Renormalisation of quantum cellular automata

Lorenzo Siro Trezzini[1,2], Alessandro Bisio[1,2], and Paolo Perinotti[1,2]

[1]Dipartimento di Fisica dell'Università di Pavia, via Bassi 6, 27100 Pavia

[2]Istituto Nazionale di Fisica Nucleare, Gruppo IV, via Bassi 6, 27100 Pavia

We study a coarse-graining procedure for quantum cellular automata on hypercubic lattices that consists in grouping neighboring cells into tiles and selecting a subspace within each tile. This is done in such a way that multiple evolution steps applied to this subspace can be viewed as a single evolution step of a new quantum cellular automaton, whose cells are the subspaces themselves. We derive a necessary and sufficient condition for renormalizability and use it to investigate the renormalization flow of cellular automata on a line, where the cells are qubits and the tiles are composed of two neighboring cells. The problem is exhaustively solved, and the fixed points of the renormalization flow are highlighted.

1 Introduction

Quantum Cellular Automata (QCA) have been developed in the years since the very advent of quantum computation [1], as the quantum counterpart of classical cellular automata [2, 3]. The notion of a QCA was made more and more rigorous over the years [4, 5, 6, 7, 8], until the algebraic formulation that is still presently used [9, 10]. The interest in QCA is due to many reasons, the main ones being that they constitute a computational model that is equivalent to a universal quantum computer [11, 12], that their geometric structure suggests their use in the simulation and modelling of quantum physical systems [13, 14, 15, 16, 17, 18, 19], and that they represent interesting many body systems [20, 21, 22], with a peculiar role in the study of the topological phases of matter [23, 24, 25, 26]. The characteristic trait of QCAs, as of cellular automata in general [27] is that they are defined by a simple local rule, and yet they can give rise to arbitrarily complex and interesting large-scale dynamics, as witnessed by their computational universality. As a consequence, the connection between the local rule and the large-scale phenomenology is clearly a problem as difficult as crucial, that calls for understanding.

A typical feature of large-scale behaviour of physical models is that it is rather insensitive to most of the details of small-scale dynamics. This statement reflects e.g. the notion of universality classes of statistical mechanical models [1], where the relevant physical parameters take on values in an astonishingly limited range, collecting very different models in the same large-scale equivalence class [28]. In this context, the renormalisation procedure due to Kadanoff [29], and perfected by Wilson [30, 31], represents the basic idea underpinning all the techniques for recovering the large-scale features of a physical model starting from its microscopic description, and is based on the progressive erasure of degrees of freedom that are supposed to be irrelevant to the large-scale picture. The validation of the approach comes from the convergence of the recursive application of the procedure, and thus to the existence of fixed point models that become stable in the process. This technique led to the discovery of critical exponents in phase transitions and is intimately connected to renormalisation techniques in quantum field theories [28]. A similar approach to the large scale behaviour of *classical* cellular automata was explored in Ref. [32, 33], where the renormalisation flow was interestingly connected to the idea that a cellular

Lorenzo Siro Trezzini: lorenzosiro.trezzini01@universitadipavia.it

Alessandro Bisio: alessandro.bisio@unipv.it

Paolo Perinotti: paolo.perinotti@unipv.it

[1]We stress that the universality notion invoked here is different from computational universality, which in simple words refers to the ability of a computational model to simulate any other conceivable computer.

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automaton A that is renormalised to another one B has a larger complexity than B, in the sense that it can simulate B and then it is at least as complex as B. Although our focus will be on such Kadanoff-like renormalisation, it is worth noting that the challenge of analysing the large-scale behaviour of QCA and Quantum Walks has already been explored in the literature (see, e.g., [34, 35]). The key distinction from previous works lies in the requirement that the model describing the large-scale behaviour of a QCA must be a QCA itself. This entails the possibility of defining a notion of "renormalisation flow" in the space of QCAs and, consequently, to identify fixed points.

The connection of small-scale dynamics with its large-scale *emergent* phenomenology is of particular interest for application of the theory of QCAs to quantum simulations, where QCAs serve both as primitive models [13] and as approximations of Floquet systems or Trotterised approximations of local Hamiltonians [36]. The paradigm of quantum computation—even more so in the present, so-called NISQ era [37]—is based on the run of algorithms on a large noisy physical qubit register where only a smaller logical nearly error-free register is considered. This is an extreme summary of the notion of *error correction*. A renormalisation technique, discarding detailed and fast variables in favour of a fraction of the total pool of degrees of freedom, where an effective dynamics occurs, can then be seen as an error correction technique for quantum simulations through QCAs: a fine-grained QCA where we do not have full control can be used to run a simulation on its renormalised QCA. redThis allows us to interpret the fine-grained QCA as a noise-resilient variant of the coarse-grained one. The issue of quantum error correction in the context of QCA has also been discussed in relation to the implementation of computational tasks, e.g. density classification on a noisy register [38]. In this respect, we observe that renormalisability can be seen as an instance of *intrinsic simulation* [39]: a QCA $\mathcal{T}$ that can be renormalised to another one $\mathcal{S}$ actually *simulates* $\mathcal{S}$—provided that one programs it in the suitable degrees of freedom. In particular, a QCA that could be renormalised to any other QCA would be *intrinsically universal* [12].

In the present work, we adapt the Kadanoff renormalisation idea to the coarse-graining of quantum cellular automata. Unlike the case of classical cellular automata, it is so hard to include irreversible evolutions in the picture of quantum cellular automata that the very definition of a QCA entails reversibility. This fact determines the consequence that, despite the initial formulation of the problem of coarse-graining is very close to that of classical cellular automata [32, 33] or even of classical networks [40], the subsequent solution has to face more constraints and has a radically different family of solutions. We start analysing the case of QCAs on a hypercubic lattice, and pose the question as to what QCAs A admit a block-tiling and, on each block, a projection P on a subspace that is equivalent to that of a single cell, with the requirement that projecting after N steps is equivalent to projecting right away, then applying one step of a different QCA B on the lattice whose elementary cells correspond to the tiles. After writing the general equations whose solution gives both the projection P and the renormalised QCA B, we restrict to the case of 1-dimensional QCAs that can be implemented as a Finite Depth Quantum Circuit (FDQC), with tiles made of two neighbouring nodes and coarse-graining two time-steps into one. The problem in this case can be manipulated, revealing a general algebraic structure that can be further simplified in the case where the cells are qubits. The main reason why the simplification occurs is that for qubit QCAs a full classification of FDQCs has been provided [9]. We then solve the problem exhaustively, providing the renormalisation flow for the 1-d qubit case, along with a classification of the fixed points.

As a remark, we can observe that very few qubit QCAs are renormalisable in this sense, and those that are, have a trivial evolution. This fact can reasonably be ascribed to the very limited choice of degrees of freedom due to the smallness of both the cells and the tiles that we are considering. The solution, however, provides technical tools that pave the way to similar analysis in more general cases, even considering special case-studies.

The manuscript is structured as follows: In section (2) we introduce the notion of Quantum Cellular Automaton following the definition proposed by Schumacher and Werner [9]. We will then focus on two important tools that we will use extensively: the Wrapping Lemma and the index. The former is a procedure to represent the evolution of a QCA defined over an infinite lattice of cells (say, for concreteness, $\mathbb{Z}^s$) on a finite lattice. The latter is a topological invariant quantity that classifies the QCA modulo FDQC. In section (3) we introduce the notion of coarse-graining and renormalisation, and we derive the renormalisability conditions. In section (4) we focus our attention on the case of one-dimensional FDQC. Finally, in section (5) we solve the above equations in the case of QCA with qubit cells. After the classification of all such QCAs that are renormalisable, we study the renormalised evolutions and the fixed points of the renormalisation flow.

2 Quantum Cellular Automata

This section is devoted to review the existing litterature about Quantum Cellular Automata. A Quantum Cellular Automaton $\mathcal{T}$ is a reversible, local and homogeneous discrete time evolution of a lattice $\mathcal{L}$ of cells, each containing a finite-dimensional quantum system. The locality condition means that, at each step of the evolution, a cell $x \in \mathcal{L}$ interacts with a finite set $\mathcal{N}_x$ of cells, called the *neighourhood* of x. Therefore, the speed of propagation of information in a QCA is bounded and we have a natural notion of future and past "light cone" or, more precisely, cone of causal influence. Homogeneity—or translation invariance—on the other hand, is the requirement that every cell is updated according to the same rule. The latter is equivalent to the requirement that the evolution commutes with the lattice translations [2].

Let us now make these concepts mathematically precise. First of all, for the purpose of the present study we will focus on the case where the lattice $\mathcal{L}$ is a square lattice, i.e.

$$\mathcal{L} \coloneqq \bigtimes_{t=1}^{s} \mathbb{Z}_{f_t}, \quad f_i \in \mathbb{N} \cup \infty, \quad s < \infty, \qquad (1)$$

where $\mathbb{Z}_{f_i}$ denotes the additive group modulo f_i, and $\mathbb{Z}_\infty = \mathbb{Z}$ [3]. Therefore, each cell of the lattice is represented by a vector $x \in \bigtimes_{t=1}^{s} \mathbb{Z}_{f_t}$ of integer coefficients.

