Dicke subsystems are entangled

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We show that all reduced states of nonproduct symmetric Dicke states of arbitrary number of qu*d*its are genuinely multipartite entangled, and of nonpositive partial transpose with respect to any subsystem.

Entanglement is the most remarkable manifestation of the nonclassical correlations in quantum systems [1, 2], still puzzling the community for almost hundred years [3]. Deciding if a mixed quantum state is entangled or separable is a notoriously difficult problem, most of the known criteria are necessary but not sufficient for separability [2, 4, 5]. A particularly strong separability criterion is the Peres-Horodecki, or partial transpose criterion [6, 7]. It states that if a quantum state is separable then its partial transposition is positive semidefinite (PPT), or contrapositively, if the partial transposition of a quantum state is not positive semidefinite (NPT) then the state is entangled. Since the reverse implication is not true in general, there are also entangled states of positive partial transpose [5, 8–11]. This PPT-entanglement is considered to be a weaker form of entanglement.

A bipartite quantum state can either be separable or entangled [1, 2, 12], while in the multipartite case entanglement shows a rich structure with many exciting features [13, 14]. A natural generalization of the separable/entangled dichotomy to multipartite systems is the partial separability classification [15–20], including the notions of *partition-separability*, *k-separability*, *k-producibility*, *k-stretchability* [19, 20], *biseparability* or *genuine multipartite entanglement* (GME). Deciding if a mixed quantum state possesses a particular multipartite entanglement property is an even more difficult problem than in the bipartite case.

A particularly interesting problem in the multipartite setting is that the entanglement inside the subsystems does not follow from the entanglement of the whole system. The paradigmatic example of this is given by the three-qubit GHZ state $\big(|000\rangle+|111\rangle\big)/\sqrt{2}$ and W state $\big(|001\rangle+|010\rangle+|100\rangle\big)/\sqrt{3}$, both of which are fully entangled, but the two-qubit subsystems of the GHZ state are separable, while those of the W state are entangled [21]. Symmetric Dicke states are the generalizations of the W state, and our result is the generalization of this.

In this work we consider mixed quantum states arising as reduced states of pure multiqu*d*it symmetric Dicke states, and we show that these states are genuinely multipartite entangled, moreover, this entanglement is also NPT (not PPT) with respect to any subsystem. This holds for all nontrivial cases, i.e., when there are at least two nonzero occupations in the Dicke state of the whole system.

Dicke states originally appeared in quantum optics [22], and later, thanks to their simple yet interesting structure, became widely used examples and tools in the theory of multipartite entanglement. Multipartite entanglement criteria in the vicinity of Dicke states were formulated [23–26], also based on quantum metrology [27–30], which made possible the experimental detection of multipartite entanglement in photonic [31] or cold atomic systems [32–35]. Entanglement was also characterized in terms of different entanglement measures in pure Dicke states and even in the mixtures of symmetric Dicke states in the qubit [36, 37] and also the qu*d*it [38–42] cases. Dicke states are also important examples in tomography [43]. Dicke states of small numbers of qubits were also prepared directly in quantum optical experiments [44–47], different methods of preparation were worked out also for quantum computers [48–51], and even the matrix product state form of qu*d*it Dicke states could be derived explicitly [52].

Let us recall first some basic notions in *partial separability* [15–18]. A quantum state ρ of a multipartite system S is *ξ-separable*, that is, separable with respect to a partition $\xi=\{X_1,X_2,\ldots\}$, if it is a statistical mixture of states being product with respect to that partition,

$$\rho=\sum_i w_i \bigotimes_{X\in\xi}\rho_{X,i}, \tag{1}$$

where the finite number of weights w_i are nonnegative and sum up to 1, $\rho_{X,i}$ are states of subsystems $X\subseteq S$, which are disjoint and cover the whole system S. We call a state *fully separable*, if it is ξ-separable and each $X\in\xi$ is an elementary subsystem; *partition-separable*, if it is ξ-separable with respect to a nontrivial partition ξ (containing at least two parts); and *bipartition-separable*, if it is ξ-separable with respect

