau_{c,d}^{-1}(k_2)}) = \bigwedge_{d\in D}\bigvee_{c\in C}(\beta^2_{\tau_{c,d}^{-1}(1)} \lhd \ldots \lhd \beta^2_{\tau_{c,d}^{-1}(k_2)}) \end{aligned}$$

for some chain types $\beta_j^1 \in \mathcal{T}_{m_j}$, $[m] = \boxplus_{j=1}^{k_1}[m_j]$, and $\beta_j^2 \in \mathcal{T}_{l_j}$, $[n-m] = \boxplus_{j=1}^{k_2}[l_j]$ and permutations $\pi_{a,b} \in \mathscr{S}_{k_1}$, $\tau_{c,d} \in \mathscr{S}_{k_2}$. Let

$$\beta_1^{a,b} := \beta^1_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta^1_{\pi_{a,b}^{-1}(k_1)}, \qquad \beta_2^{c,d} := \beta^2_{\tau_{c,d}^{-1}(1)} \lhd \ldots \lhd \beta^2_{\tau_{c,d}^{-1}(k_2)}.$$

Using the properties of the tensor product (Lemma A.2)ii), we get from Lemma 4.12

$$f \approx (\bigvee_{a\in A}\bigwedge_{b\in B}\beta_1^{a,b}) \otimes (\bigvee_{c\in C}\bigwedge_{d\in D}\beta_2^{c,d}) = \bigvee_{a,c}\bigwedge_{b,d}(\beta_1^{a,b} \otimes \beta_2^{c,d}) = \bigvee_{a,c}\bigwedge_{b,d}(\beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\beta_2^{c,d} \lhd \beta_1^{a,b})$$

On the other hand, using Lemma 4.12 and Lemma 4.11, we get

$$\begin{aligned} f &\approx (\bigwedge_{b\in B}\bigvee_{a\in A}\beta_1^{a,b}) \otimes (\bigwedge_{d\in D}\bigvee_{c\in C}\beta_2^{c,d}) \\ &= \left[(\bigwedge_{b\in B}\bigvee_{a\in A}\beta_1^{a,b}) \lhd (\bigwedge_{d\in D}\bigvee_{c\in C}\beta_2^{c,d})\right] \wedge \left[(\bigwedge_{d\in D}\bigvee_{c\in C}\beta_2^{c,d}) \lhd (\bigwedge_{b\in B}\bigvee_{a\in A}\beta_1^{a,b})\right] \\ &= (\bigwedge_{b,d}\bigvee_{a,c}\beta_1^{a,b} \lhd \beta_2^{c,d}) \wedge (\bigwedge_{b,d}\bigvee_{a,c}\beta_2^{c,d} \lhd \beta_1^{a,b}). \end{aligned}$$

We have the decomposition $[n] = \boxplus_{j=1}^{k}[n_j]$, with $k = k_1 + k_2$ and $n_j = m_j$, $j = 1, \ldots, k_1$, $n_j = l_{j-k_1}$, $j = k_1+1, \ldots, k$, and chain types $\beta_j \in \mathcal{T}_{n_j}$, $\beta_j = \beta_j^1$ for $j = 1, \ldots, k_1$ and $\beta_j = \beta^2_{j-k_1}$ for $j = k_1+1, \ldots, k$. To get the permutation sets, let $A' = A \times C$, $B' = B \times D \times \mathscr{S}_2$ and define $\pi_{a',b'}$ in $\mathscr{S}_k$ as the block permutation with respect to the decomposition $[k] = [k_1] \boxplus [k_2]$ (see Appendix A.1)

$$\pi_{(a,c),(b,d,\lambda)} = \rho_\lambda \circ (\pi_{a,b} \boxplus \tau_{c,d}).$$

This finishes the proof.

□

Remark 4.4. Note that the fact that the minima and maxima in the above theorem can be exchanged is by no means automatic and follows from the properties of the causal product $\lhd$ and the structure of type functions.

Remark 4.5. In general, it is not clear for which sets of permutations such a combination of chain types will be a chain type. Nevertheless, since all the connected chains have the same input and output indices, a function of the form as in Theorem 4.16 will always be a subtype. That is, the objects corresponding to such a function will describe a set of channels, obtained by taking pullbacks and pushouts of objects describing combs. On the other hand, Example 4.7 shows that not all subtypes are of this form.

Remark 4.6. Let us note that the set of basic chains $\beta_1, \dots, \beta_k$ in Theorem 4.16 is not given uniquely. We will consider a way to obtain some choice of this set in the next section.

4.6 The reduced poset $\mathcal{P}_f^0$

In this paragraph we will be concerned with the problem of obtaining the decomposition in the structure theorem 4.16 for a type function f. To this end, we introduce the reduced poset of f, denoted by $\mathcal{P}_f^0$, defined as the subposet in $\mathcal{P}_f$, consisting of the elements with nonempty labels and $\emptyset$ if it is contained in $\mathcal{P}_f$. We will show that any $f \in \mathcal{T}_n$ is fully determined by $\mathcal{P}_f^0$ (and n). This is convenient, because $\mathcal{P}_f^0$ is much smaller and easier to visualise that $\mathcal{P}_f$. More importantly, from $\mathcal{P}_f^0$, one can find a decomposition of f as causal products tensor products and complements of other functions. In particular, it is possible to obtain from $\mathcal{P}_f^0$ some choice of the chain types $\beta_1, \dots, \beta_k$ in the structure decomposition of f.

We start by some basic properties and examples of $\mathcal{P}_f^0$. Some further properties, and more technical parts of the proofs, can be found in Appendix C.

Recall from Section 4.2 that for all $T \in \mathcal{P}_f$, we have $T = \cup\{L_{T'},\ T' \in \mathcal{P}_f^0, T' \subseteq T\}$.

Lemma 4.17. *Let $f \in \mathcal{T}_n$.*

(i) $\mathrm{Min}(\mathcal{P}_f^0) = \mathrm{Min}(\mathcal{P}_f)$.

(ii) $\mathcal{P}_f^0$ *is a chain* $\iff$ $\mathcal{P}_f$ *is a chain* $\iff$ $\mathcal{P}_f^0 = \mathcal{P}_f$.

(iii) If $\mathcal{P}_f^0$ has a largest element, then it is the largest element in $\mathcal{P}_f$. In this case f or f^* has a free output.

Proof. (i) Obvious. For (ii), assume that $\mathcal{P}_f^0$ is a chain and let $S, T \in \mathcal{P}_f$. Let $i \in S \setminus T$ and $j \in T$, and let $S' \subseteq S$ and $T' \subseteq T$ be such that $S', T' \in \mathcal{P}_f^0$ and $i \in L_{S'}$, $j \in L_{T'}$. Since $\mathcal{P}_f^0$ is a chain, S' and T' are comparable. If $S' \subseteq T'$, then $S' \subseteq T' \subseteq T$, so that $i \in T$, which is not possible. Hence $T' \subseteq S' \subseteq S$, for all $T' \in \mathcal{P}_f^0$, $T' \subseteq T$. Hence $T \subseteq S$, and $\mathcal{P}_f$ is a chain. It is clear that then $\mathcal{P}_f^0 = \mathcal{P}_f$.

If $\mathcal{P}_f$ is not a chain, then there are some type functions f_1, f_2 such that $f = f_1 \otimes f_2$ or $f = (f_1 \otimes f_2)^*$. Moreover, the ranks of f_1 and f_2 are at least 2. It follows that both $\mathcal{P}_{f_1 \otimes f_2}$ and $\mathcal{P}_{(f_1 \otimes f_2)^*}$ contain an element $S \boxplus T$, where $S \in \mathcal{P}_{f_1}$, $T \in \mathcal{P}_{f_2}$ but none of the two elements is minimal. Then there is some $S' \in \mathcal{P}_{f_1}$ and $T' \in \mathcal{P}_{f_2}$ such that $S' \boxplus T$, $S \boxplus T' \subseteq S \boxplus T$, so that no element of $S \boxplus T$ is a label. Hence $S \boxplus T \notin \mathcal{P}_f^0$, so that $\mathcal{P}_f \neq \mathcal{P}_f^0$.

To prove (iii) let T be the largest element in P_f^0. Then

$$\cup \mathcal{P}_f = \cup L_S \subseteq T \subseteq \cup \mathcal{P}_f.$$

If follows that $T = \cup\mathcal{P}_f$ is the largest element in $\mathcal{P}_f$. If $T \neq [n]$, then clearly, f has some free outputs. If $T = [n]$, then since $[n] \in \mathcal{P}_f^0$, we have $\emptyset \neq L_{[n],f} = O_{f^*}^F$, so that f^* has free outputs.

□

The basic constructions on $\mathcal{P}_f^0$ can be easily obtained from those on $\mathcal{P}_f$.

Lemma 4.18. *Let $f \in \mathcal{T}_n$, $g \in \mathcal{T}_m$.*

(i) $\mathcal{P}_f^0$ and $\mathcal{P}_{f^*}^0$ are related as follows:

- *If $S \notin \{\emptyset, [n]\}$, then $S \in \mathcal{P}_f^0 \iff S \in \mathcal{P}_{f^*}^0$.*
- *$\emptyset \in \mathcal{P}_f^0 \iff \emptyset \notin \mathcal{P}_{f^*}^0$*
- *$[n] \in \mathcal{P}_f^0 \iff f^*$ has free outputs $\implies [n] \notin \mathcal{P}_{f^*}^0$.*

In particular, if both f and f^* have no free outputs, then $\mathcal{P}_{f^*}^0 = \mathcal{P}_f^0 \triangle \{\emptyset\}$.

(ii) $\mathcal{P}_{f\otimes g}^0 \simeq \{(S,T) : S \in \mathcal{P}_f^0,\ T \in \mathcal{P}_g^0,\ S \in \mathrm{Min}(\mathcal{P}_f) \text{ or } T \in \mathrm{Min}(\mathcal{P}_g)\}$, with ordering given as $(S,T) \leq (S',T')$ if and only if $S \leq S'$, $T \leq T'$, and with labels as in Corollary 4.7(iii).

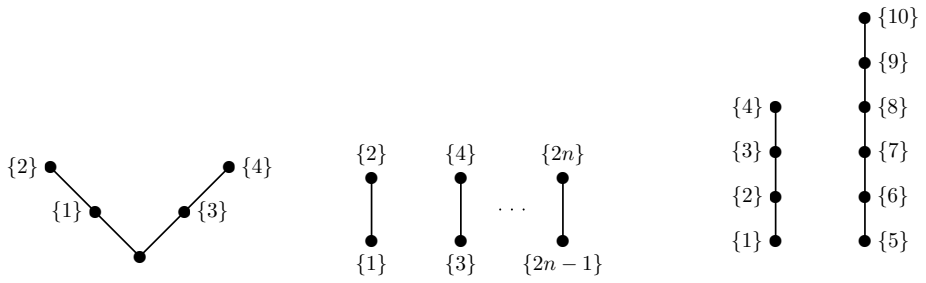
(iii) The causal product $f \lhd g$ is connected with the ordinal sum $\star$ of posets as follows. Replace the labels of $\mathcal{P}_g^0$ by their translations $L_S \mapsto n + L_S = \{n + i,\ i \in L_S\}$.

- *If $\emptyset \in \mathcal{P}_g^0$, then*
$$\mathcal{P}_{f\lhd g}^0 = \mathcal{P}_f^0 \star (\mathcal{P}_g^0 \setminus \{\emptyset\})$$
with free outputs of f added to all labels of elements covering $\emptyset$ in $\mathcal{P}_g^0$.
- *If $\emptyset \notin \mathcal{P}_g^0$ and f has no free outputs, then*
$$\mathcal{P}_{f\lhd g}^0 = (\mathcal{P}_f^0 \setminus \{[n]\}) \star \mathcal{P}_g^0$$
with labels of $[n]$ added to labels of all elements in $\mathrm{Min}(\mathcal{P}_g^0)$.
- *If $\emptyset \notin \mathcal{P}_g^0$ and f has free outputs, then*
$$\mathcal{P}_{f\lhd g}^0 = \mathcal{P}_f^0 \star \{\bullet\} \star \mathcal{P}_g^0$$
where the label set $L_\bullet$ consists of free outputs of f, all other labels remain the same.