This notation allows us to treat the case in which $\mathcal{L}$ is infinite and that where $\mathcal{L}$ is finite in a unified way. Typically, one considers the lattice $\mathcal{L}$ as a graph, the edges corresponding to neighbourhood relations. One can prove that the homogeneity requirement makes the graph of causal connections the *Cayley graph* of some group [13, 43, 27], that in the present case is $\mathcal{L}$ imagined as an abelian additive group.

Since we will deal with lattices of infinitely many quantum systems, it is convenient to formulate the QCA dynamics in the Heisenberg picture, namely defining the evolution by its action on the algebra of operators rather than on states. This approach, introduced in Ref. [9] and inspired by the algebraic approach to the statistical mechanics of spin systems [44] or to quantum field theory [45] has the advantage that the notion of local observable is always well defined, whereas defining locality through the action on the local state is highly problematic. This construction is based on the concept of *quasi-local algebra* of operators $\mathsf{A}(\mathcal{L})$. If each cell contains a finite dimensional quantum system of dimension $d \geq 2$, we can associate to each cell x the observable algebra A_x which is isomorphic to the C^*-algebra $\mathsf{M}_d(\mathbb{C})$ of $d \times d$ complex matrices.

When $\Lambda \subset \mathcal{L}$ is a finite subset of $\mathcal{L}$ we denote with $\mathsf{A}(\Lambda)$ the algebra of operators acting on all cells in Λ, namely $\mathsf{A}(\Lambda) = \bigotimes_{x\in\Lambda} \mathsf{A}_x$. If we consider two different subsets $\Lambda_1, \Lambda_2 \subset \mathcal{L}$, we can consider $\mathsf{A}(\Lambda_1)$ as a subalgebra of $\mathsf{A}(\Lambda_1 \cup \Lambda_2)$ by tensoring each element $O \in \mathsf{A}(\Lambda_1)$ with the identity over $\Lambda_2 \setminus \Lambda_1$, namely $O \otimes I_{\Lambda_2\setminus\Lambda_1} \in \mathsf{A}(\Lambda_1 \cup \Lambda_2)$. In this way the product $O_1 O_2$ with $O_i \in \mathsf{A}(\Lambda_i)$ is a well defined element of $\mathsf{A}(\Lambda_1 \cup \Lambda_2)$. Each local algebra is naturally endowed with the operator norm $\|\cdot\|$. We denote with $\mathsf{A}^{\mathrm{loc}}(\mathcal{L})$ the *local algebra* of operators, i.e. the algebra of all the operators acting non-trivially over any finite set of cells. Since tensoring with the identity does not change the norm of an operator, every local operator has a well defined norm, which is independent of the region on which we are considering the action of the operator. Thus, $\mathsf{A}^{\mathrm{loc}}(\mathcal{L})$, is a normed algebra, whose completion is the quasi-

[2]Sometimes in the literature QCAs can be defined without the homogeneity property, and what we call a QCA would then be a *homogeneous* QCA (see e.g. Ref. [41]).

[3]The most general structure on which homogeneous QCAs can be defined is the *Cayley graph* of some finitely presented group G [42, 3, 27]. However here we will restrict to the case where G is abelian.

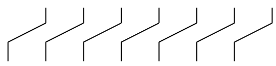
local algebra $\mathsf{A}(\mathcal{L})$. For a detailed construction of $\mathsf{A}(\mathcal{L})$ see Appendix (A). Clearly, for finite lattices $\mathsf{A}^{\mathrm{loc}}(\mathcal{L})$ is a finite dimensional algebra and $\mathsf{A}(\mathcal{L}) = \mathsf{A}^{\mathrm{loc}}(\mathcal{L})$. We could regard $\mathsf{A}(\mathcal{L})$ as the C^*-algebra of operators that can be approximated arbitrarily well by local operators.

A reversible dynamics then corresponds to an automorphism $\mathcal{T} : \mathsf{A}(\mathcal{L}) \to \mathsf{A}(\mathcal{L})$ of the quasi local algebra. The algebraic approach allows us not to chose a representation of the observable algebra. When dealing with finitely many systems (i.e. when $\mathcal{L}$ is a finite lattice) this is rather negligible advantage, since all the representation are unitarily equivalent. In this case, then $\mathcal{T}$ would be the conjugation by some unitary operator, i.e. $\mathcal{T}(O) = U^\dagger O U$. However, when $\mathcal{L}$ is an infinite lattice we have that $\mathsf{A}(\mathcal{L})$ is an infinite dimensional C^*-algebra, which generally decomposes into inequivalent representations. Thus working in a representation-independent framework is highly advantageous.

In order to define a homogeneous dynamics, we have to define the operators that implement the lattice translations.

Definition 1 (Shift)**.** *For any $x \in \mathcal{L}$ we define the shift τ_x as the extension to the whole $\mathsf{A}(\mathcal{L})$ of the isomorphism from each A_y to A_{y+x}.*

If $O \in \mathsf{A}_\Lambda$ is an operator localized in the region A_Λ, then $\tau_x(O) \in \mathsf{A}_{\Lambda+x}$ is an operator localized in the region $\Lambda + x := \{y + x | y \in \Lambda\}$. For $\mathcal{L} = \mathbb{Z}$ one can refer to the pictorial representation of e.g. τ_1:

$\tau_1 = \quad \dots$  $\dots \,.$ (2)

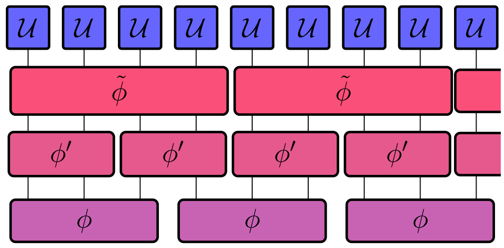
We are now in position to define a Quantum Cellular Automaton.

Definition 2. *A* Quantum Cellular Automaton *(QCA) over a lattice $\mathcal{L}$ with finite neighbourhood scheme $\mathcal{N} \subset \mathcal{L}$ is an automorphism $\mathcal{T} : \mathsf{A}(\mathcal{L}) \longrightarrow \mathsf{A}(\mathcal{L})$ of the quasi-local algebra such that*

- $\mathcal{T}(\mathsf{A}(\Lambda)) \subset \mathsf{A}(\Lambda + \mathcal{N})\ \forall \Lambda \subset \mathcal{L}$ *(locality),*
- $\mathcal{T} \circ \tau_x = \tau_x \circ \mathcal{T}\ \forall x \in \mathcal{L}$ *(homogeneity),*

where $\Lambda + \mathcal{N} := \{y + x | y \in \Lambda, x \in \mathcal{N}\}$. The homomorphism $\mathcal{T}_0 : \mathsf{A}_0 \longrightarrow \mathsf{A}(\mathcal{N})$ given by the restriction of the QCA to the cell labelled $x = 0$ is called the transition rule.

The identity and the shifts τ_x are (trivial) examples of QCA. Another example of a QCA is a finite depth quantum circuit, whose action can be decomposed in terms of local gates as follows



. (3)

The boxes represent unitary completely positive maps, e.g. those labelled $\mathcal{U}$ represent the maps $\mathcal{U}(X) := UXU^\dagger$.

Remark 1. *Notice that the algebra A_0 is isomorphic to $\mathsf{M}_d(\mathbb{C})$ (say via $\mathcal{H}_\mathcal{L} : \mathsf{A}_0 \to \mathsf{M}_d(\mathbb{C})$), and $\mathsf{A}(\mathcal{N})$ is isomorphic to $\mathsf{M}_d(\mathbb{C})^{\otimes|\mathcal{N}|}$ (via $\mathcal{K}_\mathcal{L} : \mathsf{A}(\mathcal{N}) \to \mathsf{M}_d(\mathbb{C})^{\otimes|\mathcal{N}|}$). Thus, the transition rule can be equivalently identified by the homomorphism*

$$\mathcal{T}_* := \mathcal{K}_\mathcal{L} \circ \mathcal{T}_0 \circ \mathcal{H}_\mathcal{L}^{-1} : \mathsf{M}_d(\mathbb{C}) \to \mathsf{M}_d(\mathbb{C})^{\otimes|\mathcal{N}|}. \quad (4)$$

In this way, the restriction of $\mathcal{K}_\mathcal{L}$ to $\mathcal{T}_0(\mathsf{A}_0)$ is an isomorphism to $\mathcal{T}_*(\mathsf{M}_d(\mathbb{C}))$.

It is possible to prove ([9]) that the local transition rule and the global evolution are in one-to-one correspondence. This result is stated in the following Lemma:

Lemma 1. *A homomorphism $\mathcal{T}_0 : \mathsf{A}_0 \longrightarrow \mathsf{A}(\mathcal{N})$ is the transition rule of a cellular automaton if and only if for all $x \in \mathcal{L}$ such that $\mathcal{N} \cap \mathcal{N}_x \neq \emptyset$, $\mathcal{N}_x := \{x\} + \mathcal{N}$, the algebras $\mathcal{T}_0(A_0)$ and $\tau_x(\mathcal{T}_0(A_0))$ commute element-wise. Given a local transition rule $\mathcal{T}_0$ the global homomorphism $\mathcal{T}$ is given by*

$$\mathcal{T}\Big(\bigotimes_{x\in\mathcal{L}} A_x\Big) = \prod_{x\in\mathcal{L}} \mathcal{T}_x(A_x) \quad (5)$$

where $\mathcal{T}_x(A_x) = \tau_x \mathcal{T}_0 \tau_{-x}(\mathsf{A}_x)$.