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to a bipartition ξ (containing exactly two parts). It is clear that if a state is partition-separable then it is also bipartition-separable, since it is separable with respect to all bipartitions $\upsilon = \{Y, \overline{Y}\}$ *coarser* than ξ, that is, the subsystems Y and $\overline{Y} = S \setminus Y$ are unions of subsystems $X \in \xi$. This is because we are always allowed not to take into account some of the tensor-product symbols [18], then the state ρ above can be recast as

$$\sum_i w_i \Big(\bigotimes_{\substack{X\in\xi \\ X\subseteq Y}} \rho_{X,i} \Big) \otimes \Big(\bigotimes_{\substack{X\in\xi \\ X\subseteq \overline{Y}}} \rho_{X,i} \Big) = \sum_i w_i \rho_{Y,i} \otimes \rho_{\overline{Y},i}. \tag{2}$$

A state is *biseparable*, if it is the mixture of (bi)partition-separable states, that is, mixture of ξ-separable states for *possibly different* nontrivial partitions ξ. *Genuinely multipartite entangled* (GME) states are those which are not biseparable. In this multipartite case, the *Peres-Horodecki criterion* [6, 7] is about bipartition-separability,

$$\rho \text{ is } \{Y, \overline{Y}\}\text{-sep.} \implies \rho^{\mathrm{T}_Y} \geq 0, \tag{3}$$

where the *partial transpose* T_Y is the linear map given on elementary tensors as $(A_Y \otimes B_{\overline{Y}})^{\mathrm{T}_Y} = A_Y^{\mathrm{T}} \otimes B_{\overline{Y}}$, where T is the *transpose* of the matrix of the operator A_Y in a fixed basis, that is, $\langle i|A_Y^{\mathrm{T}}|j\rangle = \langle j|A_Y|i\rangle$. Although the map T_Y is given with respect to a fixed local basis, the positivity of ρ^{T_Y} and therefore the criterion (3) are independent of this choice.

We consider *symmetric* state vectors[1], contained in the symmetric subspace of the Hilbert space [53–55]. The classification of multipartite entanglement simplifies significantly for these states, namely, these are either fully separable or GME [54, 56–58]. This is because every pure decomposition of a symmetric state consists of symmetric pure states, which are known to be either fully separable or fully entangled. Consequently, a symmetric state entangled with respect to any bipartition is GME, and a symmetric state separable with respect to any bipartition is fully separable. In this symmetric multipartite case, the Peres-Horodecki criterion (3) takes the form

$$\text{symm. } \rho \text{ is fully-sep.} \implies \forall Y \subset S : \rho^{\mathrm{T}_Y} \geq 0, \tag{4a}$$

or, contrapositively,

$$\text{symm. } \rho \text{ is GME} \impliedby \exists Y \subset S : \rho^{\mathrm{T}_Y} \not\geq 0, \tag{4b}$$

making possible to identify NPT-GME states directly. Note that the reverse implications still do not hold, and symmetric PPT-GME states were also constructed [54].

For $n \geq 1$, the n-qu*d*it symmetric Dicke state vectors are the equal weight superpositions of the permutations of the $|i_1, i_2, \ldots, i_n\rangle$ elements of a fixed local (tensor product) basis. (We number the basis vectors from $i = 1$ up to d for simplicity.) It is convenient to label the n-qu*d*it symmetric Dicke states with *excitation indices*, also called *occupation numbers*, which are *multiindices* $\boldsymbol{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$, being normalized, $||\boldsymbol{n}|| := \sum_{i=1}^d n_i = n$. Note that $n_i = 0$ is also allowed. Let us have the set of the possible occupation numbers

$$I_n^d := \big\{ \boldsymbol{n} \in \mathbb{N}_0^d \;\big|\; ||\boldsymbol{n}|| = n \big\}. \tag{5}$$