Proof. (i) The first two assertions are clear, the rest follows from the proof of Lemma 4.17 (iii). The statement (ii) follows from Corollary 4.7 (iii). For (iii) we use Proposition 4.14. If $\emptyset \in \mathcal{P}_g^0$, then parts (a) and (c) of Proposition 4.14 imply the result. If $\emptyset \notin \mathcal{P}_g^0$, we apply parts (b) and (d). If f has no free outputs, either of the situations may occur, but note that in the situation (d) the new element has an empty label, so it does not belong to $\mathcal{P}_f^0$. If f has free outputs, then only the situation in (d) is possible, with the new element now having a nonempty label. This implies the statement.

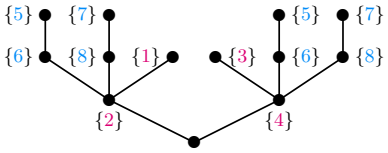
□

Example 4.12. The diagrams below show the diagrams of the restricted posets $\mathcal{P}_f^0$ for no-signalling channels $\gamma_2 \otimes \gamma_2$, n-slot process matrices $(\gamma_2 \otimes \cdots \otimes \gamma_2)^*$ and multiround process matrices $(\gamma_4 \otimes \gamma_6)^*$ (compare with the $\mathcal{P}_f$ diagrams in Example 4.8). Notice that the diagrams for process matrices are disconnected and also the sets of labels are separated, compare this with Proposition 4.20 below.



no-signalling channel n-slot process matrix multiround process matrix

Example 4.13. The next diagram shows the restricted poset $\mathcal{P}_f^0$ for the type a of adapters, see Example 4.9. Since $a = g \to g$ for the process matrix type g, the diagram can be obtained from the diagram of g in Example 4.12 and Lemma 4.18. The function of the input process matrix (pink indices in the diagram) acts on $s_1 ... s_4$, while the target process matrix (blue indices) acts on $s_5 \dots s_8$. This poset has a more complicated structure as the previous ones, in particular, notice that the labels are repeated.



adapter

4.7 Decomposition of $\mathcal{P}_f^0$

We now proceed by showing that $\mathcal{P}_f^0$ can be decomposed into a set of chains from which f is constructed by applying $\lhd$, $\otimes$ and *.

It the statement below, notice that we have $f \lhd 1_\varepsilon = 1_\varepsilon \lhd f = f$, for the trivial type function 1_ε.

Proposition 4.19. *Let $f \in \mathcal{T}_n$. There is a decomposition $[n] = [k][k+1, m][m+1, n]$, where any of the subintervals might be empty, type functions $\beta_1 \in \mathcal{T}_k$, $\beta' \in \mathcal{T}_{n-m-1}$ and $h \in \mathcal{T}_{m-k-1}$ such that β, β' are chain types, h nor h^* have any free indices and*

$$f \approx \beta \lhd h \lhd \beta'.$$

Proof. We have by (15) that $f \approx p_{k_1} \otimes h_1 \otimes 1_{l_1}$, where $k_1 = |I_f^F|$, $l_1 = |O_f^F|$ and $h_1 \in \mathcal{T}_{n_1}$, $n_1 := n - k_1 - l_1$ has no free indices. By Lemma 4.13 we get $f \approx p_{k_1} \lhd h_1 \lhd 1_{l_1}$. Assume h_1^* has free indices, then $h_1^* \approx p_{k_2} \lhd h_2 \lhd 1_{l_2}$, so that

$$f \approx p_{k_1} \lhd 1_{k_2} \lhd h_2^* \lhd p_{l_2} \lhd 1_{l_1} = \beta_2 \lhd h_2^* \lhd \beta_2'$$

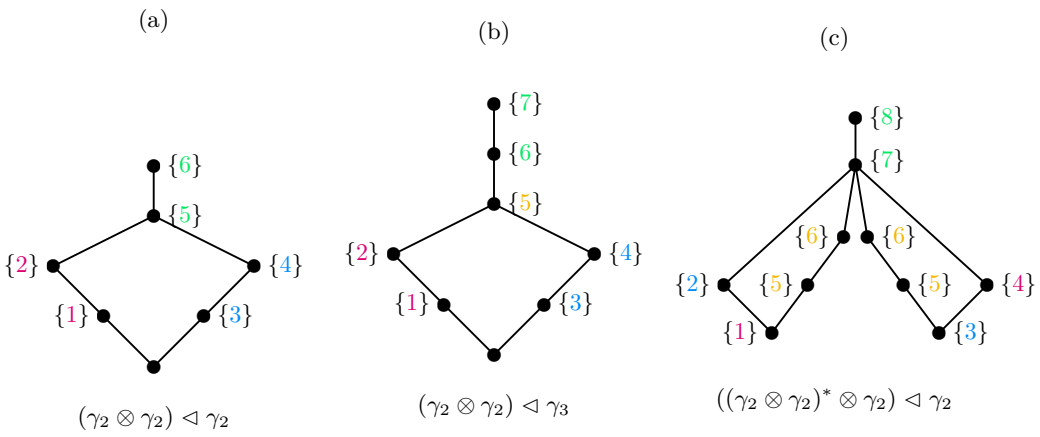
where $\beta_2 := p_{k_1} \lhd 1_{k_2}$ and $\beta_2' := p_{l_2} \lhd 1_{l_1}$ are chain types, and $h_2 \in \mathcal{T}_{n_2}$, $n_2 := n_1 - k_2 - l_2$ has no free indices. If h_2^* has again free indices, we may proceed in this way, obtaining

$$f \approx \beta_i \lhd h_i \lhd \beta_i'$$

with chain types β_i, β_i' and $h_i \in \mathcal{T}_{n_i}$ at each step. Since n_i decreases, we either get to $n_i \leq 3$, in which case f must be a chain type, and we may put $k = m = n$ or $k = m = 0$ in the above decomposition, or we get to the situation when h_i and h_i^* have no free indices. □

In the situation of the above Proposition, if f is not a chain, $\mathcal{P}_h^0$ and the chain types β_1 and β_2 can be seen from $\mathcal{P}_f^0$ as follows. Put $f_1 = \beta_1 \lhd h$, so that $f = f_1 \lhd \beta_2$. Since f_1 has no largest element and no free outputs, we obtain from Lemma 4.18 that $\mathcal{P}_f^0 = \mathcal{P}_{f_1}^0 \star (\mathcal{P}_\beta \setminus \{\emptyset\})$, with the same sets of labels. It follows that there exists a largest element in $\mathcal{P}_f^0$ with the property that it covers more than one element. Let this element be S and let $T_1, \ldots, T_k$ be the elements covered by S. There is a chain $S = S_1 \lneq \cdots \lneq S_K$, where S_K is the largest element in $\mathcal{P}_f^0$. We then have $\mathcal{P}_{f_1}^0 \simeq \cup_j T_j^{\downarrow}$. If K is even, add an element S_0 with empty label at the bottom of the chain to obtain a chain of even length. This then corresponds to the chain type β_2. Some examples are given below.

Example 4.14. The following labelled Hasse diagrams correspond to causal products of the form $f \lhd \beta$ for a chain β:



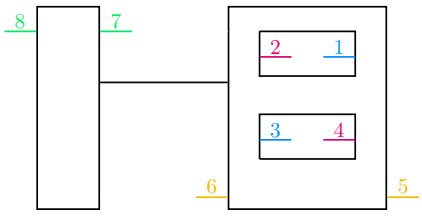
$(\gamma_2 \otimes \gamma_2) \lhd \gamma_2$ $(\gamma_2 \otimes \gamma_2) \lhd \gamma_3$ $((\gamma_2 \otimes \gamma_2)^* \otimes \gamma_2) \lhd \gamma_2$

Similarly as in Example 4.11, we have $f \lhd \beta \leq p_O^* \lhd \beta$, so that the maps of the corresponding types have the form of a comb, where this time the last tooth has a special structure given by f and β:



(a) (b)

In (a), green inputs can signal to all outputs, otherwise an input can only signal to outputs of the same color. In the case (b), the channel in the last tooth has an additional input (with index 5), which can signal to both pink and blue outputs. This is due to the fact that $\gamma_3 = p_1 \otimes \gamma_2 = p_1 \lhd \gamma_2$, so that $f \lhd \gamma_3 = (f \lhd p_1) \lhd \gamma_2$, which extends the type f by a global past. The maps of the type in (c) have a 2-comb structure as well, but the constraints are more complicated:



(c)

The last tooth is a no-signalling composition of a process matrix with a channel. The colors are related to signalling conditions as in case (a). Let us remark that the signalling relations can be seen directly from the order structure of the reduced poset, see [30].

We now decompose f_1 as $f_1 \approx \beta_1 \lhd h$. Assume first that $\mathcal{P}^0_{f_1}$ has no least element. In this case f_1^* has no free inputs, since $\mathrm{Min}(\mathcal{P}^0_{f_1^*}) = \{\emptyset\}$. We also have $I^F_f = I^F_{f_1}$, and $f_1 \approx p_k \otimes h = p_k \lhd h$, with $k = |I^F_{f_1}|$, which gives us $\beta_1 = p_k$.

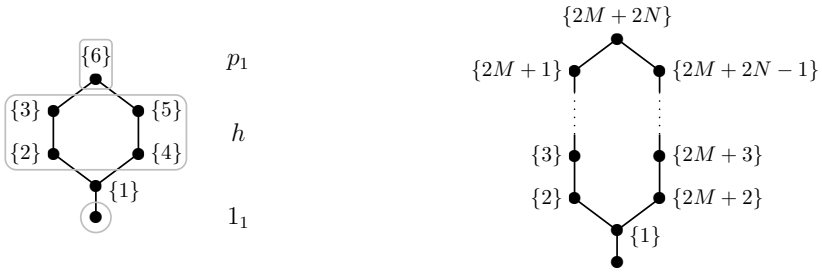
If $\mathcal{P}^0_{f_1}$ has a least element S_1, let S be the smallest element in $\mathcal{P}^0_f$ with the property that it is covered by more than one element. Let $S_1 \subsetneq \cdots \subsetneq S_K$ be a chain such that $S_K = S$ and let $T_1, \ldots, T_l$ be the elements covering S. Put $L := \cap_j L_{T_j}$. Using Lemma 4.18 as before, we have the following situations.

Case $L = \emptyset$. If K is odd, then the chain corresponds to a chain type $\beta_1 \in \mathcal{T}_k$, $k = |S_K| = \sum_j |L_{S_j}|$ and $\mathcal{P}^0_h = \cup_j T_j^{\uparrow, f_1} \cup \{\emptyset\}$. If K is even, then $\mathcal{P}^0_h = \cup_j T_j^{\uparrow, f_1}$ and β_1 is the chain type for the chain $S_1 \subsetneq \cdots \subsetneq S_{K-1}$ (with free outputs in L_{S_K}).

Case $L \neq \emptyset$. Add an element S_{K+1} at the end of the chain, with label $L_{S_{K+1}} = L$ and put $k = |S_{K+1}|$. If K is odd, then $\beta_1 \in \mathcal{T}_k$ is the chain type for the chain $S_1 \subsetneq \cdots \subsetneq S_K$ (with free outputs in L) and $\mathcal{P}^0_H = \cup_j T_j^{\uparrow, f_1} \cup \{\emptyset\}$, with labels of T_j replaced by $L_{T_j} \setminus L$. If K is even, then $\mathcal{P}^0_h = \cup_j T_j^{\uparrow, f_1}$ and β_1 is the chain type for $S_1 \subsetneq \cdots \subsetneq S_{K+1}$.

See also the diagrams in Example 4.11 (with the corresponding reduced poset $\mathcal{P}^0_f$) for some examples.

Example 4.15 (Comb to comb). The diagram on the left below depicts the function $g = (\gamma_2 \to \gamma_4) = (\gamma_2 \otimes \gamma_4^*)^* \in \mathcal{T}_6$. By Proposition 4.10 and Corollary 3.3, the corresponding higher order objects represent maps from channels to 2-combs.