Since the image $\mathcal{T}_0(\mathsf{A}_0)$ is a subalgebra of the full matrix algebra $\mathsf{A}_\mathcal{N}$, from the characterisation theorem of finite dimensional C^*-algebra (see e.g. [9]) we have

$$\mathcal{T}_0(X) = U^\dagger (I_{\mathcal{N}\setminus\{0\}} \otimes X) U, \quad \forall X \in \mathsf{A}_0 \quad (6)$$

for a suitable unitary U in $\mathsf{A}_{\mathcal{N}\cup\{0\}}$, and analogously for $\mathcal{T}_*$. In the following we will use eq. (6) both for $\mathcal{T}_0$ and $\mathcal{T}_*$, leaving to the context the specification as to which is the case.

2.1 The wrapping Lemma

Our definition of QCA encompasses both infinite lattice and finite lattices embedded in a torus. However, a result known as Wrapping Lemma (see Ref. [27]) relates the evolution of the QCAs over the infinite lattice with the one over a finite one. The idea is to "wrap" a sufficiently large finite sub-lattice of a QCA over the infinite lattice $\mathbb{Z}^s$ by imposing suitably periodic boundary conditions, obtaining a discrete torus. The "unwrapped" QCA and the "wrapped" one then have the same local transition rule. The concept of "sufficiently large" is grasped by the requirement that, after wrapping, still $\mathcal{T}_0(A)$ should commute with $\mathcal{T}_x(B)$ for all $A \in \mathsf{A}_0$ and $B \in \mathsf{A}_x$, for any x such that $\mathcal{N} \cap \mathcal{N}_x \neq \emptyset$. Therefore, a "sufficiently large" wrapping is such that the set of elements $x \in \mathcal{L}$ such that $\mathcal{N} \cap \mathcal{N}_x \neq \emptyset$ is the same as in the infinite case, i.e. no new overlapping between neighbourhoods is introduced after wrapping the lattice. Since we will heavily exploit the Wrapping Lemma in the next sections, we now provide its formal statement. However, a reader who is already familiar with this concept can easily skip this section. Let us begin defining a wrapping, i.e. a homomorphism of the infinite lattice $\mathbb{Z}^s$ into a discrete torus.

Definition 3 (Wrapping)**.** *A wrapping of the additive group $\mathbb{Z}^s$ is a homomorphism*

$$\varphi : \mathbb{Z}^s \to \mathcal{L} := \bigtimes_{t=1}^{s} \mathbb{Z}_{f_t}, \;\; f_t < \infty \;\; \forall t,$$

where $\mathbb{Z}_f$ stands for the additive group $\mathbb{Z}$ modulo f.

More concretely, if an element $x \in \mathbb{Z}^s$ is identified by the s integer numbers $x = (x_1, ..., x_s)$ the action of the wrapping is:

$$\varphi(x) = ((x_1 \bmod f_1), ..., (x_s \bmod f_s)).$$

We now define a regular neighbourhood scheme for a wrapping φ, that pins down the idea of a "sufficiently large" wrapping.

Definition 4 (Regular neighbourhood scheme)**.** *Let $\mathcal{N} \subset \mathbb{Z}^s$ be a finite set. We say that $\mathcal{N}$ is* regular *for the wrapping φ if*

$$\{\varphi^{-1}(\mathcal{N}_{\varphi(x)}) \cap \varphi^{-1}(\mathcal{N}_{\varphi(y)})\} \cap \{\mathcal{N}_x \cup \mathcal{N}_y\} = \\ = \mathcal{N}_x \cap \mathcal{N}_y$$

for every pair $x, y \in \mathbb{Z}^s$ such that $\mathcal{N}_x \cap \mathcal{N}_y \neq \emptyset$.

The above definition captures the idea that the neighbourhoods of close cells x and y, after wrapping, should have the same intersection as before wrapping. The intersection with $\mathcal{N}_x \cup \mathcal{N}_y$ on the r.h.s. in the definition is required because the inverse image of $\mathcal{N}_{\varphi(x)} \cap \mathcal{N}_{\varphi(y)}$ contains infinitely many intersections, due to periodicity of the set $\varphi^{-1}(x)$. Now, we can finally understand the meaning of "sufficiently large" wrapping, with respect to a neighbourhood scheme: the periodic boundary conditions should be such that the neighbourhood scheme is regular. For a single step of evolution of a nearest neighbour automaton, this is true for all lattices with more than four cells in each coordinate direction.

With the above notions in mind, we can now state the wrapping lemma.

Lemma 2 (Wrapping Lemma)**.** *QCAs with a regular neighbourhood scheme $\mathcal{N}$ on a lattice $\mathcal{L}$ with given periodic boundary conditions are in one-to-one correspondence with those on the infinite lattice, and the correspondence is defined by "having the same transition rule".*

Remark 2. *Notice that we did not give a formal definition of the (equivalence) relation "having the same transition rule". However, this notion is straightforward if we think of the transition rule as $\mathcal{T}_*$ rather than $\mathcal{T}_0$.*

Given a wrapping $\varphi : \mathbb{Z}^s \to \mathcal{L}$, let us now introduce the following isomorphisms, referring to those introduced in Remark (1)

$$\mathcal{U}_\varphi := \mathcal{H}_{\mathbb{Z}^s}^{-1} \circ \mathcal{H}_{\mathcal{L}} : \mathsf{A}_0^w \to \mathsf{A}_0^\infty \\ \mathcal{W}_\phi := \mathcal{K}_{\mathcal{L}}^{-1} \circ \mathcal{K}_{\mathbb{Z}^s} : \mathsf{A}_{\mathcal{N}}^\infty \to \mathsf{A}_{\mathcal{N}}^w.$$

The map $\mathcal{U}_\varphi$ maps the single cell algebra A_0^w of the wrapped lattice and into the single cell algebra A_0^∞ of the algebra of the infinite lattice, while the map $\mathcal{W}_\varphi$ maps the algebra of the neighbourhood $\mathsf{A}_{\mathcal{N}}^\infty$ of the infinite lattice into that of the wrapped lattice denoted $\mathsf{A}_{\mathcal{N}}^w$. The regularity requirement in def. (4) ensures that the application of φ to $\mathcal{N}_x$ does not change the neighbourhood structure: $\varphi(\mathcal{N}_x) = \mathcal{N}_{\varphi(x)} \leftrightarrow \mathcal{N}_x$.

If we now consider a QCA $\mathcal{T}$ over $\mathbb{Z}^s$, with transition rule $\mathcal{T}_0^{(\infty)}$, we can define its wrapped version as follows.

Definition 5 (Wrapped evolution)**.** *The wrapped evolution $\mathcal{T}_w$ on the wrapped lattice $\mathcal{L} = \varphi(\mathbb{Z}^s)$ of* *a QCA $\mathcal{T}$ over $\mathbb{Z}^s$ is the QCA over the lattice $\mathcal{L}$ with local rule given by:*

$$\mathcal{T}_0^{(w)} = \mathcal{W}_\varphi \mathcal{T}_0^{(\infty)} \mathcal{U}_\varphi. \tag{7}$$

Notice that, in this case,

$$\mathcal{K}_\mathcal{L} \circ \mathcal{T}_0^{(w)} \circ \mathcal{H}_\mathcal{L}^{-1} = \mathcal{K}_{\mathbb{Z}^s} \circ \mathcal{T}_0^{(\infty)} \circ \mathcal{H}_{\mathbb{Z}^s}^{-1} = \mathcal{T}_*.$$

Consequently, we can say that wrapped QCAs over different lattices share the same local rule if their local rule can be described as in eq.(7) starting from the same infinite QCA $\mathcal{T}$, namely $\mathcal{T}^{(w)}$ and $\mathcal{T}^{(w')}$ share the same local rule if:

$$\mathcal{K}_\mathcal{L} \circ \mathcal{T}_0^{(w)} \circ \mathcal{H}_\mathcal{L}^{-1} = \mathcal{K}_{\mathcal{L}'} \circ \mathcal{T}_0^{(w')} \circ \mathcal{H}_{\mathcal{L}'}^{-1} = \mathcal{T}_*.$$

2.2 One dimensional QCA: Index Theory

One key feature of QCAs on $\mathbb{Z}$ is their *index*, which was introduced in Ref. [41] for QCAs without the translation-invariance hypothesis. The index is a locally computable invariant, meaning it can be determined from a finite portion of the automaton and will yield the same result regardless of the portion chosen, even if the QCA is not translation-invariant. Intuitively, the index theory for one dimensional QCAs suggests that information can be thought as an incompressible fluid, with the index representing its flow rate.

Let us now review the main results of index theory. The whole construction is based on the notion of support algebras [41] (also called interaction algebras [46]).