For such a multiindex $\boldsymbol{n} \in I_n^d$, let us have the (non-normalized) Dicke vector

$$|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle := |\underbrace{11\ldots1}_{n_1}\underbrace{22\ldots2}_{n_2}\ldots\underbrace{dd\ldots d}_{n_d}\rangle + \text{`perms.'}, \tag{6}$$

where taking all the possible *different* orderings of the basis vectors is understood. The norm-square of this is the multinomial coefficient $||\tilde{\mathrm{D}}_{\boldsymbol{n}}||^2 = \langle \tilde{\mathrm{D}}_{\boldsymbol{n}}|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \binom{n}{\boldsymbol{n}} = \frac{n!}{\prod_{i=1}^d n_i!}$, leading to the (normalized) *n-qudit symmetric Dicke state vector of occupation* $\boldsymbol{n} \in I_n^d$,

$$|\mathrm{D}_{\boldsymbol{n}}\rangle := \binom{||\boldsymbol{n}||}{\boldsymbol{n}}^{-1/2} |\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle. \tag{7}$$

The vectors $|\tilde{\mathrm{D}}\rangle_{\boldsymbol{n}}$ and $|\mathrm{D}_{\boldsymbol{n}}\rangle$ are also called elementary symmetric tensors and symmetric basis states [40], respectively, since $\big\{ |\mathrm{D}_{\boldsymbol{n}}\rangle \;\big|\; \boldsymbol{n} \in I_n^d \big\}$ is an orthonormal set, $\langle \mathrm{D}_{\boldsymbol{n}}|\mathrm{D}_{\boldsymbol{n}'}\rangle = \delta_{\boldsymbol{n},\boldsymbol{n}'}$, and those span the symmetric subspace of the Hilbert space of the n-partite composite system [53]. The dimension of the symmetric subspace is the $|I_n^d|$ number of the possible occupation numbers, which is given by the binomial coefficient as $|I_n^d| = \binom{n+d-1}{d-1} = \binom{n+d-1}{n} = \frac{(n+d-1)!}{n!(d-1)!}$, coming from elementary combinatorics (see the 'stars and bars problem' [59]). Note that we also cover the extreme case of one single system, $n = 1$, then it is just the computational basis itself, that is, say $n_i = 1$ and $n_{j\neq i} = 0$ for an i, then $|\mathrm{D}_{\boldsymbol{n}}\rangle = |i\rangle$. Note also that if there is only one nonzero occupation in a composite system, say $n_i = n$ and $n_{j\neq i} = 0$ for an i, then $|\mathrm{D}_{\boldsymbol{n}}\rangle = |i, i, \ldots, i\rangle$, which is fully separable.

For any $1 \leq m < n$, the nonnormalized Dicke vectors (6) can be decomposed into two subsystems of sizes m and $n - m$ as

$$|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \sum_{\boldsymbol{m}\in I_{m,\boldsymbol{n}}^d} |\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle, \tag{8}$$

[1] Symmetric state vectors are state vectors $|\phi\rangle$ for which $P_n|\phi\rangle = |\phi\rangle$, where $P_n = \frac{1}{n!}\sum_{\sigma\in \mathrm{S}_n} R_\sigma$ is the projection operator, projecting onto the symmetric subspace, where R_σ is the representation of the symmetric group S_n on the multipartite Hilbert space, permuting the elementary subsystems. The m-partite reduced states $\rho_m = \mathrm{Tr}_{n-m}(|\phi\rangle\langle\phi|)$ of an n-partite symmetric pure state $|\phi\rangle\langle\phi|$ given by the symmetric state vector $|\phi\rangle$ is a *symmetric operator*, that is, it acts only on the symmetric subspace, $P_m\rho_m P_m = \rho_m$. (Not to be confused with *permutation invariant* operators, defined as $R_\sigma A R_\sigma^\dagger = A$.) All the possible pure convex decompositions of a symmetric state contain symmetric pure states given by symmetric state vectors. (Note that the convex decompositions of permutation invariant states is a more difficult problem, the extremal points are not necessarily pure states.) See also Refs. [53–55].