Applying Proposition 4.19 together with the remarks below it, we obtain that $g \approx 1_1 \lhd h \lhd p_1$, as indicated on the diagram. It is easily checked that h is the process matrix type $h \simeq (\gamma_2 \otimes \gamma_2)^*$ (see Example 4.8), and g is the type of process matrices with global past and future (Example 4.10). In this way, we decomposed g into chain types, with respect to the decomposition of $[6]$ into four intervals $[6] = [1][2,3][4,5][6,6]$. Let $s = s^1 \ldots s^4$ be the corresponding concatenation. We indicate the component on which any of the chain types is acting by an upper index, so that $g \approx 1_1^1 \lhd (\gamma_2^2 \otimes \gamma_2^3)^* \lhd p_1^4$. By Lemmas 4.11 and 4.12, we have

$$\begin{aligned} g &\approx 1_1^1 \lhd ((\gamma_2^2 \lhd \gamma_2^3) \wedge (\gamma_2^3 \lhd \gamma_2^2))^* \lhd p_1^4 = 1_1^1 \lhd ((\gamma_2^2 \lhd \gamma_2^3)^* \vee (\gamma_2^3 \lhd \gamma_2^2)^*) \lhd p_1^4 \\ &= (1_1 \lhd (\gamma_2^2)^* \lhd (\gamma_2^3)^* \lhd p_1^4) \vee (1_1 \lhd (\gamma_2^3)^* \lhd (\gamma_2^2)^* \lhd p_1^4). \end{aligned}$$

It can be checked that the last expression is equal to $\gamma_6 \vee (\gamma_6 \circ \rho_\lambda)$, where $\lambda \in \mathscr{S}_4$ acts as $(1234) \mapsto (1324)$ and ρ_λ is the block permutation corresponding to λ and the above decomposition of $[6]$, see Appendix A.1. Similarly, for any M, N, the type describing higher order objects corresponding to mappings of M-combs into N-combs (see the diagram on the right) is an element of $\mathcal{T}2(M+N)$, and with respect to the decomposition $[2(M+$

$N)] = [1][M][N-2][1]$, it can be decomposed as

$$1_1 \lhd (\gamma_{2M} \otimes \gamma_{2N-2})^* \lhd p_1 = (1_1 \lhd \gamma_{2M}^* \lhd \gamma_{2N-2}^* \lhd p_1) \vee (1_1 \lhd \gamma_{2N-2}^* \lhd \gamma_{2M}^* \lhd p_1)$$
$$= \gamma_{2(M+N)} \vee (\gamma_{2(M+N)} \circ \rho_\lambda).$$

In the case of quantum objects, this corresponds to results of [10].

We will now deal with the case when f and f^* have no free indices. Let $\mathcal{P}$ be a poset with labels in $[n]$ and let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}$ be nonempty. We will say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are independent components of $\mathcal{P}$ if $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ (direct sum of posets) and $L_S \cap L_T = \emptyset$ for any $S \in \mathcal{P}_1$ and $T \in \mathcal{P}_2$. In this case, we will write

$$\mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2.$$

Proposition 4.20. *Let $f \in \mathcal{T}_n$ be such that f and f^* have no free indices. Assume that $\emptyset \notin \mathcal{P}_f$.*

(i) If $f^* \approx f_1 \otimes f_2$ for some type functions f_1 and f_2, then

$$\mathcal{P}_f^0 \simeq (\mathcal{P}_{f_1}^0 \setminus \emptyset) \uplus (\mathcal{P}_{f_2}^0 \setminus \emptyset).$$

(ii) If $\mathcal{P}_f^0 = \mathcal{P}_1 \uplus \mathcal{P}_2$ for some labelled subposets $\mathcal{P}_1$ and $\mathcal{P}_2$, then there are some type functions f_1 and f_2 such that $\mathcal{P}_1 = (\mathcal{P}_{f_1}^0 \setminus \emptyset)$, $\mathcal{P}_2 = (\mathcal{P}_{f_2}^0 \setminus \emptyset)$ and $f^* \approx f_1 \otimes f_2$.

(iii) If $f \approx f_1 \otimes f_2$ for type functions f_1 and f_2, then no decomposition of $\mathcal{P}_f^0$ into independent components exists.

Proof. Under the assumptions, $\emptyset \in \mathcal{P}_{f^*}$ and $\mathcal{P}_f^0 = \mathcal{P}_{f^*}^0 \setminus \{\emptyset\}$. Assume that $f_1 \in \mathcal{T}_{n_1}$ and $f_2 \in \mathcal{T}_{n_2}$ are such that $f^* = f_1 \otimes f_2$ for the decomposition $[n] = [n_1] \boxplus [n_2]$. Since $\emptyset \in \mathcal{P}_{f^*}^0 = \mathcal{P}_{f_1 \otimes f_2}^0$, we see by Lemma 4.18 (ii) that both $\mathcal{P}_{f_1}^0$ and $\mathcal{P}_{f_2}^0$ must contain $\emptyset$, and $\mathcal{P}_{f^*}^0$ consists of $\mathcal{P}_{f_1}^0$ and $\mathcal{P}_{f_2}^0$ glued at $\emptyset$, with labels of $\mathcal{P}_{f_2}^0$ translated by n_1. Since $\mathcal{P}_f^0 = \mathcal{P}_{f^*}^0 \setminus \{\emptyset\}$, the assertion (i) follows.

For (iii), assume that $f = f_1 \otimes f_2$ and $\mathcal{P}_f^0 = \mathcal{P}_1 \uplus \mathcal{P}_2$. Note that none of the functions can be a 1-element chain, since then f would have free inputs or outputs. Let $\mathrm{Min}(\mathcal{P}_{f_1}) = \{U_1, \ldots, U_k\}$, $\mathrm{Min}(\mathcal{P}_{f_2}) = \{V_1, \ldots, V_l\}$. By Lemma 4.18, we have $\mathrm{Min}(\mathcal{P}_f) = \mathrm{Min}(\mathcal{P}_{f_1}) \times \mathrm{Min}(\mathcal{P}_{f_2})$. For some i and j, let $(U_i, V_j) \in \mathcal{P}_1$. Since (U_i, V_j) and (T, V_j) are comparable for any $T \in \mathcal{P}_{f_1}^0$, $U_i \leq T$, we must have $(T, V_j) \in \mathcal{P}_1$ for all such T. By Lemma C.1, there is some T that covers U_i. But then $L_{(T,V_j)} = L_{T,V_{j'}} = L_T$ for all j', so that $(T, V_{j'}) \in \mathcal{P}_1$ for all j'. Since $(U_i, V_{j'}) \leq (T, V_{j'})$ for all j', this implies that $(U_i, V_{j'}) \in \mathcal{P}_1$ for all j'. By the same reasoning with V_j, we get that all $(U_i, V_j) \in \mathcal{P}_1$, which is not possible.

For (ii), assume that $\mathcal{P}_f^0 = \mathcal{P}_1 \uplus \mathcal{P}_2$. Since either f or f^* is, up to a permutation, a tensor product of type functions and we cannot have $f \approx f_1 \otimes f_2$ by (iii), it must hold that $f^* \approx f_1 \otimes f_2$. But then by (i) $\mathcal{P}_f^0 = \mathcal{P}_1' \uplus \mathcal{P}_2'$, with $\mathcal{P}_i' = \mathcal{P}_{f_i}^0 \setminus \{\emptyset\}$. Let

$$\mathcal{P}_f^0 = \mathcal{Q}_1 \uplus \ldots \uplus \mathcal{Q}_M$$

be the finest decomposition into independent components. Then there are some $C, D \subset [M]$ such that

$$\mathcal{P}_1 = \biguplus_{i \in C} \mathcal{Q}_i, \quad \mathcal{P}_2 = \biguplus_{i \in [M] \setminus C} \mathcal{Q}_i, \quad \mathcal{P}_1' = \biguplus_{i \in D} \mathcal{Q}_i, \quad \mathcal{P}_2' = \biguplus_{i \in [M] \setminus D} \mathcal{Q}_i.$$

Assume that $M \geq 3$, otherwise each $\mathcal{P}_i$ is one of $\mathcal{P}_j'$ and we are done. Then D or $[M] \setminus D$ has at least two elements. So assume $|D| \geq 2$. This implies that $\mathcal{P}_1'$ has no largest element

and since we have $\mathcal{P}_1' = \mathcal{P}_{f_1}^0 \setminus \{\emptyset\}$, $\mathcal{P}_{f_1}^0$ has no largest element either. By Lemma 4.18 (i), this implies that f_1^* has no free outputs, and since f_1 has no free outputs by the assumptions, the same Lemma implies that $\mathcal{P}_1' = \mathcal{P}_{f_1^*}^0$. It follows that f_1^* satisfies the assumptions in (iii). Hence $f_1 \approx g_1 \otimes g_2$ for some type functions g_1 and g_2 and we obtain that

$$\mathcal{P}_{g_1}^0 \setminus \{\emptyset\} = \biguplus_{i \in D'} \mathcal{Q}_i, \quad \mathcal{P}_{g_2}^0 \setminus \{\emptyset\} = \biguplus_{i \in D \setminus D'} \mathcal{Q}_i$$

for some $D' \subset D$. Continuing in this way, we obtain that for any $i \in [M]$ there is some type function g_i such that $\emptyset \in \mathcal{P}_{g_i}^0$ and $\mathcal{Q}_i = \mathcal{P}_{g_i}^0 \setminus \{\emptyset\}$. It follows that

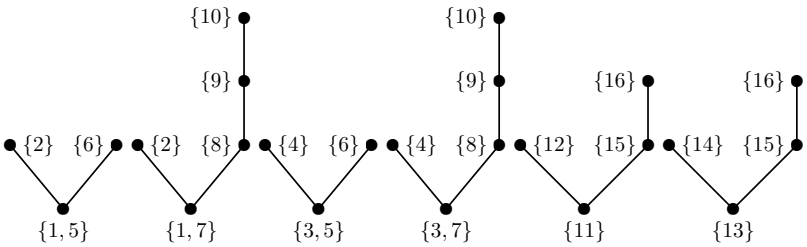
$$\mathcal{P}_1 = \biguplus_{i \in C} (\mathcal{P}_{g_i}^0 \setminus \{\emptyset\}) = \mathcal{P}^0_{\otimes_{i \in D} g_i} \setminus \{\emptyset\}$$

and $f_1 \approx \otimes_{i \in D} g_i$, similarly, $\mathcal{P}_2 = \mathcal{P}^0_{\otimes_{i \in [M] \setminus D} g_i} \setminus \{\emptyset\}$ and $f_2 \approx \otimes_{i \in [M] \setminus D} g_i$.

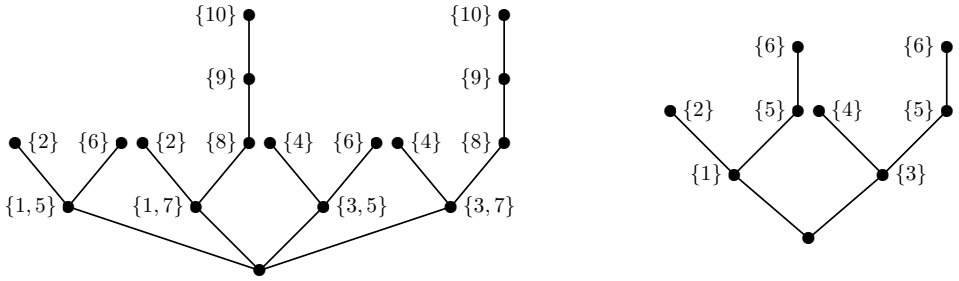
□

It follows from the above proof that in the situation of the above Proposition, if $f^* \approx g_1 \otimes \cdots \otimes g_k$, we can identify $\mathcal{P}_{g_l}^0$ by looking at the independent components of $\mathcal{P}_f^0$.

Example 4.16. The next diagram shows $\mathcal{P}_f^0$ for a type function $f \in \mathcal{T}_{16}$:



Note that $\mathcal{P}_f^0$ has no largest or smallest element, and also no free indices, so the assumptions of Proposition 4.20 are satisfied. As a poset, $\mathcal{P}_f^0$ is a direct sum of 6 posets, but there are only two independent components: one with labels ≤ 10, and one with labels > 10. It follows that $f^* \approx g_1 \otimes g_2$. We can obtain $\mathcal{P}_{g_1}^0$ and $\mathcal{P}_{g_2}^0$ by adding $\emptyset$, that is, an unlabelled minimal element to each component, and relabelling if necessary:



We can also see that both g_1^* and g_2^* are products. Using the procedure described in Appendix C, we obtain that $g_1 \approx ((\gamma_2 \otimes \gamma_2)^* \otimes (\gamma_2 \otimes \gamma_4)^*)^*$ and $g_2 \approx ((\gamma_2 \otimes \gamma_2)^* \otimes \gamma_2)^*$.

The case when $\mathcal{P}_f^0$ has no independent components is somewhat more complicated. By the proposition above, this is the case when $f \approx f_1 \otimes \cdots \otimes f_k$. The proof of the following result is in Appendix C.