Definition 6 (Support algebra)**.** *Let B_1 and B_2 be C^*-algebras, and consider a $*$-subalgebra $\mathsf{A} \subset \mathsf{B}_1 \otimes \mathsf{B}_2$. Each element $a \in \mathsf{A}$ can be expanded uniquely in the form $a = \sum_\mu a_\mu \otimes e_\mu$, where $\{e_\mu\}$ is a fixed basis of B_2. The* support algebra *$\mathsf{S}(\mathsf{A}, \mathsf{B}_1)$ of A in B_1 is the smallest C^*-algebra generated by all a_μ in such an expansion* [4].

Let us now consider a nearest neighbour QCA on $\mathbb{Z}$, i.e. with $\mathcal{N}_x = \{x-1, x, x+1\}$ for all x. Two comments are now in order. First, any QCA on $\mathbb{Z}$ can be expressed as a nearest-neighbour QCA by grouping multiple cells in a single one. In the following discussion we will assume that all the QCAs that we introduce are nearest-neighbour QCAs. Second, since translation invariance is not assumed, local algebras at different sites may not be isomorphic. Let us now define the left and right support algebras around cells $2x$ and $2x+1$ for a QCA with evolution $\mathcal{T}$ as

$$\begin{aligned} \mathsf{L}_{2x} &= \mathsf{S}(\mathcal{T}(\mathsf{A}_{2x} \otimes \mathsf{A}_{2x+1}), (\mathsf{A}_{2x-1} \otimes \mathsf{A}_{2x})), \\ \mathsf{R}_{2x} &= \mathsf{S}(\mathcal{T}(\mathsf{A}_{2x} \otimes \mathsf{A}_{2x+1}), (\mathsf{A}_{2x+1} \otimes \mathsf{A}_{2x+2})). \end{aligned} \tag{8}$$

The first result of index theory is the following theorem.

Proposition 1 (Index of a QCA)**.** *For any QCA $\mathcal{T}$ on $\mathbb{Z}$, the quantity*

$$\operatorname{ind}(\mathcal{T}) = \sqrt{\frac{\dim[\mathsf{L}_{2x}]}{\dim[\mathsf{A}_{2x}]}} \tag{9}$$

does not depend on x. We call $\operatorname{ind}(\mathcal{T})$ the index *of the QCA.*

The following Proposition presents the most important consequences of index theory.

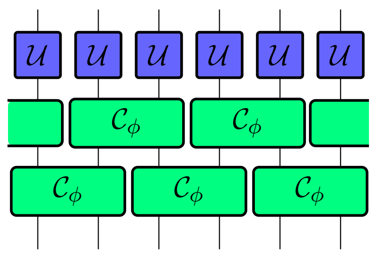
Proposition 2. *For any QCAs $\mathcal{T}$ and $\mathcal{S}$ we have that $\operatorname{ind}[\mathcal{T}\mathcal{S}] = \operatorname{ind}[\mathcal{T}]\operatorname{ind}[\mathcal{S}]$ and $\operatorname{ind}[\mathcal{T}] = 1$ iff $\mathcal{T}$ has a* Margolus partitioning scheme, *namely*

$$\begin{aligned} &\mathcal{T} = \mathcal{W} \circ \mathcal{V} \\ &\mathcal{V}(\cdot) = (\prod_{n\in\mathbb{Z}} V_{2n}^\dagger)(\cdot)(\prod_{m\in\mathbb{Z}} V_{2m}) \\ &\mathcal{W}(\cdot) = (\prod_{k\in\mathbb{Z}} W_{2k+1}^\dagger)(\cdot)(\prod_{l\in\mathbb{Z}} W_{2l+1}) \end{aligned} \tag{10}$$

where V_{2n} are unitaries acting on sites $2n$ and $2n+1$, and W_{2m+1} are unitaries acting on sites $2m+1$ and $2m+2$.

2.3 One Dimensional Nearest Neighbour Qubit Automata

We now restore the translation invariance assumption. It is possible to show [9] that a nearest-neighbor (i.e. $\mathcal{N} = \{-1, 0, 1\}$) QCA $\mathcal{T}$ on $\mathbb{Z}$ where each cell is a qubit, has $\operatorname{ind}(\mathcal{T}) = 1$ if and only if

$\mathcal{T} =$


(11)

[4] It is easy to prove that the support algebra does not depend on the choice of the basis $\{e_\mu\}$

with $\mathcal{U}(X) := U^\dagger X U$, U an arbitrary unitary gate acting on a single cell, and $\mathcal{C}_\phi(X) := C_\phi^\dagger X C_\phi$, where C_ϕ is a controlled-phase gate, i.e.

$$C_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}.$$

The expression (6) of the local rule of the QCA is

$$\begin{aligned} \mathcal{T}_0(A_0) &= X^\dagger (I \otimes U^\dagger A_0 U \otimes I) X, \\ X &= (C_\phi \otimes I_3)(I_1 \otimes C_\phi). \end{aligned} \tag{12}$$

From now on, we will set the basis in which C_ϕ is diagonal as the basis $\{|a\rangle\}_{a=0,1}$ such that $\sigma_z|a\rangle = (-1)^a|a\rangle$. We will refer to this basis as the *computational basis.*

3 QCA renormalisation

Following the idea presented in [33] we now define a procedure to provide a coarse-grained description of the dynamics of a QCA, in such a way that the large scale description that we eventually obtain is still a QCA. Intuitively speaking, the procedure consists in choosing a subset of degrees of freedom of the original evolution through a projection on a subspace of the Hilbert space of N neighbouring cells, and requiring that the restriction of N steps of the original QCA to such a subspace is still a QCA. Naively speaking, one could try to restrict the domain of the QCA to the operators whose support is contained in the support of $\bigotimes_y \Pi_y$, where each Π_y identifies a subset of degrees of freedom of $\mathsf{A}_{2y} \otimes \mathsf{A}_{2y+1}$. However, this infinite tensor product is ill-defined, lying outside the quasi local algebra. We overcome such problem exploiting the wrapping lemma to reduce the coarse-graining problem of an infinite lattice to the corresponding problem over all possible wrappings $\mathcal{L}$. In order to do that, we have to define a partition $\{\Lambda_x\}_{x\in\mathcal{L}'}$ of the original lattice $\mathcal{L}$. The indices x of such a partition will be the cells of a new lattice $\mathcal{L}'$: this will establish the relation between the wrapped lattice $\mathcal{L}$ and the coarse-grained one $\mathcal{L}'$. Once this is done, we have a formulation of the problem in terms of coarse-graining of the family of evolution rules over all the possible wrapped lattices $\mathcal{L}$. We will then find conditions for the coarse-graining to exist, and show that these conditions do not depend on the choice of the lattice $\mathcal{L}$. This means that our coarse-graining condition ultimately boils down to a single condition over a minimal regular lattice $\mathcal{L}$. In particular, in the first subsection we will introduce the technical tools to properly formulate the problem of coarse-graining a finite lattice $\mathcal{L}$ in another finite lattice $\mathcal{L}'$, while in the second subsection we will apply these tools to the study of the coarse-graining condition for a QCA over an infinite lattice.

A QCA that admits a solution to the coarse-graining equation is *renormalisable.* The coarse-graining of QCAs determines a displacement in the space of QCAs as specified by their defining parameters, that represents a *renormalisation flow*, whose fixed points play a distinguished role as in any renormalisation scenario.

3.1 Coarse-grained algebra

Let $\mathcal{T}$ be a QCA on $\mathbb{Z}^s$ such that the algebra of each cell is isomorphic to the C^*-algebra of $d\times d$ complex matrices, and let us fix some $N \in \mathbb{N}$. Let us now consider a regular tessellation of $\mathbb{Z}^s$ such that each tile is a Ns-dimensional cube i.e.

$$\bigcup_{x\in\mathbb{Z}^s} \Lambda_x = \mathbb{Z}^s, \tag{13}$$

$$\Lambda_0 := [0, N-1]^s, \qquad \Lambda_x := \Lambda_0 + Nx, \tag{14}$$

where $[a, b]$ denotes the set $\{a, a+1, \ldots, b-1, b\} \subset \mathbb{Z}$ (see Fig. (1)). From Eq. (14) we have that the cubes $\{\Lambda_x\}$ are related to each other by a translation. Consider now a projection Π_y in the algebra $\mathsf{A}(\Lambda_y)$ of each of the cubic tiles. As already noted, the projector $\bigotimes_y \Pi_y$ lies outside the quasi local algebra. We then have to exploit the wrapping lemma and consider the wrapped evolution $\mathcal{T}_w$ on a finite lattice

$$\mathcal{L} = \mathbb{Z}^s_{Nf} \quad f \in \mathbb{N} \tag{15}$$

where f is suffieciently large such that the regular neighborhood condition is satisfied. Clearly, the lattice $\mathcal{L}$ admits a tessellation in terms of the same s-dimensional hypercubes. Then we have

$$\bigcup_{y\in\mathcal{L}'} \Lambda_y = \mathcal{L}, \quad \mathcal{L}' := \mathbb{Z}^s_f \tag{16}$$

If Π_0 is a given a projection belonging to the local algebra of the tile Λ_0, we can define the projection Π_x on the tile Λ_x by translating Π_0,