where the summation runs over the multiindices in the restricted index set

$$I^d_{m,\boldsymbol{n}} := \left\{\boldsymbol{m} \in \mathbb{N}_0^d \;\middle|\; ||\boldsymbol{m}|| = m, \boldsymbol{m} \leq \boldsymbol{n}\right\} \subseteq I^d_m, \quad (9)$$

where the relation $\leq$ is understood elementwisely. $I^d_{m,\boldsymbol{n}}$ is the intersection of a rectangular hypercuboid specified by $\boldsymbol{0} := (0, 0, \ldots, 0)$ and $\boldsymbol{n}$ and the hyperplane $||\boldsymbol{m}|| = m$, which is difficult to walk through sequentially if $d \geq 3$. (We provide a proof of the decomposition (8) in the Appendix. For earlier proofs, see Sec. 3 of Ref. [39] or Appendix A of Ref. [60].) Then the (8)-like decomposition of the Dicke state vector (7) is

$$|\mathrm{D}_{\boldsymbol{n}}\rangle = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \sqrt{\eta^{\boldsymbol{n}}_{\boldsymbol{m}}} |\mathrm{D}_{\boldsymbol{m}}\rangle \otimes |\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}}\rangle, \quad (10a)$$

where

$$\eta^{\boldsymbol{n}}_{\boldsymbol{m}} := \frac{\binom{||\boldsymbol{m}||}{\boldsymbol{m}}\binom{||\boldsymbol{n}-\boldsymbol{m}||}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{||\boldsymbol{n}||}{\boldsymbol{n}}} = \eta^{\boldsymbol{n}}_{\boldsymbol{n}-\boldsymbol{m}}. \quad (10b)$$

Since the Dicke state vectors of the subsystems $\left\{|\mathrm{D}_{\boldsymbol{m}}\rangle \;\middle|\; \boldsymbol{m} \in I^d_m\right\}$ and $\left\{|\mathrm{D}_{\boldsymbol{m}'}\rangle \;\middle|\; \boldsymbol{m}' \in I^d_{n-m}\right\}$ form orthonormal bases, the formula (10a) is just the Schmidt decomposition of the $|\mathrm{D}_{\boldsymbol{n}}\rangle$ Dicke state vector with the Schmidt coefficients $\eta^{\boldsymbol{n}}_{\boldsymbol{m}}$ (10b) and Schmidt rank $|I^d_{m,\boldsymbol{n}}|$ (9). The reduced states[2] of subsystems of size m are then

$$\rho_{\boldsymbol{n},m} := \mathrm{Tr}_{n-m}\left(|\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}|\right) = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} |\mathrm{D}_{\boldsymbol{m}}\rangle\langle\mathrm{D}_{\boldsymbol{m}}|. \quad (11)$$

(It also follows that $\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} = 1$, being just $\mathrm{Tr}(\rho_{\boldsymbol{n},m})$, which is the multinomial generalization of the Vandermonde identity for binomials [59].) For later use, we would like to cover also the $m = n$ trivial reduction, then (10a) does not make sense, but (11) still holds with $I^d_{n,\boldsymbol{n}} = \{\boldsymbol{n}\}$ and $\eta^{\boldsymbol{n}}_{\boldsymbol{n}} = 1$ by the original definitions (9) and (10b), noting that $0! = 1$, and $\rho_{\boldsymbol{n},n} = |\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}|$ (11), as it has to be.