Proposition 4.21. *Let $f \in \mathcal{T}_n$ be such that f and f^* have no free indices and $\emptyset \notin \mathcal{P}_f^0$. Assume $f \approx f_1 \otimes \cdots \otimes f_k$ is a finest decomposition of f as a product. Then the labelled posets $\mathcal{P}_{f_l}^0$, $l = 1, \ldots, k$ can be obtained from the structure of $\mathcal{P}_f^0$, up to a permutation on the labels.*

Theorem 4.22. *Every type function $f \in \mathcal{T}_n$ is fully determined by the labelled poset $\mathcal{P}_f^0$.*

Proof. We will proceed by induction on n. If f is a chain type, the assertion follows from Lemma 4.17 and Proposition 4.9, so that the assertion holds for $n \leq 3$. Assume it is true for all $m < n$ and let $f \in \mathcal{T}_n$. By Proposition 4.19 and remarks below it, if f or f^* has some free indices, then we have $f \approx \beta_1 \lhd h \lhd \beta_2$, where β_1, β_2 are chain types and $h \in \mathcal{T}_m$ is such that h and h^* do not have any free indices. Moreover, the chain types and $\mathcal{P}_h^0$ can be obtained from $\mathcal{P}_f^0$. Since $m < n$, h is determined by $\mathcal{P}_h^0$ by the induction assumptions, so we are done.

If both f and f^* have no free indices, we may assume that $\emptyset \notin \mathcal{P}_f$, otherwise we replace f by f^*. Then if $\mathcal{P}_f^0$ has independent components, we have $f^* \approx f_1 \otimes f_2$ for some type functions $f_i \in \mathcal{T}_{n_i}$, $i = 1, 2$ and $n = n_1 + n_2$, such that the components have the form $\mathcal{P}_{f_i}^0 \setminus \{\emptyset\}$. Since $n_1, n_2 < n$, we are done. If $\mathcal{P}_f^0$ has no independent components, then $f \approx f_1 \otimes \cdots \otimes f_k$ for some $f_i \in \mathcal{T}_{m_i}$, $i = 1, \ldots, k$, $m_1 + \cdots + m_k = n$, and we obtain $\mathcal{P}_{f_l}^0$ for all $l = 1, \ldots, k$ from $\mathcal{P}_f^0$ by Proposition 4.21. Again, the assertion follows by applying induction assumption on each f_l.

□

Using repeatedly the procedure described in the above proof, we get to the situation when all the obtained components are necessarily chains. In this way, we get a decomposition of f into chain types, together with a recipe for a construction of f from these chain types by using tensor products, complements and causal products. It is then clear, using also the proof of Theorem 4.16, that these chain types give us a choice of $\beta_1, \ldots, \beta_k$ in the structure theorem 4.16.

5 Conclusions

We have studied the category Af of affine subspaces in finite dimensional vector spaces, endowed with a *-autonomous structure inherited from the compact closed category FinVect. We used this structure to define a hierarchy of higher order objects, which, restricted to certain objects we call quantum, contains the higher order quantum maps. The quantum objects have the space M_n^h as the underlying vector space and satisfy the condition that both the affine subspace and its affine dual contains a multiple of the identity. This restriction satisfies the assumptions of the setting in [13].

We used the combinatorial characterization of types as in [1, 10, 12] to relate functions in $\mathcal{F}_n$ to certain objects in Af, constructed over a fixed sequence of first order objects $X_1, \ldots, X_n$. We proved that this relation defines a representation of the boolean algebra $\mathcal{F}_n$ in the lattice of subspaces of the tensor product of the underlying vector spaces $V_{X_1} \otimes \cdots \otimes V_{X_n}$. We also proved that all higher order objects can be obtained in this way, the corresponding functions are called the type functions.

We then studied the type functions and their properties. We defined a labelled poset $\mathcal{P}_f^0$ for each type function f and proved that f is related to combs if and only if $\mathcal{P}_f^0$ is a chain. We showed that f is fully determined by $\mathcal{P}_f^0$ and devised a procedure to decompose $\mathcal{P}_f^0$ to a set of basic chains, from which the type function f is constructed. We proved a structure theorem, saying that f can be obtained by taking maxima and minima over concatenations of these chains in different orders.

There is a number of questions left for future research. The first set of open questions pertains the type functions themselves. It is still not clear how the type functions, or, equivalently, the corresponding posets $\mathcal{P}_f$ or $\mathcal{P}_f^0$, can be characterized among elements in

$\mathcal{F}_n$. In this respect, it might be interesting to study some general notion of labelled posets, their structure and categorical constructions over them.

In connection to higher order objects, it will be shown in a subsequent work [30] that the signalling relations of a type can be obtained from the order structure in $\mathcal{P}_f^0$ (or $\mathcal{P}_f$). As shown in [1], the signalling relations restrict the way how the higher order maps can be connected so that the resulting map is given by a subtype. This should correspond to some construction over the posets of the involved type functions.

In the context of the structure theorem, one may ask which sets of permutations over connections of chains would lead to a type function after taking maxima and minima. By the proof of Theorem 4.16, it seems that these should be of a block form, coming from the operadic structure on permutations, see A.1.

Another important problem is a deeper study of causal order of higher order objects, related to causal separability, [31]. Causal nonseparability cannot be the property of a type function itself, but involves also the positive cone. Nevertheless, it is an interesting question how the problem of witnessing causal nonseparability [17] fits into the framework of Af. There are other important properties that are not expressed as affine constraints, such as LOSR or LOSE channels, or constructions such as quantum circuits with classical or quantum constraints [19]. Finding necessary additional structures that would enable us to study such questions in Af is another direction of future investigations.

Finally, as already mentioned in the introduction, one may think of building a similar theory of higher order maps in a general probabilistic theory, see e.g. [21]. In this framework, state spaces are represented by compact convex sets that can be obtained by an intersection of a positive cone in a vector space by a hyperplane given by an interior element of the dual positive cone. In this way, it is related to a first order object in Af and we may think of channels as linear maps transforming between hyperplanes, satisfying certain positivity properties. Here the choice of positive cones and their tensor products plays an important role.

Besides the quantum theory, one of the basic examples is the classical theory, where the state spaces are probability simplexes. The underlying positive cones form a compact closed category $\mathbf{Mat}(\mathbb{R}_+)$, which enables us to identify the higher order classical theory with the classical objects in our framework. This situation appears also in the quantum case, where the underlying compact closed category is **CPM**, the category of completely positive maps.

The higher order theory over classical state spaces is far from trivial, since it is known that there are purely classical higher order processes (e.g. the Lugano process [32]) that exhibit indefinite causality. As mentioned above, such properties cannot be explained by type functions themselves, but our approach enables us to study both quantum and classical higher order theory on an equal footing.

For general GPTs, we cannot expect to have a compact closed category at our disposal. But if a theory can be constructed based of ordered vector spaces, with tensor product giving it a structure of a closed monoidal or even *-autonomous category, then one can expect the higher order maps to be described by higher order objects in Af, intersected by a positive cone.

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A Some basic definitions

For $m \leq n \in \mathbb{N}$, we will denote the corresponding interval $\{m, m+1, \ldots, n\}$ by $[m, n]$. For $m = 1$, we will simplify to $[n] := [1, n]$. Let $\mathscr{S}_n$ denote the set of all permutations of $[n]$.

A.1 Block permutations

For $n_1, n_2 \in \mathbb{N}$, $n_1 + n_2 = n$, we will denote by $[n] = [n_1] \boxplus [n_2]$ the decomposition of $[n]$ as a concatenation of two intervals

$$[n] = [n_1][n_1 + 1, n_1 + n_2].$$

Similarly, for $n = \sum_{j=1}^{k} n_j$, we have the decomposition

$$[n] = \boxplus_{j=1}^{k}[n_j] = [m_1, m_1 + n_1][m_2, m_2 + n_2] \dots [m_k, m_k + n_k],$$

where $m_j := \sum_{l=1}^{j-1} n_j$ (so $m_1 = 0$). Note that the order of $n_1, \dots, n_k$ in this decomposition is fixed.

We have two kinds of special permutations related to the above decomposition. For $\sigma_j \in \mathscr{S}_{n_j}$, we denote by $\boxplus_j \sigma_j \in \mathscr{S}_n$ the permutation that acts as

$$m_j + l \mapsto m_j + \sigma_j(l), \qquad l = 1, \dots, n_j,\ j = 1, \dots, k.$$

On the other hand, we have for any $\lambda \in \mathscr{S}_k$ a unique permutation $\rho_\lambda \in \mathscr{S}_n$ such that ρ_λ^{-1} acts as

$$[m_1, m_1+n_1][m_2, m_2+n_2]...[m_k, m_k+n_k] \mapsto [m_{\lambda(1)}+n_{\lambda(1)}][m_{\lambda(2)}+n_{\lambda(2)}] \dots [m_{\lambda(k)}+n_{\lambda(k)}]$$

Note that we have

$$\rho_\lambda \circ (\boxplus_j \sigma_j) = (\boxplus_j \sigma_{\lambda(j)}) \circ \rho_\lambda.$$

(These permutations come from the operadic structure on the set of all permutations $\mathscr{S}_*$. See [33] for the definition of and operad.)

A.2 Partially ordered sets

An overall reference for this section is [34].

A partially ordered set, or a poset, is a set $\mathcal{P}$ endowed with a reflexive, antisymmetric and transitive relation $\leq$, called the partial order. We will only encounter the situation when $\mathcal{P}$ is finite. A basic example of a poset is the set $\mathcal{P}(X)$ of all subsets of a finite set X, ordered by inclusion. If $X = [n]$, we will denote $\mathcal{P}(X)$ by 2^n.

A subposet in a poset $\mathcal{P}$ is a $\mathcal{Q} \subseteq \mathcal{P}$ endowed with the partial order relation inherited from $\mathcal{P}$. For any subset $\mathcal{R} \subseteq \mathcal{P}$, we define two special subposets in $\mathcal{P}$ as

$$\mathcal{R}^{\downarrow} = \{p \in \mathcal{P},\ p \leq r \text{ for some } r \in \mathcal{R}\},\ \mathcal{R}^{\uparrow} = \{p \in \mathcal{P},\ p \geq r \text{ for some } r \in \mathcal{R}\}.$$

The set of minimal elements in $\mathcal{P}$ will be denoted by $\mathrm{Min}(\mathcal{P})$. For elements $p, q \in \mathcal{P}$, we say that q covers p, in notation $p \lessdot q$, if $p \leq q$ and for any r such that $p \leq r \leq q$ we have $r = p$ or $r = q$. If p covers a minimal element, we will say that p is a minimal covering element.

A totally ordered subposet $\mathcal{C} \subseteq \mathcal{P}$ is called a chain in $\mathcal{P}$. Such a chain is maximal if it is not contained in any other chain in $\mathcal{P}$. The length of a chain $\mathcal{C}$ is defined as $|\mathcal{C}| - 1$.

We say that a poset $\mathcal{P}$ is graded of rank k if every maximal chain of P has the same length k. In this case, there is a (unique) rank function $\rho : \mathcal{P} \to \{0, 1, \dots, k\}$ such that $\rho(p) = 0$ if p is a minimal element of $\mathcal{P}$ and $\rho(q) = \rho(p) + 1$ if $p \lessdot q$. Basic examples of graded posets are chains, antichains and 2^n.

If $\mathcal{P}$ and $\mathcal{Q}$ are posets with disjoint sets, their direct sum $\mathcal{P} + \mathcal{Q}$ is a poset defined as the disjoint union $\mathcal{P} \cup \mathcal{Q}$, such that the order is preserved in each component and elements in different components are incomparable. Another way to compose $\mathcal{P}$ and $\mathcal{Q}$ is the ordinal sum $\mathcal{P} \star \mathcal{Q}$, where the underlying set is again the union $\mathcal{P} \cup \mathcal{Q}$ and the order in each component is preserved, but for $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ we have $p \leq q$. A third way to compose posets that we will use is the direct product $\mathcal{P} \times \mathcal{Q}$, where the underlying set is the cartesian product $\mathcal{P} \times \mathcal{Q}$, with $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ in $\mathcal{P}$ and $q_1 \leq q_2$ in $\mathcal{Q}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are

graded posets with rank functions ρ_1 and ρ_2, then $\mathcal{P}_1 \times \mathcal{P}_2$ is graded as well, with rank function ρ given as

$$\rho(p_1, p_2) = \rho_1(p_1) + \rho_2(p_2).$$

The Hasse diagram of a finite poset $\mathcal{P}$ is a graph whose vertices are elements of $\mathcal{P}$ and there is an edge between p and q if $p \ll q$, and if $p \lneq r$, then r is drawn above p. Two posets are isomorphic if and only if they have the same Hasse diagrams.