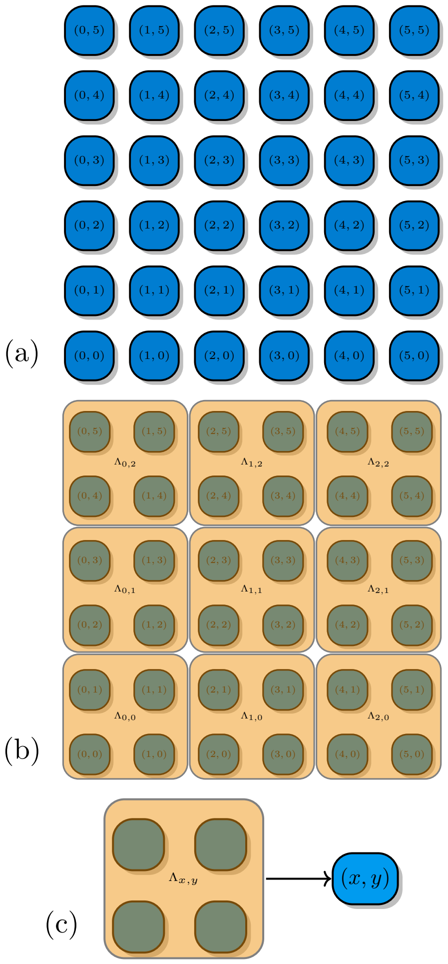


Figure 1: Example of tessellation of $\mathcal{L}$ and mapping in $\mathcal{L}'$ in $\mathbb{Z}^2$. (a) The wrapped lattice $\mathcal{L} \simeq \mathbb{Z}_6 \times \mathbb{Z}_6$. (b) Tessellation $\mathcal{L}' \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ of the lattice $\mathcal{L}$ through $\{\Lambda_x\}$ with $\Lambda_{0,0} = [0,1]^2$. (c) The new lattice $\mathcal{L}'$ is given by the identification $\Lambda_{\mathbf{x}} \to \mathbf{x} = (x, y)$.

namely $\Pi_x := \tau_{Nx}(\Pi_0)$. We now define the coarse-grained algebra $\mathsf{A}|_\Pi(\mathcal{L})$ over $\mathcal{L}$ as follows:

$$\begin{aligned} &\mathsf{A}|_\Pi(\mathcal{L}) := \{A \in \mathsf{A}(\mathcal{L}) \text{ s.t. } \Pi A \Pi = A\}, \\ &\Pi := \bigotimes_{x \in \mathcal{L}'} \Pi_x. \end{aligned} \tag{17}$$

i.e. $\mathsf{A}|_\Pi(\mathcal{L})$ is the algebra of operators with support and range contained in those of Π. The algebra $\mathsf{A}|_\Pi(\mathcal{L})$ is isomorphic to the algebra $\mathsf{A}(\mathcal{L}')$ on the lattice $\mathcal{L}'$ where each local system has dimension $d' := \operatorname{rank}(\Pi)$. The node $x \in \mathcal{L}'$ corresponds to the tile Λ_x in $\mathcal{L}$, and operators $A \in \mathsf{A}(\mathcal{L}')$ are in one-to-one correspondence with operators in $\mathsf{A}|_\Pi(\mathcal{L})$. Let us explicitly construct such isomorphisms. Let $\{|\psi_k\rangle\}$ be a basis on the Hilbert space of the tile Λ_x such that

$$\Pi_x = \sum_{k=0}^{d'-1} |\psi_k\rangle\langle\psi_k|. \tag{18}$$

We then define

$$J := \left(\bigotimes_{x \in \mathcal{L}'} J_{\Lambda_x} \right), \quad J_{\Lambda_x} := \sum_{k=0}^{d'-1} |k\rangle\langle\psi_k|, \tag{19}$$

where $\{|k\rangle\}_{k=0}^{d'-1}$ is a basis of $\mathbb{C}^{d'}$, i.e. the Hilbert space of the cell $x \in \mathcal{L}'$. This operator applies the J_{Λ_x} over the support of each Λ_x. It is easy to verify that

$$J_{\Lambda_x}^\dagger J_{\Lambda_x} = \Pi_x, \; J_{\Lambda_x} J_{\Lambda_x}^\dagger = I, \; J^\dagger J = \Pi, \; J J^\dagger = I. \tag{20}$$

Then we define

$$\begin{aligned} &\mathcal{V} : \mathsf{A}(\mathcal{L}') \to \mathsf{A}(\mathcal{L}), \quad \mathcal{V}^\dagger : \mathsf{A}(\mathcal{L}) \to \mathsf{A}(\mathcal{L}'), \\ &\mathcal{V}(A_r) := J^\dagger A_r J \\ &\mathcal{V}^\dagger(A) := J \mathcal{P}(A) J^\dagger, \quad \mathcal{P}(A) := \Pi A \Pi \end{aligned} \tag{21}$$

and from Equation (20) we have

$$\mathcal{V}^\dagger \circ \mathcal{V} = \mathcal{I}, \quad \mathcal{V} \circ \mathcal{V}^\dagger = \mathcal{P} \tag{22}$$

i.e. $\mathcal{V}$ is an ismomorphism between $\mathsf{A}(\mathcal{L}')$ and its image $\mathsf{A}|_\Pi(\mathcal{L})$ and the restriction of $\mathcal{V}^\dagger$ to $\mathsf{A}|_\Pi(\mathcal{L})$ is the inverse isomorphism. With the choice of basis in which each Π_x is diagonal, the matrices representing operators $A \in \mathsf{A}|_\Pi(\mathcal{L})$, are obtained by padding with zeros the matrices that represent their correspondent $A_r \in \mathsf{A}(\mathcal{L}')$, and conversely, given the matrix of an operator $A \in \mathsf{A}|_\Pi(\mathcal{L})$, the corresponding operator on $A_r \in \mathsf{A}(\mathcal{L}')$ is obtained by removing the null blocks with image or

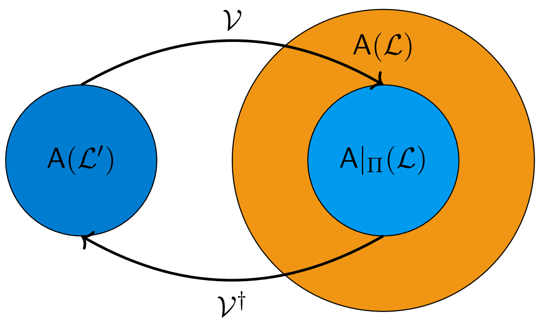


Figure 2: Representation of the maps $\mathcal{V}^\dagger, \mathcal{V}$.

support on the kernel of Π. In other words, we have:

$$A_r \in \mathsf{A}(\mathcal{L}') \leftrightarrow \begin{pmatrix} A_r & 0 \\ 0 & 0 \end{pmatrix} \in \mathsf{A}|_\Pi(\mathcal{L}). \tag{23}$$

We can now define the map $(\mathcal{T})_r$ induced by $\mathcal{T}_w$ on $\mathsf{A}(\mathcal{L}')$ as follows

Definition 7 (Induced map)**.** *Let $\mathcal{T}$ be a QCA on $\mathbb{Z}^s$ and let $\mathcal{L} = \mathbb{Z}^s_{Nf}$, be a wrapped lattice such that the regular neighbourhood condition is satisfied. Let $\mathcal{L}' = \mathbb{Z}^s_f$ and let $\mathcal{V}^\dagger$ and $\mathcal{V}$ be defined as in Eq.* (21)*. The* induced map *on the lattice $\mathcal{L}'$ is defined as follows:*

$$\mathcal{T}_r : \mathsf{A}(\mathcal{L}') \to \mathsf{A}(\mathcal{L}'), \quad \mathcal{T}_r := \mathcal{V}^\dagger \circ \mathcal{T}_w \circ \mathcal{V}. \tag{24}$$

Notice that the induced map depends on the choice of the lattices $\mathcal{L}$ and $\mathcal{L}'$, and we have a *family* of induced maps $\{\mathcal{T}_r\}_{\mathcal{L},\mathcal{L}'}$. In order to lighten the notation, we will not explicitly write dependance of the induced map on the lattices $\mathcal{L}$ and $\mathcal{L}'$. This simplification is also justified by the fact that that there is no dependence of the local rule on the choice of the lattices, as we will prove later. We remark that, in general, we need to differentiate between the map induced by k steps of the QCA $\mathcal{T}_w$ and k subsequent applications of the map induced by $\mathcal{T}_w$. The above two transformations of the quasi-local algebra are given by

$$(\mathcal{T}^k)_r = \mathcal{V}^\dagger \circ \mathcal{T}_w^k \circ \mathcal{V} \quad (\mathcal{T}_r)^k = (\mathcal{V}^\dagger \circ \mathcal{T}_w \circ \mathcal{V})^k, \tag{25}$$

respectively. In the following we will see how the induced map pins down the idea of coarse-graining.

3.2 Renormalisable QCAs

We are now in position to provide a definition of renormalisable QCA.