For example, for qubits ($d = 2$), the second component of the $\boldsymbol{n} = (n_1, n_2) =: (n-e, e)$ occupation number multiindex, considered as the number of excitations $|2\rangle$ over the ground state $|1\rangle$, is usually used to label the states [4, 36]. Then the multinomial coefficients boil down to the binomial ones, $\binom{n}{\boldsymbol{n}} = \binom{n}{e} = \frac{n!}{e!(n-e)!}$, and $|\mathrm{D}^n_e\rangle := |\mathrm{D}_{(n-e,e)}\rangle = \binom{n}{e}^{-1/2}(|\underbrace{11\ldots1}_{n-e}\underbrace{22\ldots2}_{e}\rangle + \text{`perms.'})$. The Schmidt decomposition (10a) in the qubit case is then $|\mathrm{D}^n_e\rangle = \sum_{l=l_{\min}}^{l_{\max}} \sqrt{\binom{m}{l}\binom{n-m}{e-l}/\binom{n}{e}} |\mathrm{D}^m_l\rangle \otimes |\mathrm{D}^{n-m}_{e-l}\rangle$, where l is the second component of the $\boldsymbol{m} = (m-l, l)$ multiindex in (8), by which one could walk through the index

[2]The reduced states of all subsystems of a given size m are of the same form, because of the permutation symmetry of the vector (6). Accordingly, Tr_{n-m} simply denotes partial trace over any subsystem of size $n-m$.

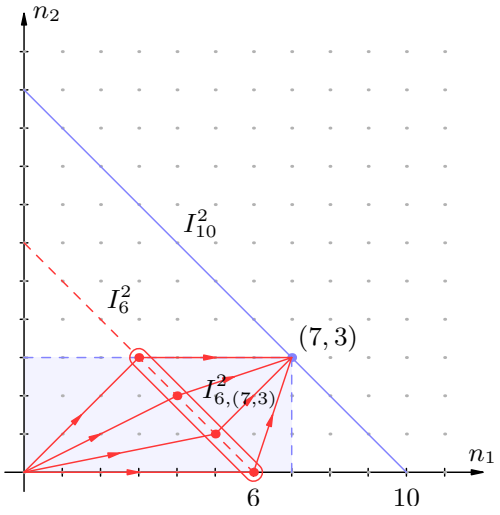


Figure 1: Example of the index sets I^d_n (5) and $I^d_{m,\boldsymbol{n}}$ (9) for $n = 10$ qubits ($d = 2$), of occupation $\boldsymbol{n} = (7, 3)$ for subsystem size $m = 6$. Arrows illustrate the occupation number multiindices $\boldsymbol{m}$ and $\boldsymbol{n} - \boldsymbol{m}$, appearing in the decompositions (8) and (10a), adding up to $\boldsymbol{n}$.

set (9) sequentially, $I^2_{m,(n-e,e)} = \left\{(m-l, l) \;\middle|\; l_{\min} \leq l \leq l_{\max}\right\}$, where $l_{\min} = \max\{0, e-(n-m)\}$ and $l_{\max} = \min\{m, e\}$. (For illustrations, see Figure 1.) As a concrete example, the doubly excited three-qubit Dicke state is $|\mathrm{D}^3_2\rangle = |\mathrm{D}_{(1,2)}\rangle = \frac{1}{\sqrt{3}}\left(|122\rangle + |212\rangle + |221\rangle\right)$, which is equivalent to the W state [21]. We can also illustrate the Schmidt decomposition (10a) of this for $m = 2$ as $|\mathrm{D}^3_2\rangle = \sqrt{2/3}|\mathrm{D}^2_1\rangle \otimes |\mathrm{D}^1_1\rangle + \sqrt{1/3}|\mathrm{D}^2_2\rangle \otimes |\mathrm{D}^1_0\rangle$, which is much more expressive with the general multiindex labeling, $|\mathrm{D}_{(1,2)}\rangle = \sqrt{2/3}|\mathrm{D}_{(1,1)}\rangle \otimes |\mathrm{D}_{(0,1)}\rangle + \sqrt{1/3}|\mathrm{D}_{(0,2)}\rangle \otimes |\mathrm{D}_{(1,0)}\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) \otimes |2\rangle + |22\rangle \otimes |1\rangle\right)$, which is indeed $\frac{1}{\sqrt{3}}\left(|122\rangle + |212\rangle + |221\rangle\right)$.