A.3 Binary strings

A binary string of length n is a sequence $s = s_1 \dots s_n$, where $s_i \in \{0, 1\}$. Such a string can be interpreted as an element $\{0, 1\}^n$, but also as a map $[n] \to \{0, 1\}$, or a subset in $[n] := \{1, \dots, n\}$. It will be convenient to use all these interpretations, but we will distinguish between them. The strings in $\{0, 1\}^n$ will be denoted by small letters, whereas the corresponding subsets of $[n]$ will be denoted by the corresponding capital letters. More specifically, for $s \in \{0, 1\}^n$ and $T \subseteq [n]$, we denote

$$S := \{i \in [n],\ s_i = 0\}, \qquad t := t_1 \dots t_n,\ t_j = 0 \iff j \in T. \tag{A.1}$$

As usual, the set of all subsets of $[n]$ will be denoted by 2^n. With the inclusion ordering and complementation $S^c := [n] \setminus S$, 2^n is a boolean algebra, with the smallest element $\emptyset$ and largest element $[n]$.

The group $\mathscr{S}_n$ has an obvious action on $\{0, 1\}^n$. Indeed, for a string s interpreted as a map $[n] \to 2$, we may define the action of $\sigma \in \mathscr{S}_n$ by precomposition as

$$\sigma(s) := s \circ \sigma^{-1} = s_{\sigma^{-1}(1)} \dots s_{\sigma^{-1}(n)}.$$

Note that in this way we have $\rho(\sigma(s)) = (\rho \circ \sigma)(s)$. For a decomposition $[n] = \boxplus_{j=1}^k [n_j]$, we have a corresponding decomposition of any string $s \in \{0, 1\}^n$ as a concatenation of strings

$$s = s^1 \dots s^k, \qquad s^j \in \{0, 1\}^{n_j}.$$

For permutations $\sigma_j \in \mathscr{S}_{n_j}$ and $\lambda \in \mathscr{S}_k$, we have

$$\rho_\lambda \circ (\boxplus_j \sigma_j)(s^1 \dots s^k) = \rho_\lambda(\sigma_1(s^1) \dots \sigma_k(s^k)) = \sigma_{\lambda(1)}(s^{\lambda(1)}) \sigma_{\lambda(2)}(s^{\lambda(2)}) \dots \sigma_{\lambda(k)}(s^{\lambda(k)}).$$

A.4 Boolean functions and the Möbius transform

A function $f : \{0, 1\}^n \to \{0, 1\}$ is called a boolean function. The set of boolean functions, with pointwise ordering and complementation given by the negation $\bar{f} = 1 - f$, is a boolean algebra that can be identified with 2^{2^n}. We will denote the maximal element (the constant 1 function) by 1_n. Similarly, we denote the constant zero function by 0_n. For boolean functions f, g, the pointwise minima and maxima will be denoted by $f \wedge g$ and $f \vee g$. It is easily seen that we have

$$f \vee g = f + g - fg, \qquad f \wedge g = fg, \tag{A.2}$$

all the operations are pointwise. We now introduce and important example.

Example A.1. For $S \subseteq [n]$, we define

$$p_S(t) = \Pi_{j \in S}(1 - t_j), \qquad t \in \{0, 1\}^n.$$

That is, $p_S(t) = 1$ if and only if $S \subseteq T$. In particular, $p_\emptyset = 1_n$ and $p_{[n]}$ is the characteristic function of the zero string. Clearly, for $S, T \subseteq [n]$ we have $p_{S \cup T} = p_S p_T = p_S \wedge p_T$, in particular, $p_S = \Pi_j p_{\{j\}}$.

By the Möbius transform, all boolean functions can be expressed as combinations of the functions p_S, $S \subseteq [n]$ from the previous example.

Theorem A.1. *Any $f : \{0,1\}^n \to 2$ can be expressed in the form*

$$f = \sum_{S \subseteq [n]} \hat{f}_S p_S$$

in a unique way. The coefficients $\hat{f}_S \in \mathbb{R}$ obtained as

$$\hat{f}_S = \sum_{\substack{t \in \{0,1\}^n \\ t_j = 1, \forall j \in S^c}} (-1)^{\sum_{j \in S} t_j} f(t).$$

Proof. By the Möbius inversion formula (see [34, Sec. 3.7] for details), functions $f, g : 2^n \to \mathbb{R}$ satisfy

$$f(S) = \sum_{T \subseteq S} g(T), \qquad S \in 2^n$$

if and only if

$$g(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f(T).$$

We now express this in terms of the corresponding strings s and t. It is easily seen that $T \subseteq S$ if and only if $s_j = 0$ for all $j \in T$, equivalently, $t_j = 1$ for all $j \in S^c$. Moreover, in this case we have $|S \setminus T| = \sum_{j \in S} t_j$. This shows that $g(S) = \hat{f}_S$, as defined in the statement. The first equality now gives

$$f(s) = f(S) = \sum_{T \subseteq S} g(T) = \sum_{T : s_i = 0, \forall i \in T} \hat{f}_T = \sum_{T : p_T(s) = 1} \hat{f}_T = \sum_{T \subseteq [n]} \hat{f}_T p_T.$$

For uniqueness, assume that $f = \sum_{T \subseteq [n]} c_T p_T$ for some coefficients $c_T \in \mathbb{R}$. Then

$$f(s) = \sum_{T : p_T(s) = 1} c_T = \sum_{T \subseteq S} c_T.$$

Uniqueness now follows by uniqueness in the Möbius inversion formula.

□

A.5 The boolean algebra $\mathcal{F}_n$

Let us introduce the subset of boolean functions

$$\mathcal{F}_n := \{f : \{0,1\}^n \to 2,\ f(\theta_n) = 1\},$$

where we use θ_n to denote the zero string $00\ldots0$. In other words, $\mathcal{F}_n$ is the interval of all elements greater than $p_{[n]}$ in the boolean algebra 2^{2^n} of all boolean functions. With the pointwise ordering, $\mathcal{F}_n$ is a distributive lattice, with top element 1_n and bottom element $p_n := p_{[n]}$. We also define complementation in $\mathcal{F}_n$ as

$$f^* := 1_n - f + p_n.$$

It can be easily checked that with these structures $\mathcal{F}_n$ is a boolean algebra, though it is not a subalgebra of 2^{2^n}.

Note that $p_S \in \mathcal{F}_n$ for any $S \subseteq [n]$, so in particular for $[k]$, with $k \leq n$. If $k < n$, we denote these functions as before by $p_{[k]}$, using the notation p_n only for the distinguished bottom element.

We now introduce some more operations in $\mathcal{F}_n$. For $f \in \mathcal{F}_n$ and any permutation $\sigma \in \mathscr{S}_n$, we clearly have $f \circ \sigma \in \mathcal{F}_n$. Further, let $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$. With the decomposition $[n_1 + n_2] = [n_1] \uplus [n_2]$ and the corresponding concatenation of strings $s = s^1 s^2$, we define the function $f \otimes g \in \mathcal{F}_{n_1+n_2}$ as

$$(f \otimes g)(s^1 s^2) = f(s^1) g(s^2), \qquad s^1 \in \{0,1\}^{n_1},\ s^2 \in \{0,1\}^{n_2}.$$

Let $\lambda \in \mathscr{S}_2$ be the transposition, then we have for any $f \in \mathcal{F}_{n_1}$ and $g \in \mathcal{F}_{n_2}$

$$(g \otimes f) = (f \otimes g) \circ \rho_\lambda,$$

where ρ_λ is the block permutation defined in Section A.1.

If $f, g \in \mathcal{F}_n$ are such that $g = f \circ \sigma$ for some $\sigma \in \mathscr{S}_n$, we write $f \approx g$. It is easily observed that if $f_1 \approx g_1$ and $f_2 \approx g_2$, then $f_1 \otimes f_2 \approx g_1 \otimes g_2$ and if $f \approx g$ then also $f^* \approx g^*$.

We now show some further important properties of these operations.

Lemma A.2. *For $f \in \mathcal{F}_{n_1}$ and $g, h \in \mathcal{F}_{n_2}$, we have*

(i) $f \otimes g \leq (f^* \otimes g^*)^*$, with equality if and only if either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{n_1}$ and $g = p_{n_2}$.

(ii) $f \otimes (g \vee h) = (f \otimes g) \vee (f \otimes h)$, $f \otimes (g \wedge h) = (f \otimes g) \wedge (f \otimes h)$.

Proof. The inequality in (i) is easily checked, since $(f \otimes g)(s^1 s^2)$ can be 1 only if $f(s^1) = g(s^2) = 1$. If both s^1 and s^2 are the zero strings, then $s^1 s^2 = \theta_{n_1+n_2}$ and both sides are equal to 1. Otherwise, the condition $f(s^1) = g(s^2) = 1$ implies that $(f^* \otimes g^*)(s^1 s^2) = 0$, so that the right hand side must be 1. If f and g are both constant 1, then

$$(1_{n_1} \otimes 1_{n_2})^* = 1^*_{n_1+n_2} = p_{n_1+n_2} = p_{n_1} \otimes p_{n_2} = 1^*_{n_1} \otimes 1^*_{n_2},$$

in the case when both f and g are the minimal elements equality follows by duality. Finally, assume the equality holds and that $f \neq 1_{n_1}$, so that there is some s^1 such that $f(s^1) = 0$. But then $s^1 \neq \theta_{n_1}$, so that $f^*(s_1) = 1$ and for any s^2,

$$0 = (f \otimes g)(s^1 s^2) = (f^* \otimes g^*)^*(s^1 s^2) = 1 - f^*(s^1) g^*(s^2) + p_{n_1+n_2}(s^1 s^2) = 1 - g^*(s^2),$$

which implies that $g(s^2) = 0$ for all $s^2 \neq \theta_{n_2}$, that is, $g = p_{n_2}$. By the same argument, $f = p_{n_1}$ if $g \neq 1_{n_2}$, which implies that either $f = 1_{n_1}$ and $g = 1_{n_2}$, or $f = p_{n_1}$ and $g = p_{n_2}$.

The statement (ii) is easily proved from (A.2).

□

Consider the decomposition $[n] = [n_1] \uplus [n_2]$ and let $S \subseteq [n_1]$, $T \subseteq [n_2]$. We then denote by $S \uplus T$ the disjoint union

$$S \uplus T := S \cup (n_1 + T) = S \cup \{n_1 + j,\ j \in T\}. \tag{A.3}$$

We summarize some easy properties of the basic functions p_S, $S \subseteq [n]$.

Lemma A.3. *(i) For $S, T \subseteq [n]$, we have $S \subseteq T \iff p_T \leq p_S \iff p_S p_T = p_S$.*

(ii) For $S \subseteq [n]$, $\sigma \in \mathscr{S}_n$, $p_S \circ \sigma = p_{\sigma^{-1}(S)}$.

(iii) For $S \subseteq [n_1]$ and $T \subseteq [n_2]$, $p_S \otimes p_T = p_{S \boxplus T}$.

Let $f \in \mathcal{F}_n$ and let $\hat{f}$ be the Möbius transform. Note that since f has values in $\{0,1\}$, we have by the proof of Theorem A.1

$$\forall S \in 2^n, \quad \sum_{T \subseteq S} \hat{f}_T = f(s) \in \{0,1\}; \qquad \sum_{T \in 2^n} \hat{f}_T = f(\theta_n) = 1.$$

Proposition A.4. *(i) For $f \in \mathcal{F}_n$ and $\sigma \in \mathscr{S}_n$, $\widehat{(f \circ \sigma)}_S = \hat{f}_{\sigma(S)}$, $\quad S \subseteq [n]$.*

(ii) For $f \in \mathcal{F}_n$, $\widehat{f^*}_S = \begin{cases} 1 - \hat{f}_S & S = \emptyset \text{ or } S = [n], \\ -\hat{f}_S & \text{otherwise.} \end{cases}$

(iii) For $f \in \mathcal{F}_{n_1}$, $g \in \mathcal{F}_{n_2}$, we have $\widehat{(f \otimes g)}_{S \boxplus T} = \hat{f}_S \hat{g}_T$, $S \subseteq [n_1]$, $T \subseteq [n_2]$.

Proof. All statements follow easily from Lemma A.3 and the uniqueness part in Theorem A.1.

□

B Affine subspaces

Let V be a finite dimensional real vector space. A subset $A \subseteq V$ is an affine subspace in V if for any choice of $a_1, \ldots, a_k \in A$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that $\sum_i \alpha_i = 1$, we have $\sum_i \alpha_i a_i \in A$. It is clear that $A = \emptyset$ is trivially an affine subspace. Moreover, any linear subspace in V is an affine subspace, and an affine subspace A is linear if and only if $0 \in A$. If $A \neq \emptyset$ and also $0 \notin A$, we say that A is proper.