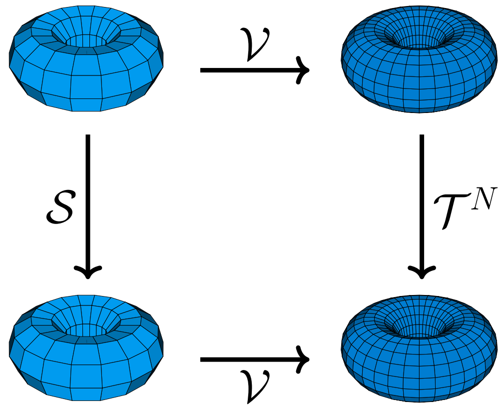


Figure 3: The condition in eq.(26) amounts to ask the commutativity of the above diagram: if $\mathcal{S}$ is a renormalisation of $\mathcal{T}$ then evolving with a single step of $\mathcal{S}$ in the coarse-grained algebra and then embed the coarse-grained algebra in the original one through the map $\mathcal{V}$ is the same than embed the coarse-grained algebra in the original one and consider N steps of $\mathcal{T}$. In this way, a single step of $\mathcal{S}$ mimics the behaviour of $\mathcal{T}^N$ if reduced to the chosen degrees of freedom.

Definition 8 (Renormalisable QCA)**.** *Let $\mathcal{T}$ and $\mathcal{S}$ be two QCAs on $\mathbb{Z}^s$, N be a natural number, and $\Pi_0 \in \mathsf{A}(\Lambda_0)$ be a projection on the local algebra of the region $\Lambda_0 := [0, N-1]^s$. We say that $\mathcal{T}$ is* renormalisable *through the coarse-graining (Π_0, Λ_0) to $\mathcal{S}$ if*

$$\mathcal{T}_w^N \circ \mathcal{V} = \mathcal{V} \circ \mathcal{S}_w \tag{26}$$

holds for any any wrapping $\mathcal{T}_w$ on $\mathcal{L} = \mathbb{Z}^s_{Nf}$ such that the neighbourhood of $\mathcal{T}^N$ is regular over $\mathcal{L}$. We say that $\mathcal{S}$ is a size-N renormalisation *of $\mathcal{T}$ through the coarse-graining (Π_0, Λ_0). We will also say that $\mathcal{T}$* admits a size-N renormalisation *if there exists a QCA $\mathcal{S}$ and a coarse-graining (Π_0, Λ_0) such that $\mathcal{S}$ is a size-N renormalisation of $\mathcal{T}$ through (Π_0, Λ_0).*

Remark 3. *Notice that this definition involves the map $\mathcal{V}$, whose role is to embed the single cell algebra into the N-cell algebra. This definition may seem in contrast with the one in [33], where the coarse-graining map takes the state of the N-supercell and projects it into the state of a single cell. However, we have to remember that we are working in the Heisenberg picture here. The above definition represent the faithful transposition of that in [33] to the quantum case.*

This definition is modelled on the analogous one from [33]. However it is not particularly amenable for the analysis of the problem due to the reference to all possible wrappings $\mathcal{T}_w^N$. The following lemmas will show that it is sufficient to consider just one wrapping and check the renormalisation condition there. Z

Lemma 3. *If a QCA $\mathcal{T}$ on $\mathbb{Z}^s$ is renormalisable through (Π_0, Λ_0) to some QCA $\mathcal{S}$ as in eq.(26), then every induced map $(\mathcal{T}^N)_r$ is a QCA.*

Proof. Let us begin remembering the expression of the induced map of N-steps of $\mathcal{T}$:

$$(\mathcal{T}^N)_r = \mathcal{V}^\dagger \circ \mathcal{T}_w^N \circ \mathcal{V}. \tag{27}$$

Composing (26) on the left with $\mathcal{V}^\dagger$ we have

$$\mathcal{V}^\dagger \circ \mathcal{T}_w^N \circ \mathcal{V} = \mathcal{V}^\dagger \circ \mathcal{V} \circ \mathcal{S}_w = \mathcal{S}_w, \tag{28}$$

which reads

$$\mathcal{S}_w = (\mathcal{T}^N)_r, \tag{29}$$

for every possible choice of the lattice $\mathcal{L}$. This means that, if $\mathcal{S}$ is a renormalisation of $\mathcal{T}$, then $(\mathcal{T}^N)_r$ is a QCA. ■

Notice that, if $(\mathcal{T}^N)_r$ is a QCA for every choice of the lattice $\mathcal{L}$, the Wrapping Lemma allows us to extend it to a QCA $\mathcal{S}$ on the *quasi-local* algebra over the infinite lattice $\mathsf{A}(\mathbb{Z}^s)$. However, the local rule of $\mathcal{S}$ may depend on the specific wrapping. In the following, we prove that this is not the case.

Let us now study when the condition (26) holds for a given wrapped lattice $\mathcal{L}$. First of all, since $\mathcal{T}_w$ is a C^*-algebra automorphism, we can always find a unitary $U_\mathcal{T}$ such that

$$\mathcal{T}_w(A) = (U_\mathcal{T})^\dagger A (U_\mathcal{T}) \quad \forall A \in \mathsf{A}(\mathcal{L}), \tag{30}$$

and the same holds for $\mathcal{S}_{\mathcal{L}'}$:

$$\mathcal{S}_w(A) = (U_\mathcal{S})^\dagger A (U_\mathcal{S}) \quad \forall A \in \mathsf{A}(\mathcal{L}'). \tag{31}$$

Moreover, eq.(27) allows us to write:

$$(\mathcal{T}^N)_r(A) = (U_{\mathcal{T}^N})_r^\dagger A (U_{\mathcal{T}^N})_r \tag{32}$$
$$(U_{\mathcal{T}^N})_r := J U_{\mathcal{T}^N} J^\dagger \tag{33}$$

Let us begin providing some trivial examples of renormalisation to develop the intuition.

- **Local Unitaries**: Let $\mathcal{T}$ be a QCA over $\mathbb{Z}$ whose effect is to apply the same local transformation U_x on each cell x, namely:

$$\mathcal{T}(\mathsf{A}_\Lambda) = \left(\bigotimes_{x\in\Lambda} U_x^\dagger \right) \mathsf{A}_\Lambda \left(\bigotimes_{y\in\Lambda} U_y \right).$$

Wrapping $\mathcal{T}$ over a lattice $\mathcal{L}$ we obtain the induced map:

$$(U_{\mathcal{T}^N})_r = J \left(\bigotimes_{x\in\mathcal{L}} U_x^N \right) J^\dagger =$$
$$= \left(\bigotimes_{x\in\mathcal{L}'} J_{\Lambda_x} \right) \left(\bigotimes_{x\in\mathcal{L}} U_x^N \right) \left(\bigotimes_{y\in\mathcal{L}'} J_{\Lambda_y} \right)^\dagger.$$

Since the transformation is completely factorized, we can evaluate the expression over the support of a single tile Λ_x, i.e.:

$$((U_{\mathcal{T}^N})_r)_x =$$
$$= J_{\Lambda_x} \left(\bigotimes_{x'\in\Lambda_x} U_{x'}^N \right) J_{\Lambda_x}^\dagger = \sum_{k,k'} u_{k,k'} |k\rangle\langle k'|,$$

with

$$u_{k,k'} := \langle \psi_k^{(x)} | \left(\bigotimes_{x'\in\Lambda_x} U_{x'}^N \right) | \psi_{k'}^{(x)} \rangle.$$

The total matrix is then $(U_{\mathcal{T}^N})_r = \bigotimes_{x\in\mathcal{L}'} ((U_{\mathcal{T}^N})_r)_x$. In particular, if we choose the $\{|\psi_k^{(x)}\rangle\}_k$ to be eigenvectors of $\left(\bigotimes_{x\in\Lambda_x} U_x^N\right)$ we have:

$$((U_{\mathcal{T}^N})_r)_x = \sum_k u_{k,k} |k\rangle\langle k|.$$

That is, the renormalised evolution is still a local unitary with eigenvalues given by a subset of the eigenvalues of $\left(\bigotimes_{x\in\Lambda_x} U_x^N\right)$ and eigenvector corresponding to the chosen basis $\{|k\rangle\}_{k=0}^{d'-1}$. A pictorial representation of the size-2 renormalisation is shown in fig.(4).

- **Shift**: Consider the shift $\tau_{\pm 1}$ as in Definition (1). Following the same steps as before we can compute the matrix implementing the induced map as:

$$(U_{\mathcal{T}^N})_r = J \left(\bigotimes_{x\in\mathcal{L}} \tau_{\pm 1}^N \right) J^\dagger =$$
$$= \left(\bigotimes_{x\in\mathcal{L}'} J_{\Lambda_x} \right) \left(\bigotimes_{x\in\mathcal{L}} \tau_{\pm N} \right) \left(\bigotimes_{y\in\mathcal{L}'} J_{\Lambda_y} \right)^\dagger.$$

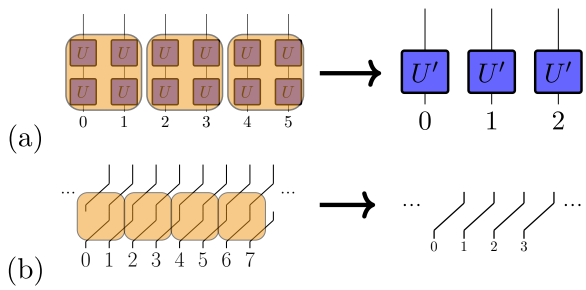


Figure 4: Trivial examples of size-2 renormalisation over one dimensional lattices. (a) Example of renormalisation of a local transformation. Here we have $\Lambda_x = \{2x, 2x+1\}$, two steps of the unitary matrix U over two cells are mapped in a single application of the new matrix U'. (b) Example of renormalisation of two steps of a right shift with $\Lambda_x = \{2x, 2x+1\}$. After two steps, all the content of Λ_x is moved in Λ_{x+1}. Thus the renormalised evolution is a shift $x \to x+1$ in the $\mathcal{L}'$ lattice.