Let us consider now the n-partite Dicke state $|\mathrm{D}_{\boldsymbol{n}}\rangle$ ($n \geq 2$) for an arbitrary occupation number multiindex $\boldsymbol{n} \in I^d_n$. We are interested in the bipartite entanglement inside its m-partite subsystems ($2 \leq m \leq n$), with respect to the split into k and $(m-k)$-partite subsystems ($1 \leq k \leq m-1$). We have already seen the rather special property of Dicke states that their Schmidt vectors are also Dicke states (10a), helping our derivations a lot. Exploiting this, the reduced state (11) takes the form

$$\rho_{\boldsymbol{n},m} = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \sum_{\boldsymbol{k},\boldsymbol{k}'\in I^d_{k,\boldsymbol{m}}} \sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}'}} \; |\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}| \otimes |\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}|, \quad (12)$$

convenient for the calculation of the partial transpose. To detect entanglement (4), we have to confirm the nonpositivity of the partial transpose[3] $\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}$, which

[3]The partial transposed states in all subsystems of a given size k are of the same form, because of the permutation symmetry of the vector (6). Accordingly, T_k simply denotes partial transpose over any subsystem of size k.

holds if we find a vector $|\psi\rangle$ giving $\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle < 0$. For the role of $|\psi\rangle$, let us have the educated guess

$$|\psi\rangle := \alpha|\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle \otimes |\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle + \beta|\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle \otimes |\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle \quad (13)$$

with free parameters $\hat{\boldsymbol{m}} \in I^d_m$, $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$ and $\alpha, \beta \in \mathbb{C}$. Noting that $\big(|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}|\big)^{\mathrm{T}} = |\mathrm{D}_{\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{k}}|$ in the computational basis, we calculate the sandwich $\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle$ with the state (12) and the ansatz (13). For this first we calculate the sandwich $\langle\psi|\big(|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}| \otimes |\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\big)|\psi\rangle$ for the partial transpose of the basis part of the state (12),

$$\begin{aligned}
&\big(\overline{\alpha}\langle\mathrm{D}_{\hat{\boldsymbol{k}}}| \otimes \langle\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}| + \overline{\beta}\langle\mathrm{D}_{\hat{\boldsymbol{k}}'}| \otimes \langle\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}|\big)\Big(|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}| \otimes |\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\Big)\big(\alpha|\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle \otimes |\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle + \beta|\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle \otimes |\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle\big)\\
&= \overline{\alpha}\alpha\langle\mathrm{D}_{\hat{\boldsymbol{k}}}|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}|\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle\langle\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}|\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle + \overline{\alpha}\beta\langle\mathrm{D}_{\hat{\boldsymbol{k}}}|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}|\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle\langle\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}|\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle\\
&+ \overline{\beta}\alpha\langle\mathrm{D}_{\hat{\boldsymbol{k}}'}|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}|\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle\langle\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}|\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle + \overline{\beta}\beta\langle\mathrm{D}_{\hat{\boldsymbol{k}}'}|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}|\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle\langle\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}|\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle.
\end{aligned}$$