A proper affine subspace $A \subseteq V$ can be determined in two ways. Let

$$\mathrm{Lin}(A) := \{a_1 - a_2,\ a_1, a_2 \in A\}.$$

It is easily verified that $\mathrm{Lin}(A)$ is a linear subspace, moreover, for any $a \in A$, we have

$$\mathrm{Lin}(A) = \{a_1 - a,\ a_1 \in A\}, \qquad A = a + \mathrm{Lin}(A). \tag{B.1}$$

We put $\dim(A) := \dim(\mathrm{Lin}(A))$, the dimension of A.

Let V^* be the vector space dual of V and let $\langle \cdot, \cdot \rangle$ be the duality. For a subset $C \subseteq V$, put

$$C^* := \{v^* \in V^*,\ \langle v^*, a \rangle = 1,\ \forall a \in C\}.$$

Let $\tilde{a} \in A^*$ be any element and let $\mathrm{Span}(A)$ be the linear span of A in V. We then have

$$A = \mathrm{Span}(A) \cap \{\tilde{a}\}^*, \tag{B.2}$$

independently of $\tilde{a}$. The relation between the two expressions for A, given by (B.1) and (B.2) is obtained as

$$\mathrm{Span}(A) = \mathrm{Lin}(A) + \mathbb{R}\{a\}, \qquad \mathrm{Lin}(A) = \mathrm{Span}(A) \cap \{\tilde{a}\}^\perp, \tag{B.3}$$

independently of $a \in A$ or $\tilde{a} \in A^*$. Here $+$ denotes the direct sum of the vector spaces and $C^\perp$ denotes the annihilator of a set C.

Example B.1. A basic example of an affine subspace is a hyperplane in $\mathbb{R}^n$, given by a vector $x \in \mathbb{R}^n$ and a constant $c \in \mathbb{R}$ as

$$H = \{y \in \mathbb{R}^n,\ \langle x, y \rangle = c\}.$$

H is a proper affine subspace iff $x \neq 0$ and $c \neq 0$, we then may assume that $c = 1$ by properly normalizing x. In this case, $L_H = \{x\}^\perp$, $S_H = \mathbb{R}^n$ and $H^* = \{x\}$.

In general, an affine subspace $A \subseteq \mathbb{R}^n$ is the intersection of (finitely many) hyperplanes, which shows that A can be represented as the set of all solutions of a system of linear equations $My = b$, for some $m \times n$ matrix M and a vector $b \in \mathbb{R}^m$. Clearly, A is proper iff $b \neq 0$ and a solution of the system exists. In this case L_A is the space of solutions of the homogeneous system. Using a change of basis if necessary, we may assume that $b = (1, 0, \ldots, 0)$. Then S_A is the set of solutions of the homogeneous system $\tilde{M}y = 0$, where $\tilde{M}$ is the $(m-1) \times n$ matrix obtained from M by removing the first row, and the space A is the intersection of S_A by the hyperplane determined by the first row of M.

A simple geometric example is a straight line in $\mathbb{R}^3$, given by a point x and a vector u as

$$A = \{x + tu,\ t \in \mathbb{R}\}.$$

This affine subspace is proper iff $x \neq 0$. Here $L_A = \mathbb{R}u$ and S_A is the plane containing the origin and A. Any other plane containing A is determined as $\{z\}^*$ for some vector $z \perp u$ such that $\langle z, x \rangle = 1$, this gives the representation $A = S_A \cap \{z\}^*$.

The following lemma is easily proven.

Lemma B.1. *Let $C \subseteq V$ be any subset. Then C^* is an affine subspace in V^* and we have*

$$0 \in C^* \iff C = \emptyset, \qquad C^* = \emptyset \iff 0 \in \mathrm{Aff}(C).$$

Assume $C \neq \emptyset$ and $0 \notin \mathrm{Aff}(C)$. Then

(i) C^* is proper and we have $\mathrm{Lin}(C^*) = C^\perp = \mathrm{Span}(C)^\perp$,

(ii) $\mathrm{Aff}(C) = C^{**}$ and for any $c_0 \in C$, we have

$$\mathrm{Lin}(C) := \mathrm{Span}\{c_1 - c_2,\ c_1, c_2 \in C\} = \mathrm{Span}\{c - c_0,\ c \in C\} = \mathrm{Lin}(C^{**}).$$

Corollary B.2. *Let $A \subseteq V$ be a proper affine subspace. Then*

(i) A^* is a proper affine subspace in V^* and $A^{**} = A$.

(ii) $\mathrm{Lin}(A^*) = \mathrm{Span}(A)^\perp$, $\mathrm{Span}(A^*) = \mathrm{Lin}(A)^\perp$.

(iii) $\dim(A^*) = \dim(V) - \dim(A) - 1$.

The proper affine subspace A^* in the above Corollary will be called the affine dual of A. Note that the dual depends on the choice of the ambient vector space V.

B.1 A proof of Lemma 2.9 (ii)

Let us restate the statement (ii) of the lemma.

Lemma B.3. *Let Z be any object in* Af *and let F_0, F_1, F_2 be first order objects. Then we have*

$$(Z \multimap F_1) \multimap (F_0 \multimap F_2) = ((F_0 \multimap Z) \multimap F_1) \multimap F_2.$$

Proof. It is clear by the definition of $\multimap$ that we need to prove that

$$(Z \multimap F_1) \otimes F_0 \simeq (F_0 \multimap Z) \multimap F_1.$$

Put $X := F_0 \otimes (Z \multimap F_1)$ and $Y := (F_0 \multimap Z) \otimes F_1^*$, then it is enough to show that $X = Y^*$. We first check that $V_X = V_{F_0} \otimes V_Z^* \otimes V_{F_1} = V_{Y^*}$. Let now $w \in A_X$ and $v \in A_Y$. Since $A_{F_1^*} = \{\tilde{a}_{F_1}\}$, we have $v = u \otimes \tilde{a}_{F_1}$ for some $u \in A_{F_0 \multimap Z}$ and

$$\langle w, v \rangle = \langle w, u \otimes \tilde{a}_{F_1} \rangle = \langle (id \otimes \tilde{a}_{F_1})(w), u \rangle.$$

Using Lemma 2.9 (iii) we get $(id \otimes \tilde{a}_{F_1})(w) \in A_{F_0 \otimes Z^*}$, which implies that $\langle w, v \rangle = 1$. Hence $A_X \subseteq A_Y^*$. Note that, in fact, in any *-autonomous category, we have a canonical embedding $(A \multimap B) \otimes C \hookrightarrow (A^* \otimes C) \parr B \simeq (C \multimap A) \multimap B$.

It is now enough to prove that $d_X = d_{Y^*}$, or equivalently, $\dim(S_X) = \dim(S_{Y^*})$. We have

$$S_X = S_{F_0} \otimes S_{(Z \otimes F_1^*)^*} = V_{F_0} \otimes L^{\perp}_{Z \otimes F_1^*},$$

so that

$$\dim(S_X) = D_{F_0}(D_Z D_{F_1} - d_{Z \otimes F_1^*}) = D_{F_0} D_Z D_{F_1} - D_{F_0} d_Z,$$

where we used Lemma 2.1 and the fact that F_1 is first order, so that $d_{F_1^*} = 0$. Further, since $S_{Y^*} = L_Y^{\perp}$, we have

$$\dim(S_{Y^*}) = D_{F_0} D_Z D_{F_1} - d_Y = D_{F_0} D_Z D_{F_1} - d_{(F_0 \otimes Z^*)^*},$$

where we again have used the fact that F_1 is first order. We have

$$d_{(F_0 \otimes Z^*)^*} = \dim(S^{\perp}_{F_0 \otimes Z^*}) = D_{F_0} D_Z - D_{F_0} \dim(L_Z^{\perp}) = D_{F_0} D_Z - D_{F_0}(D_Z - d_Z) = D_{F_0} d_Z,$$

so that indeed $d_X = d_{Y^*}$.

□

C Labelled posets and type functions

We start by showing some basic properties of the reduced poset $\mathcal{P}_f^0$, $f \in \mathcal{T}_n$. By definition, $\mathcal{P}_f^0$ is obtained from $\mathcal{P}_f$ by removing any element that is not minimal and has an empty label, hence it is a poset whose elements are labelled by subsets in $[n]$. The elements in $\mathcal{P}_f^0$ will be denoted by capital letters $S, T, R, \ldots$, but they will not be viewed as subsets of $[n]$. The label set of T will be denoted by L_T. The order relation in $\mathcal{P}_f^0$ will be denoted as $\leq$.

The labelling in $\mathcal{P}_f^0$ has some immediate properties: if $S, T \in \mathcal{P}_f^0$, then $S \leq T$ implies that $L_S \cap L_T = \emptyset$ and if $S, T \in \mathrm{Min}(\mathcal{P}_f^0)$, then $L_S \subseteq L_T$ implies $S = T$. Also, the smallest element, if present, is the only element in $\mathcal{P}_f^0$ that may have an empty label set.

Lemma C.1. *If $\mathcal{P}_f^0$ has more than one element, then any $S \in \mathrm{Min}(\mathcal{P}_f^0)$ is covered by at least one element.*

Proof. We will proceed by induction on n. The assertion is clearly true for chains, so for $n \leq 3$. Assume it holds for all $m < n$ and let $f \in \mathcal{T}_n$. Assume that $S \in \mathrm{Min}(\mathcal{P}_f)$ is not covered by any element. If S is the least element, then $\mathcal{P}_f^0 = \{S\}$. Otherwise, $\mathcal{P}_f^0$ does not contain $\emptyset$ and has no largest element, so that $\emptyset \in \mathcal{P}_{f^*}^0$ and f^* has no free outputs.

If $f \approx f_1 \otimes f_2$, then $S = (S_1, S_2)$ for $S_i \in \mathrm{Min}(\mathcal{P}_{f_i}^0)$ and it is clear that both S_1 and S_2 must be coverless. By the induction assumption, $\mathcal{P}_{f_1}^0 = \{S_1\}$ and $\mathcal{P}_{f_2}^0 = \{S_2\}$, so that $\mathcal{P}_f^0 = \{(S_1, S_2)\}$.

Assume next that $f^* \approx f_1 \otimes f_2$. In this case, both $\mathcal{P}^0_{f_i}$ contain $\emptyset$ and f_i have no free outputs. Since S covers $\emptyset$ in $\mathcal{P}^0_{f^*}$, we must have the same situation in one of $\mathcal{P}^0_{f_i}$, that is, say in $\mathcal{P}^0_{f_1}$, there is some S_1 such that $\emptyset \ll S_1$ and $S = (S_1, \emptyset)$.

Note that S cannot be covered in $\mathcal{P}^0_{f^*}$ unless $[n] \in \mathcal{P}^0_{f^*}$, since any other cover in $\mathcal{P}^0_{f^*}$ would be also present in $\mathcal{P}^0_f$. But it is easy to see from Lemma 4.18 (ii) that this is impossible. Indeed, assume that $\mathcal{P}^0_{f^*}$ has a largest element T, then it must have one of the forms $T = (T_1, \emptyset)$ or $(\emptyset, T_2)$. But then, say, $(T_1, \emptyset) = T \geq (\emptyset, S_2)$ for any $S_2 \in \mathcal{P}^0_{f_2}$, we conclude that $\mathcal{P}_{f_2} = \mathcal{P}^0_{f_2} = \{\emptyset\}$, in which case f^* would have free outputs.

It follows that S has no cover in $\mathcal{P}^0_{f^*}$ and we must have the same situation for S_1 in $\mathcal{P}^0_{f_1}$. Again, $\mathcal{P}^0_{f_1}$ has no largest element, so that f_1^* has no free outputs. Since also f_1 has no free outputs, $\mathcal{P}^0_{f_1^*} = \mathcal{P}^0_{f_1} \setminus \{\emptyset\}$, so that S_1 is a minimal element in $\mathcal{P}^0_{f_1^*}$ that has no cover. By the induction assumption, $\mathcal{P}^0_{f_1^*} = \{S_1\}$, which is not possible with the assumed structure of $\mathcal{P}^0_{f_1}$.

□

Lemma C.2. *Any element $T \in \mathcal{P}^0_f$ can cover at most one minimal element.*

Proof. We will proceed by induction on n. Since the assertion is trivial for chains, it holds for $n \leq 3$. Assume it is true for $m < n$ and let $f \in \mathcal{T}_n$. Let T be an element that covers $T_1, \ldots, T_k \in \mathrm{Min}(\mathcal{P}^0_f)$, $k > 1$, so that we must have $\emptyset \notin \mathcal{P}^0_f$. Then T is not the largest element, otherwise by Lemma 4.17, T would be the largest element in $\mathcal{P}_f$ in which case the rank of f would be 1, which is not possible. It follows that T and $T_1, \ldots, T_k$ are all contained in $\mathcal{P}^0_{f^*}$.