Exploiting the fact that the size of each Λ_x is N, we have $\tau_{\pm N}|\psi_k^{(x)}\rangle = |\psi_k^{(x\pm1)}\rangle$, i.e. $\tau_{\pm N}$ shifts the whole Λ_x in $\Lambda_{x\pm1}$ and, consequently, $\tau_{\pm N} J_{\Lambda_x} = J_{\Lambda_{x+1}}$. In this way we have:

$$\begin{aligned}&(U_{\mathcal{T}^N})_r = \\ &= \left(\bigotimes_{x\in\mathcal{L}'} J_{\Lambda_x}\right)\left(\bigotimes_{y\in\mathcal{L}'} J_{\Lambda_{y+1}}\right)^\dagger = \tau_{\pm1}.\end{aligned}$$

In other words, N steps of a shift in $\mathcal{L}$ are renormalised to a single step of a shift in $\mathcal{L}'$. A pictorial representation of the size-2 renormalisation is shown in fig.(4).

Notice that, in principle, when we deal with more complex automata, we have no reason to believe that $(U_{\mathcal{T}^N})_r$ is a unitary matrix. However, condition (26) implies that $(U_{\mathcal{T}^N})_r = U_{\mathcal{S}} e^{i\phi}$ is a unitary matrix for every suitable lattice $\mathcal{L}$. If this is the case eq.(32) guarantees that $(\mathcal{T}^N)_r$ is an automorphism.

The following Lemma imposes a necessary and sufficient condition on $U_{\mathcal{T}}$ for eq.(26) to hold:

Lemma 4. *Eq.(26) holds with $\mathcal{S}_{\mathcal{L}} = (\mathcal{T}^N)_r$ being an automorphism iff*

$$[U_{\mathcal{T}^N}, \Pi] = 0. \tag{34}$$

Proof. If (26) holds, we can compose it on the right with $\mathcal{V}^\dagger$ and get

$$\mathcal{T}_w^N \circ \mathcal{P} = \mathcal{V} \circ \mathcal{S}_w \circ \mathcal{V}^\dagger. \tag{35}$$

Now, composing the r.h.s. of the latter expression with $\mathcal{P} = \mathcal{V}\circ\mathcal{V}^\dagger$ on the left gives $\mathcal{P}\circ\mathcal{V}\circ\mathcal{S}_w\circ\mathcal{V}^\dagger = \mathcal{V}\circ\mathcal{S}_w\circ\mathcal{V}^\dagger = \mathcal{T}_w^N\circ\mathcal{P}$, while on the other hand, considering l.h.s., we obtain $\mathcal{P}\circ\mathcal{T}_w^N\circ\mathcal{P} = \mathcal{T}_w^N\circ\mathcal{P}$.

In terms of the unitary matrix $U_{\mathcal{T}^N}$ this means: $\Pi U_{\mathcal{T}^N}\Pi = \Pi U_{\mathcal{T}^N}$, i.e. $U_{\mathcal{T}^N}$ is block-diagonal over Supp(Π), and the commutation relation (34) trivially follows. To prove the converse, we now remind that $(U_{\mathcal{T}^N})_r = JU_{\mathcal{T}^N}J^\dagger$. Remembering that $JJ^\dagger = I$ and $J^\dagger J = \Pi_{\mathcal{L}}$, we then have

$$\begin{aligned}&(U_{\mathcal{T}^N})_r(U_{\mathcal{T}^N})_r^\dagger = JU_{\mathcal{T}^N}J^\dagger JU_{\mathcal{T}^N}^\dagger J^\dagger = \\ &= JU_{\mathcal{T}^N}\Pi U_{\mathcal{T}^N}^\dagger J^\dagger = J\Pi U_{\mathcal{T}^N}U_{\mathcal{T}^N}^\dagger J^\dagger = \\ &= JJ^\dagger JJ^\dagger = I_{\mathcal{L}'},\end{aligned}$$

where we exploited the commutation of $U_{\mathcal{T}^N}$ with Π. The same holds for $(U_{\mathcal{T}^N})_r^\dagger(U_{\mathcal{T}^N})_r$ and thus $(U_{\mathcal{T}^N})_r$ is unitary over $\mathsf{A}(\mathcal{L}')$. We can thus make the identification $U_{\mathcal{S}} = (U_{\mathcal{T}^N})_r$, and obtain the identity $U_{\mathcal{S}}J = JU_{\mathcal{T}^N}$, which yelds $\mathcal{V}\circ\mathcal{S}_w = \mathcal{T}_w^N\circ\mathcal{V}$, with $\mathcal{S}_w$ an automorphism of the lattice algebra $\mathsf{A}(\mathcal{L}')$. ■

Notice that eq. (34) is necessary and sufficient for condition (26) for *some* automorphism $\mathcal{S}_{\mathcal{L}}$, which by now is not necessarily the wrapped version of a fixed QCA $\mathcal{S}$ on $\mathbb{Z}^s$. Thus, we do not have yet a necessary and sufficient condition for renormalisability.

It is easy to see that the commutation relation (34) can be formulated as follows

$$U_{\mathcal{T}^N}\Pi U_{\mathcal{T}^N}^\dagger = \Pi \iff \mathcal{T}_w^N(\Pi) = \Pi, \tag{36}$$

for all the possible lattices. We now show that all maps $\mathcal{S}_{\mathcal{L}}$ share the same local rule. This has two important consequences. The first one is that eqs. (27) and (34) become necessary and sufficient for renormalisability through (Π_0, Λ_0). The second and most powerful one is that we can check the condition in eq.(36) only on the smallest lattice such that the neighborhood of $\mathcal{T}^N$ is regular. We first state the following two lemmas. The first one relates the translations in $\mathcal{L}$ with the ones in the coarse-grained lattice $\mathcal{L}'$:

Lemma 5. *$(\mathcal{T}^N)_r$ is invariant under shifts on $\mathcal{L}'$*

Proof. A shift τ_x^w along the vector x in $\mathcal{L}'$ corresponds to a shift τ_{Nx}^w along the vector Nx in $\mathcal{L}$. Since

$$\begin{aligned}\tau_{Nx}^w\mathcal{V} = \mathcal{V}\tau_x^w, \\ \tau_x^w\mathcal{V}^\dagger = \mathcal{V}^\dagger\tau_{Nx}^w,\end{aligned}$$

and the evolution $\mathcal{T}^N$ commutes with τ^w_{Nx}, considering their composition $(\mathcal{T}^N)_r$, we have:

$$[(\mathcal{T}^N)_r, \tau_x] = 0 \quad \forall x \in \mathcal{L}.$$

■

The other Lemma we have to prove is about the form of the unitary matrix $U_{\mathcal{T}^N}$.

Lemma 6. *$U_{\mathcal{T}^N}$ can be written in the following form:*

$$\begin{aligned} U_{\mathcal{T}^N} &= [(I_{\Lambda_0} \otimes V)(U \otimes I_{\Lambda_{\mathcal{W}}})], \\ \mathcal{W} &:= \mathcal{L} \setminus (\Lambda_0 \cup \mathcal{N}^N_{\Lambda_0}), \end{aligned}$$

where U is independent of the wrapping, and $\mathcal{N}^N_{\Lambda_0}$ denotes the neighbourhood of Λ_0 for $\mathcal{T}^N$.