To obtain $\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle$, we have to deal with the summations in the state (12) and the Kronecker-δs, coming from the orthonormality of the Dicke state vectors above. (Remember that $\boldsymbol{m}, \boldsymbol{k}, \boldsymbol{k}'$ are summation indices in the state (12), and $\hat{\boldsymbol{m}}, \hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}'$ are free-to-choose parameters of the ansatz (13).)
For the first term, we have the factors $\delta_{\hat{\boldsymbol{k}},\boldsymbol{k}}\delta_{\boldsymbol{k}',\hat{\boldsymbol{k}}}\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}',\boldsymbol{m}-\boldsymbol{k}'}\delta_{\boldsymbol{m}-\boldsymbol{k},\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}$, this is nonzero iff $\boldsymbol{k} = \boldsymbol{k}' = \hat{\boldsymbol{k}}$ and $\boldsymbol{m} = \hat{\boldsymbol{m}} - \hat{\Delta}$, where $\hat{\boldsymbol{\Delta}} = \hat{\boldsymbol{k}}' - \hat{\boldsymbol{k}}$. That is, the parameters $\hat{\boldsymbol{m}}, \hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}'$ are needed to be such that $\hat{\boldsymbol{m}} - \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}$ and $\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}$ to meet the indices $\boldsymbol{m}, \boldsymbol{k}, \boldsymbol{k}'$ in the sum to make this term to be nonzero.
For the second term, we have the factors $\delta_{\hat{\boldsymbol{k}},\boldsymbol{k}}\delta_{\boldsymbol{k}',\hat{\boldsymbol{k}}'}\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}',\boldsymbol{m}-\boldsymbol{k}'}\delta_{\boldsymbol{m}-\boldsymbol{k},\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}$, this is nonzero iff $\boldsymbol{k} = \hat{\boldsymbol{k}}$, $\boldsymbol{k}' = \hat{\boldsymbol{k}}'$ and $\boldsymbol{m} = \hat{\boldsymbol{m}}$. That is, the parameters $\hat{\boldsymbol{m}}, \hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}'$ are needed to be such that $\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}$ and $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$ to meet the indices $\boldsymbol{m}, \boldsymbol{k}, \boldsymbol{k}'$ in the sum to make this term to be nonzero.
The third and the fourth terms can be dealt with similarly, and finally we have

$$\begin{aligned}
\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle &= \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \sum_{\boldsymbol{k},\boldsymbol{k}'\in I^d_{k,\boldsymbol{m}}} \sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}'}}\langle\psi|\Big(|\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}| \otimes |\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}|\Big)|\psi\rangle\\
&= \overline{\alpha}\alpha\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}\eta^{\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}_{\hat{\boldsymbol{k}}}\delta\big(\hat{\boldsymbol{m}} - \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}\big) + \overline{\alpha}\beta\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}}\sqrt{\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}\big)\\
&+ \overline{\beta}\alpha\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}}\sqrt{\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}\big) + \overline{\beta}\beta\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}\eta^{\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}} + \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}\big),
\end{aligned}$$

where the symbol $\delta(\boldsymbol{a} \in A)$ gives 1 if $\boldsymbol{a} \in A$ and 0 otherwise. This expression is a Hermitian form of nonnegative coefficients in the two complex variables α and β, which can take negative values if and only if its discriminant

$$\begin{aligned}
&\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}\eta^{\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}} - \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{m}} + \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}\big)\\
&\qquad - (\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}})^2\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}\big)
\end{aligned}$$

is negative. For this, the second term is necessarily nonvanishing, so the parameters in (13) are restricted to $\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}$ and $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$. It is easy to see that in this case $\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}$ and $\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}$ also hold (e.g., if $\hat{\boldsymbol{k}}' \le \hat{\boldsymbol{m}}$ then $\boldsymbol{0} \le \hat{\boldsymbol{m}} - \hat{\boldsymbol{k}}'$, so $\hat{\boldsymbol{k}} \le \hat{\boldsymbol{m}} - \hat{\boldsymbol{k}}' + \hat{\boldsymbol{k}}$), so the negativity condition on the discriminant reads as

$$\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}\eta^{\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}} - \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{m}} + \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big) < (\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}})^2\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'},$$

assuming $\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}$ and $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$. Substituting the Schmidt coefficients (10b), the inequality is simplified,

$$\binom{n-m}{\boldsymbol{n}-\hat{\boldsymbol{m}}+\hat{\boldsymbol{\Delta}}}\binom{n-m}{\boldsymbol{n}-\hat{\boldsymbol{m}}-\hat{\boldsymbol{\Delta}}}\delta\big(\hat{\boldsymbol{m}} - \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{m}} + \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}\big) < \binom{n-m}{\boldsymbol{n}-\hat{\boldsymbol{m}}}^2.$$