Assume that $f \approx f_1 \otimes f_2$, then any T_i is of the form $T_i = (T_1^i, T_2^i)$, with $T_j^i \in \mathrm{Min}(\mathcal{P}^0_{f_j})$. If $i \neq i'$, we may assume that, say, $T_1^i \neq T_1^{i'}$. Since T covers both T_i and $T_{i'}$, we must have $T_2^i = T_2^{i'}$ and $T = (S, T_2^i)$ for some $S \in \mathcal{P}^0_{f_1}$ such that $T_1^i, T_1^{i'} \ll S$. By the induction assumption, this is not possible.

If $f^* \approx f_1 \otimes f_2$, then we may assume that there are $S, S_1, \ldots, S_k \in \mathcal{P}^0_{f_1}$ such that $\emptyset \ll S_i \ll S$. Then $S_1, \ldots, S_k$ are minimal elements in $\mathcal{P}^0_{f_1^*}$ which are either uncovered or all covered by S. The former case is excluded by Lemma C.1 and the latter by the induction assumption.

□

Lemma C.3. *Any minimal covering element in $\mathcal{P}^0_f$ is minimal covering in $\mathcal{P}_f$.*

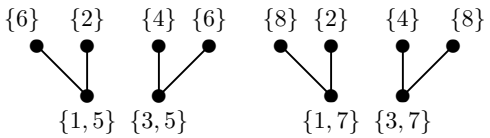
Proof. Let $S \in \mathcal{P}^0_f$ be a minimal covering element. By Lemma C.1, S covers exactly one minimal element $U \in \mathrm{Min}(\mathcal{P}^0_f)$. Assume that S is not minimal covering in $\mathcal{P}_f$. Since $U \in \mathrm{Min}(\mathcal{P}_f)$ by Lemma 4.17 and $U \leq S$ in $\mathcal{P}_f$, this can only happen if there is some $T \in \mathcal{P}_f$ such that $U \ll T \leq S$. In this case, $T \notin \mathcal{P}^0_f$, so that $L_T = \emptyset$. Since $T = \cup_{T' \leq T} L_{T'}$, this implies that there must be some $U' \neq U$ strictly below T and since $\mathcal{P}_f$ is graded, U' must be a minimal element and $U' \ll T$. This implies that $U' \leq S$, but since S cannot cover U' in $\mathcal{P}^0_f$, there must be, by Lemma C.2, some minimal covering $S' \in \mathcal{P}^0_f$ such that $U' \ll S' \leq S$. In $\mathcal{P}_f$, this implies that $U \leq T \leq S' \leq S$, which implies that $S' = S$ and hence S covers both U and U', a contradiction.

□

We now introduce the following relations. For $f \in \mathcal{T}_n$ and indices $i, j \in [n]$, we say that j covers i in f, in notation $i <_f j$, if there are some $S, T \in \mathcal{P}^0_f$, $S \ll T$, such that $i \in L_S$

and $j \in L_T$. We say that j strongly covers i in f, in notation $i \ll_f j$, if for all $T \in \mathcal{P}_f^0$ such that $j \in L_T$, there is some $S \in \mathcal{P}_f^0$ with $i \in S \ll T$.

To clarify these notions, consider the following diagram of the reduced poset $\mathcal{P}_f^0$ for the product of process matrices:



Here we have $1 \ll_f 2$, $5 <_f 2$ but $5 \not\ll_f 2$, and $2 \not<_f 3$.

We will say that an index $i \in [n]$ is minimal in f if $i \in U$ for some $U \in \mathrm{Min}(\mathcal{P}_f^0)$ and that $j \in [n]$ is minimal covering if $j \in L_T$ for some minimal covering element $T \in \mathcal{P}_f^0$. It is clear from Proposition 4.8 and Lemmas 4.17 and C.3 that any element $S \in \mathcal{P}_f^0$ is minimal (minimal covering) if its label set contains a minimal (minimal covering) index.

Lemma C.4. *If $\mathcal{P}_f^0$ is not a singleton, any minimal index i of f is strongly covered by some minimal covering index j.*

Proof. We will once more proceed by induction. The statement is quite obvious for chains, so for $n \leq 3$. Assume it holds for all $m < n$ and let $f \in \mathcal{T}_n$. Let $i \in U$ for some $U \in \mathrm{Min}(\mathcal{P}_f^0)$.

Let $f = f_1 \otimes f_2$. Then $U = (U_1, U_2)$ for $U_1 \in \mathrm{Min}(\mathcal{P}_{f_1}^0)$, $U_2 \in \mathrm{Min}(\mathcal{P}_{f_2}^0)$ and we may assume that $i \in U_1$. If $\mathcal{P}_{f_1}^0$ is a singleton, then $\mathcal{P}_{f_1}^0 = \mathcal{P}_{f_1} = \{U_1\}$. It follows that i is a free input of f, contained in any minimal element of $\mathcal{P}_f^0$ and it is easily seen that i is strongly covered in f by any minimal covering index of f. If $\mathcal{P}_{f_1}^0$ is not a singleton, then $i \ll_{f_1} j$ for some minimal covering index j of f_1, by the induction assumption. Let $T \in \mathcal{P}_f^0$, then $j \in L_T$ if and only if $T = (T_1, U_2)$ for $U_2 \in \mathrm{Min}(\mathcal{P}_{f_2}^0)$ and $j \in L_{T_1}$. Since $i \ll_{f_1} j$, there is some V_1 such that $i \in V_1$ and $V_1 \ll T_1$, so that $i \in (V_1, U_2) \ll T$.

Let now $f = (f_1 \otimes f_2)^*$. Since the statement is trivial if $\emptyset \in \mathcal{P}_f^0$ and easily seen to be true for free inputs, we may assume that f has no free inputs and that $\emptyset \notin \mathcal{P}_f^0$, which implies that f^* has no free inputs either. Then $f \simeq h \lhd \beta$ for a chain β and h and h^* have no free indices. Since by our assumptions $\mathcal{P}_f$ has no least element, we see from Lemmas C.1 and C.2 that there must be at least two minimal covering elements, which are mutually incomparable and hence they must all belong to $\mathcal{P}_h^0$, as well as all the minimal elements. If β is nontrivial, the statement follows from the induction assumption. If β is trivial, then $f = h$ and by Proposition 4.20, $\mathcal{P}_f^0 = \mathcal{P}_1 \uplus \mathcal{P}_2$, with $\mathcal{P}_k = \mathcal{P}_{f_k} \setminus \{\emptyset\}$, $k = 1, 2$. A minimal index i of f is contained in the labels of exactly one of the components, say $\mathcal{P}_1$, and is a label of a minimal element in $\mathcal{P}_1$, hence i is a minimal index of f_1^*. If $\mathcal{P}_{f_1^*}^0$ is a singleton, then $\mathcal{P}_1$ is a two-element chain, so that i is covered by all the labels of the top element T in $\mathcal{P}_1$, which are easily seen to be minimal covering indices of f. Moreover, any label set L_S of f containing some of these indices must be related to $\mathcal{P}_1$ and hence $S = T$.

If f_1^* is not a singleton, then by the induction assumption, $i \ll_{f_1^*} j$ for some minimal covering index j of f_1^*. By similar arguments as above, it can be seen that j is a minimal covering index of f and that $i \ll_f j$.

□

We now proceed towards the proof of Proposition 4.21. Let $f \in \mathcal{T}_n$ be such that f and f^* have no free indices and $\emptyset \notin \mathcal{P}_f$. Assume that $\mathcal{P}_f^0$ has no independent component, so that $f \approx f_1 \otimes \cdots \otimes f_k$ for some type functions $f_1, \ldots, f_k$ and $[n] = [n_1] \boxplus \cdots \boxplus [n_k]$.

Assume that this is a finest decomposition of this form, so that no f_l is a product. These assumptions will be kept throughout this section. Note that for any l, f_l cannot have free indices, since these would be also free indices of f.

By Lemma 4.18, any element in $\mathcal{P}_f^0$ can be written as $T = (T_1, \dots, T_k)$, with all $T_l \in \mathrm{Min}(\mathcal{P}_{f_l}^0)$ except possibly one index l_0. Application of a permutation is manifested only on the label sets.

Lemma C.5. *There is a decomposition $C_1, \dots, C_k$ of $[n]$ and a bijection $\varphi_l : [n_l] \to C_l$ such that for any $T = (T_1, \dots, T_k) \in \mathcal{P}_f^0$*

$$L_{(T_1,\dots,T_k)} = \begin{cases} \varphi_l(L_{T_l}) & \text{if } \exists\, l, T_l \notin \mathrm{Min}(\mathcal{P}_{f_l}) \\ \cup_l \varphi_l(L_{T_l}), & \text{otherwise.} \end{cases}$$

Proof. Let $\sigma \in \mathscr{S}_n$ be such that $f \circ \sigma = f_1 \otimes \cdots \otimes f_k$. By Corollary 4.7,

$$L_{(T_1,\dots,T_k),f} = \sigma^{-1}(L_{(T_1,\dots,T_k),\otimes_l f_l}) = \begin{cases} \sigma^{-1}(m_l + L_{T_l}) & \text{if } \exists\, l, T_l \notin \mathrm{Min}(\mathcal{P}_{f_l}) \\ \cup_l \sigma^{-1}(m_l + L_{T_l}), & \text{otherwise,} \end{cases}$$

where $m_l = \sum_{i=1}^{l-1} n_i$. Put $C_l = \sigma^{-1}(m_l + [n_l])$ and $\varphi_l(i) = \sigma^{-1}(m_l + i)$.

□

To ease the subsequent notations, we will replace the labels of $T_l \in \mathcal{P}_{f_l}^0$ with $\varphi_l(L_{T_l}) \subseteq C_l$. Any $i \in C_l$ is thus connected to f_l. We will refer to the inclusion of an index i in C_l as coloring i by a color $l \in \{1, \dots, k\}$.

Lemma C.6. *Assume that the coloring is known for all minimal and minimal covering indices of f. Then we can reconstruct all $\mathcal{P}_{f_l}^0$ from $\mathcal{P}_f^0$.*

Proof. Let $U \in \mathrm{Min}(\mathcal{P}_f^0)$, $U = (Z_1, \dots, Z_k)$ with $Z_l \in \mathrm{Min}(\mathcal{P}_{f_l})$. By the assumption, we know the coloring of any $i \in L_U = \cup_l L_{Z_l}$. Since L_{Z_l} is a label set of f_l if and only if $L_{Z_l} \subseteq C_l$, we obtain all label sets of minimal elements in $\mathcal{P}_{f_l}^0$.

For any $U = (Z_1, \dots, Z_k) \in \mathrm{Min}(\mathcal{P}_f)$, let

$$\tilde{L}_U^l := L_U \cap (\cup_{l' \neq l} C_{l'}) = \{i \in L_U, i \notin C_l\} = \cup_{l' \neq l} L_{Z_{l'}},$$

i.e. $\tilde{U}^l$ is the set of all indices in L_U of color different from l. It follows by the properties of the minimal label sets that $\tilde{L}_U^l \subseteq L_{U'}$ if and only if $\tilde{L}_U^l = \tilde{L}_{U'}^l$, so that $U' = (Z_1, \dots, Z_l', \dots, Z_k)$ for some $Z_l' \in \mathrm{Min}(\mathcal{P}_{f_l}^0)$.

Fix a minimal element U and consider the subposet in $\mathcal{P}_f^0$, given as

$$\mathcal{P}_l := \{U' \in \mathrm{Min}(\mathcal{P}_f^0),\ \tilde{L}_U^l \subseteq L_{U'}\}^{\uparrow}.$$

From Lemma 4.18 (ii) we see that after removing the minimal elements of $\mathcal{P}_l$, the poset decomposes into independent components, one of which corresponds to $\mathcal{P}_{f_l}^0$ with removed minimal elements. This component can be recognized by the labels of minimal covering elements (which are now minimal elements in the component), colored by l. To this component, we add back the minimal elements in $\mathcal{P}_l$, with the order relations as in $\mathcal{P}_f^0$, but removing the labels that are in $\tilde{L}_U^l$.

□

We now show how to obtain the coloring of labels of minimal and minimal covering elements.