This obviously holds if any of the two conditions $\hat{\boldsymbol{m}} \pm \hat{\boldsymbol{\Delta}} \in I^d_{m,\boldsymbol{n}}$ is violated, however, in some cases this may not be guaranteed, so we proceed in a different way. If there are at least two nonzero occupations in $\boldsymbol{n}$, say $n_i, n_j \neq 0$, then we can always choose $\boldsymbol{m} \in I^d_{m,\boldsymbol{n}}$ such that $m_i, m_j \neq 0$, and then $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$ such that they differ only in those two positions, so $\hat{\Delta}_i = 1$, $\hat{\Delta}_j = -1$ and 0 elsewhere, for which the inequality reads as

$$\frac{(n-m)!}{(n_i-\hat{m}_i+1)!(n_j-\hat{m}_j-1)!}\frac{(n-m)!}{(n_i-\hat{m}_i-1)!(n_j-\hat{m}_j+1)!} < \left(\frac{(n-m)!}{(n_i-\hat{m}_i)!(n_j-\hat{m}_j)!}\right)^2,$$

which is

$$\frac{n_i - \hat{m}_i}{n_i - \hat{m}_i + 1} \frac{n_j - \hat{m}_j}{n_j - \hat{m}_j + 1} < 1,$$

which holds for every $\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}$. This concludes the proof.

Summing up, we have that if there are at least two nonzero occupations in $\boldsymbol{n}$, then the m-partite reduced Dicke state $\rho_{\boldsymbol{n},m} = \mathrm{Tr}_{n-m}\big(|\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}|\big)$ for all $2 \leq m \leq n$ is NPT for all splits, $\rho^{\mathrm{T}_k}_{\boldsymbol{n},m} \not\geq 0$ for all $1 \leq k \leq m-1$, then the Peres-Horodecki criterion for symmetric states (4) tells us that it is NPT-GME. (Note that this holds also for the original Dicke state $|\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}|$, which is the $m = n$ case, although this is easier to see directly by the Schmidt rank $|I^d_{k,\boldsymbol{n}}| > 1$, see (9).) Otherwise, if there is only one occupation in $\boldsymbol{n}$, then the Dicke state and also all the reduced Dicke states are fully separable pure states $|i,i,\ldots,i\rangle\langle i,i,\ldots,i|$.

Note that for some small systems, mixtures of Dicke states are known to be entangled *if and only if* NPT, so there is no PPT-entanglement, however, this does not hold in general [61–63]. PPT-entanglement is considered to be a weaker form of entanglement, based on that PPT-entangled states are bound-entangled, that is, no pure entangled states can be distilled from them [64], and it is conjectured that their Schmidt number, although scaling linearly with the local dimension [9–11], cannot be maximal. (For qutrits, this conjecture [65] has been proven [66].) Although we have seen that the reduced Dicke states are NPT-entangled, there are further important open questions about the strength of this entanglement compared to the weakness of PPT-entanglement, e.g., regarding Schmidt number or distillability.

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Appendix: To see the decomposition (8), we have by construction that (i) for any $\boldsymbol{m} \in I^d_{m,\boldsymbol{n}}$, the vector $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ is the linear combination of basis vectors $|i_1, i_2, \ldots, i_d\rangle$ with coefficients $+1$ (there are no repetitions); (ii) for any different $\boldsymbol{m}, \boldsymbol{m}' \in I^d_{m,\boldsymbol{n}}$, the vectors $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ and $|\tilde{\mathrm{D}}_{\boldsymbol{m}'}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}'}\rangle$ contain different basis vectors; (iii) every basis vector in $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle$ is contained in a $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ for an $\boldsymbol{m} \in I^d_{m,\boldsymbol{n}}$; (iv) every basis vector in every $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ is contained in $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle$.

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