Let $L_1, \dots, L_M$ be all the different label sets for minimal covering elements in $\mathcal{P}_f^0$ and let us denote $\mathbb{T}_a := \{T \in \mathcal{P}_f^0,\ L_T = L_a\}$, $a = 1, \dots, M$. Note that all $T \in \mathbb{T}_a$ are labeled by minimal covering indices and hence must be minimal covering elements. Also, by Lemma C.6, each such T covers exactly one minimal element U. By Lemma 4.18, we have $T = (T_1, \dots, T_k)$, where all $T_{l'}$ are minimal in $\mathcal{P}_{f_{l'}}$ except a single index l, for which T_l is minimal covering in $\mathcal{P}_{f_l}^0$. Moreover, $L_a = L_T = L_{T_l}$, in particular, all indices in L_a have the same color l.

We introduce the following sets. Let $\mathcal{U}$ be the set of minimal indices of f and

$$\mathcal{V}_a := \{i \in \mathcal{U},\ i \ll_f j \text{ for some } j \in L_a\}$$
$$\mathcal{W}_a := \{i \in \mathcal{U},\ i \not< _f j,\ \forall j \in L_a\}.$$

Then all indices in $\mathcal{V}_a \cup \mathcal{W}_a \cup L_a$ must have the same color. Indeed, we have already seen that $L_a \subseteq C_l$ for some l. Assume that a minimal index $i \in C_{l'}$ for $l' \neq l$. Since i cannot be a free input of $f_{l'}$, there must be some minimal elements $U_{l'}$ and $U'_{l'}$ in $\mathcal{P}_{f_{l'}}^0$ such that $i \in U_{l'}$ and $i \notin U'_{l'}$. Choose any $j \in L_a$ and let T_l be a minimal covering element in $\mathcal{P}_{f_l}^0$ such that $j \in L_{T_l}$. Let also $U_j \in \mathrm{Min}(\mathcal{P}_{f_j}^0)$ for $j \neq l'$.

Put $T = (U_1, \dots, U_{l'}, \dots, T_l, \dots U_k)$ and $T' = (U_1, \dots, U'_{l'}, \dots, T_l, \dots U_k)$. Then $j \in L_T = L_{T'}$ and $i \in (U_1, \dots, U_k) \ll T$, so that $i <_f j$, while T' cannot cover any element containing i, so that $i \not\ll_f j$. Hence $p \notin \mathcal{V}_i \cup \mathcal{W}_i$.

For all a, let us denote $C'_a := \mathcal{V}_a \cup \mathcal{W}_a \cup L_a$ and define $a \sim b$ if $C'_a \cap C'_b \neq \emptyset$ or if there are some $T \in \mathbb{T}_a$ and $S \in \mathbb{T}_b$ that have a common upper bound in $\mathcal{P}_f^0$. Take the transitive closure of this relation (also denoted by $\sim$). We next prove several claims:

Claim 1. For any $i \in \mathcal{U}$, there is some $a \in [M]$ such that $i \in \mathcal{V}_a$, so that all minimal indices are in some set C'_a. Indeed, by Lemma C.4, any $i \in \mathcal{U}$ is strongly covered by some minimal covering element j, which must belong to some L_a.

Claim 2. If $a \sim b$, then $L_a, L_b \in C_l$ for some l. To see this, assume that a and b are such that $C'_a \cap C'_b \neq \emptyset$. Since $C'_a \subseteq C_l$ if $L_a \subseteq C_l$ and C_l and $C_{l'}$ are disjoint for $l \neq l'$, this implies that L_a, L_b belong to the same component. Similarly, if there are some $S \in \mathbb{T}_a$, $T \in \mathbb{T}_b$ with a common upper bound, then L_a and L_b must be connected to the same function f_l. Hence both relations imply that $L_a, L_b \subseteq C_l$ for some l, and this is clearly preserved by the transitive closure.

Claim 3. If $\emptyset \notin \mathcal{P}_{f_l}$ and $L_a, L_b \subseteq C_l$, then $a \sim b$. Assume first that $\mathcal{P}_{f_l}^0$ has a largest element, then we can easily construct elements $S \in \mathbb{T}_a$ and $T \in \mathbb{T}_b$ that have a common upper bound. Hence $a \sim b$. Assume that $\mathcal{P}_{f_l}$ has no largest element, then one can check that f_l and f_l^* have no free indices. Since the decomposition $f = f_1 \otimes \cdots \otimes f_k$ is the finest decomposition of f as a product, f_l cannot be a product. By Proposition 4.20, we obtain that $\mathcal{P}_{f_l}^0$ must have independent components, $\mathcal{P}_{f_l}^0 = \mathcal{P}_1 \uplus \mathcal{P}_2$. Note that none of the $\mathcal{P}_i$ can be a singleton by Lemma C.2, so that either of the components has at least one of the label sets $L_{a'}$ associated with it. Assume that L_a belongs to $\mathcal{P}_1$, then any minimal index i of f_l such that $i <_f j$ for $j \in L_a$ must belong to $\mathcal{P}_1$. Hence $\mathcal{W}_a$ contains all minimal indices of $\mathcal{P}_2$. In particular, $\mathcal{V}_{a'} \subseteq \mathcal{W}_a$ for all $L_{a'}$ in $\mathcal{P}_2$, so that $a \sim a'$ for all such a'. If L_b is connected with $\mathcal{P}_2$, we are done, otherwise we have $a \sim a' \sim b$ by repeating the argument with a replaced by a'.

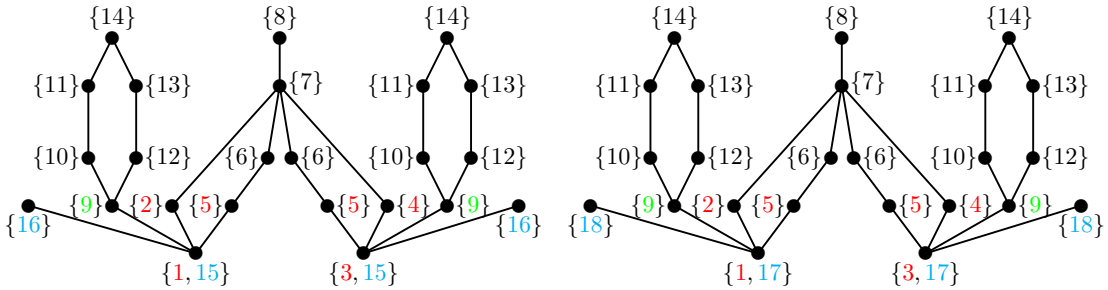
Claim 4. If $\emptyset \in \mathcal{P}_{f_l}$, and $L_a \subseteq C_l$, then $L_a = C'_a$ (so that $\mathcal{V}_a = \mathcal{W}_a = \emptyset$) and $a \not\sim b$ for any b with $\mathcal{V}_b$ or $\mathcal{W}_b$ nonempty. If L_a and L_b are two such label sets, then they are

connected with the same f_l if and only if they are in the same independent component of the labelled poset $U^{\uparrow} \setminus \{U\}$, here U is an arbitrary minimal element in $\mathrm{Min}(\mathcal{P}_f^0)$. Indeed, the first statement is clear from the definition of $\mathcal{V}_a$ and $\mathcal{W}_a$, and Claim 2. For the second statement, note that by Lemma 4.18, $U^{\uparrow} \setminus \{U\}$ contains $\mathcal{P}_{f_l}^0 \setminus \{\emptyset\}$ for any $\mathcal{P}_{f_l}$ containing $\emptyset$ as one of its independent components. If $\mathcal{P}_{f_l}^0$ has a largest element, then $\mathcal{P}_{f_l}^0 \setminus \{\emptyset\}$ cannot have independent components. Otherwise we have $\mathcal{P}_{f_l}^0 \setminus \{\emptyset\} = \mathcal{P}_{f_l^*}^0$ (since f_l has no free outputs) and since f_l is not a product, $\mathcal{P}_{f_l^*}^0$ cannot have any independent components by Proposition 4.20.

Proof of Proposition 4.21. Let C'_a, $a = 1, \ldots, M$ be as described above. Assume that the equivalence relation $\sim$ has k' equivalence classes, then pick some of the colors $1, \ldots, k'$ for each equivalence class $[a]_\sim$ and use it for all indices in $\cup_{b \in [a]_\sim} C'_b$. Take all L_a such that $\mathcal{V}_a = \mathcal{W}_a = \emptyset$ that are not related to any b with nonempty $\mathcal{V}_b$ or $\mathcal{W}_b$, and use the procedure described in Claim 4 to merge some of the equivalence classes if necessary. Claims 1-4 show that for all label indices for minimal and minimal covering elements, we obtain a coloring that matches the decomposition of f as a tensor product $f \approx f_1 \otimes \cdots \otimes f_k$. Using Lemma C.6, we get all the labelled posets $\mathcal{P}_{f_l}^0$, with labels transformed by the bijections φ_l. Applying any bijection $C_l \to [n_j]$ on the label sets, we obtain $\mathcal{P}_{f_l}^0$ up to a permutation of the labels.

□

Example C.1. The following example shows the diagram of $\mathcal{P}_f^0$ for a function $f \in \mathcal{T}_{18}$:

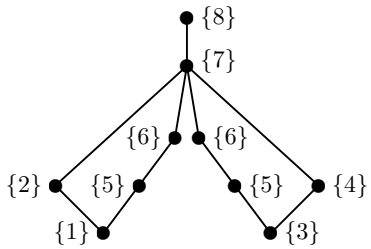


It is clear that f has no free outputs, since all indices in $[18]$ are contained in the label sets. Also f^* has no free output, since $\mathcal{P}_f^0$ has no largest element. It is also clear that f and f^* have no free inputs, and $\emptyset \notin \mathcal{P}_f^0$. The poset consists of two direct summands, but the labels are repeating, so there are no independent components. Hence $f \approx f_1 \otimes \cdots \otimes f_k$ and the assumptions of Proposition 4.21 are satisfied. We will now show how to find the functions using the procedure described in the proof of Proposition 4.21. There are 6 different labels of the minimal covering elements. The label sets L_a and the sets $\mathcal{V}_a$ and $\mathcal{W}_a$ are shown in the following table:

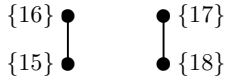
a	1	2	3	4	5	6
L_a	$\{2\}$	$\{4\}$	$\{5\}$	$\{9\}$	$\{16\}$	$\{18\}$
$\mathcal{V}_a$	$\{1\}$	$\{3\}$	$\emptyset$	$\emptyset$	$\{15\}$	$\{17\}$
$\mathcal{W}_a$	$\{3\}$	$\{1\}$	$\emptyset$	$\emptyset$	$\{17\}$	$\{15\}$

It is immediately clear that $1 \sim 2$ and $5 \sim 6$, since $C'_1 \cap C'_2 \neq \emptyset$ and $C'_5 \cap C'_6 \neq \emptyset$. We also have $1 \sim 2 \sim 3$, since vertices labeled by $L_1 = \{2\}$, $L_2 = \{4\}$ and $L_3 = \{5\}$ have a common upper bound (labelled by $\{7\}$). It follows that we have three color classes: $C_1 = \{1, 2, 3, 4, 5\}$ (colored red in the diagram), $C_2 = \{15, 16, 17, 18\}$ (colored cyan) and $C_3 = \{9\}$ (colored green). Since C_3 does not contain any labels of minimal elements, the

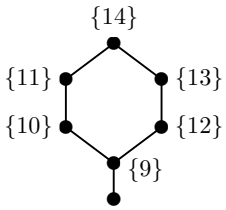
corresponding function f_3 satisfies $\emptyset \in \mathcal{P}^0_{f_3}$. We now apply Lemma C.6 to obtain $\mathcal{P}^0_{f_1}$. Let $U = \{1, 15\}$, then $\tilde{L}^1_U = \{15\}$ and $\mathcal{P}_1 = \{U' \in \mathrm{Min}(\mathcal{P}^0_f), 15 \in L_{U'}\}^\uparrow$ is then the left half of the diagram. Continuing similarly for C_2 and C_3, we obtain the following three factors of $\mathcal{P}^0_f$:



$((\gamma_2 \otimes \gamma_2)^* \otimes \gamma_2) \lhd \gamma_2$



$(\gamma_2 \otimes \gamma_2)^*$



$(\gamma_2 \otimes \gamma_4^*)^*$

The first factor here corresponds to a more complicated type (see Example 4.14 (c)), while the second and third factors are the types of process matrices (Examples 4.8, 4.12 and 2.4) and process matrices with global past and future (Examples 4.10 and 4.15).