Proof. Since $\mathcal{T}(\mathsf{A}(\Lambda_0)) \subseteq \mathsf{A}(\mathcal{N}_{\Lambda_0})$, this means that $\mathcal{T}^N(\mathsf{A}(\Lambda_0)) \subseteq \mathsf{A}(\mathcal{N}^N_{\Lambda_0})$. In other words, the algebra $\mathcal{T}^N(\mathsf{A}(\Lambda_0))$ is an homomorphic representation of the full matrix algebra $\mathsf{A}(\Lambda_0)$ in $\mathsf{A}(\mathcal{N}^N_{\Lambda_0})$, which means that there exists a unitary $U \in \mathsf{A}(\Lambda_0 \cup \mathcal{N}^N_{\Lambda_0})$ such that, for every $X \in \mathsf{A}(\Lambda_0)$,

$$\mathcal{T}^N(X) = U^\dagger (X \otimes I_{\mathcal{N}^N_{\Lambda_0} \setminus \Lambda_0}) U \in \mathsf{A}(\Lambda_0 \cup \mathcal{N}_{\Lambda_0}).$$

Consider now the wrapped evolution $\mathcal{T}^N_w$. This is implemented over the wrapped lattice as a unitary matrix (see eq. (30)), i.e.

$$\mathcal{T}^N_w(X) = U^\dagger_{\mathcal{T}^N}(X \otimes I_{\mathcal{L}\setminus\mathcal{N}_X}) U_{\mathcal{T}^N}. \tag{37}$$

Since $\mathcal{T}$ and $\mathcal{T}_w$ share the same local rule, we then have

$$\begin{aligned} \mathcal{T}^N_w(X) &= U^\dagger_{\mathcal{T}^N}(X \otimes I_{\mathcal{L}\setminus\Lambda_0}) U_{\mathcal{T}^N} \\ &= U^\dagger (X \otimes I_{\mathcal{N}^N_{\Lambda_0}\setminus\Lambda_0}) U \otimes I_{\mathcal{L}\setminus(\Lambda_0 \cup \mathcal{N}^N_{\Lambda_0})}. \end{aligned}$$

One can easily prove that this implies

$$U_{\mathcal{T}^N}(U \otimes I_{\mathcal{L}\setminus(\Lambda_0\cup\mathcal{N}^N_{\Lambda_0})})^\dagger = (I_{\Lambda_0} \otimes V),$$

for some unitary $V \in \mathsf{A}(\mathcal{L} \setminus \Lambda_0)$, and finally

$$U_{\mathcal{T}^N} = (I_{\Lambda_0} \otimes V)(U \otimes I_{\mathcal{L}\setminus(\Lambda_0\cup\mathcal{N}^N_{\Lambda_0})}). \tag{38}$$

Since the neighbourhood of Λ_0 after N steps is $\Lambda_{\mathcal{N}}$, this gives the above decomposition. ■

We now prove that the commutation condition is independent of the lattice $\mathcal{L}$.

Lemma 7. *If condition in (36) is satisfied for a lattice $\mathcal{L}$ where $\mathcal{T}^N$ is regular, then it is true for every other such lattice.*

Proof. Let us divide the chain $\Pi_{\mathcal{L}}$ over $\mathcal{L}$ in

$$\begin{aligned} &\Pi = \Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}}, \\ &\mathcal{M} := \mathcal{N} \setminus \{0\}, \\ &\mathcal{W} := \mathcal{L} \setminus (\mathcal{N} \cup \{0\}), \end{aligned} \tag{39}$$

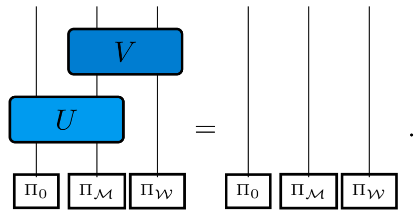
with the convention:

$$\Pi_R = \left(\bigotimes_{x \in R} \Pi_x \right).$$

The commutation relation (34) can then be expressed as

$$U_{\mathcal{T}^N}(\Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}}) U^\dagger_{\mathcal{T}^N} = \Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}}. \tag{40}$$

We can represent eq. (40) diagrammatically, with the decomposition (38) of $U_{\mathcal{T}^N}$, as follows

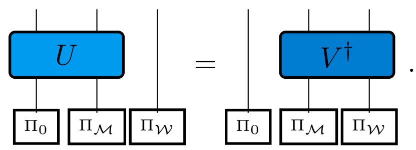


(41)

Exploiting the decompositon in eq.(38) and conjugating both members of eq. (40) with $(I_{\Lambda_0} \otimes V)^\dagger$ we get to:

$$\begin{aligned} &(U \otimes I_{\Lambda_{\mathcal{W}}})(\Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}})(U \otimes I_{\Lambda_{\mathcal{W}}})^\dagger = \\ &= (I_{\Lambda_0} \otimes V)^\dagger (\Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}})(I_{\Lambda_0} \otimes V). \end{aligned} \tag{42}$$

Diagrammatically:

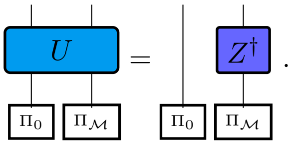


(43)

The two members of eq. (43) are factorised in two different ways. This implies that both factorisations must hold, and then the transformation V on the right produces a factorised projection map, i.e.

$$\begin{aligned} &(I_0 \otimes V)^\dagger (\Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}})(I_0 \otimes V) = \\ &= \Pi_0 \otimes Z^\dagger \Pi_{\mathcal{M}} Z \otimes \Pi_{\mathcal{W}}, \end{aligned} \tag{44}$$

for some unitary $Z \in \mathsf{A}(\Lambda_{\mathcal{M}})$. In particular, we have the following diagrammatic identity



(45)

The latter is independent of the lattice $\mathcal{L}$, and proves that the image of Π under $U_{\mathcal{T}^N}|_{\Pi}$ is a projection of the form $\tilde{\Pi}_{\mathcal{L}' \setminus \{0\}} \otimes \Pi_0$. Finally, by translation invariance we can conclude that for every $\mathcal{L}$ such that $\mathcal{T}^N$ is regular over $\mathcal{L}$ one has

$$U_{\mathcal{T}^N}^{\dagger} \Pi U_{\mathcal{T}^N} = \Pi,$$

namely condition (36) holds. ■

This immediately implies the following corollary

Corollary 1. *If condition in (36) is satisfied for a lattice $\mathcal{L}$ where $\mathcal{T}^N$ is regular, then*

$$\begin{aligned}&(U \otimes I_{\mathcal{W}})^{\dagger}(\Pi_0 \otimes Z^{\dagger}\Pi_{\mathcal{M}} Z \otimes \Pi_{\mathcal{W}})(U \otimes I_{\mathcal{W}}) \\ &= \Pi_0 \otimes \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{W}}.\end{aligned} \tag{46}$$

The above result allows us to show that condition (34) is necessary and sufficient for renormalisability through (Π_0, Λ_0).

Lemma 8. *Let $\mathcal{T}^N$ be regular on the lattices $\mathcal{L}'$ and $\tilde{\mathcal{L}}'$, and let condition (34) be satisfied for both. Then the local rule of $(\mathcal{T}^N)_r$ over $\mathcal{L}'$ and $\tilde{\mathcal{L}}'$ is the same.*

Proof. It is sufficient to notice that a local operator A_x in $\mathsf{A}|_{\Pi}(\mathcal{L})$ is obtained as $\Pi A_{\Lambda_x} \Pi$ for some A_{Λ_x} on the macro-cell Λ_x. As a consequence, one has

$$\begin{aligned}&U_{\mathcal{T}^N}^{\dagger} A_x U_{\mathcal{T}^N} = U_{\mathcal{T}^N}^{\dagger} \Pi A_{\Lambda_x} \Pi U_{\mathcal{T}^N} \\ &= U^{\dagger}(\Pi_x A_{\Lambda_x} \Pi_x \otimes Z^{\dagger} \Pi_{\mathcal{N}_{\Lambda_x} \setminus \Lambda_x} Z) U \otimes \Pi_{\mathcal{L} \setminus (\Lambda_x \cup \mathcal{N}_{\Lambda_x})},\end{aligned}$$

where U is independent of $\mathcal{L}'$. Finally, reminding eq. (32) and (33), one has

$$(\mathcal{T}^N)_r|_{\tilde{\mathcal{L}}'}(A_x) = (\mathcal{T}^N)_r|_{\mathcal{L}'}(A_x),$$

for every $\tilde{\mathcal{L}}'$, where we indicated with $(\mathcal{T}^N)_r|_{\mathcal{L}}$ the evolution $(\mathcal{T}^N)_r$ restricted to the particular lattice $\mathcal{L}$. ■

We have then proved the main result of this section

Proposition 3. *Let $\mathcal{T}$ be a QCAs on $\mathbb{Z}^s$, N be a natural number and let $\Pi_0 \in \mathsf{A}(\Lambda_0)$ be a projection on the local algebra of the region $\Lambda_0 := [0, N-1]^s$. The following conditions are equivalent:*

1. *there exists a QCA $\mathcal{S}$ on $\mathbb{Z}^s$ which is a size-N renormalisation of $\mathcal{T}$ through (Π_0, Λ_0).*

2. *there exist a lattice $\mathcal{L} = \mathbb{Z}^s_{Nf}$ such that the regular neighborhood condition is satisfied for $\mathcal{T}^N$ and $[U_{\mathcal{T}^N}, \Pi] = 0$, where $U_{\mathcal{T}^N}$ is the unitary matrix defining the QCA on the lattice $\mathcal{L}$.*

This proves that the renormalisability condition of Eq.(26), reduces to a finite dimensional equation in terms of the unitary matrix defining the QCA on a wrapped lattice. In particular, the renormalisability condition can be checked by considering the smallest wrapped lattice such that the regular neighbourhood condition is satisfied.

Remark 4. *The chain of projections Π in eq.(34) identifies the degrees of freedom of the observable algebra of the lattice that we are preserving in the coarse-graining—i.e. its support—while the discarded ones correspond to the kernel of Π. However, the choice of Π does not fix univocally the operators J.Indeed, the operators J and J' give rise to the same projection if :*

$$J' = \bigotimes_{x \in \mathcal{L}'} U_x J \quad U_x \in SU(d).$$

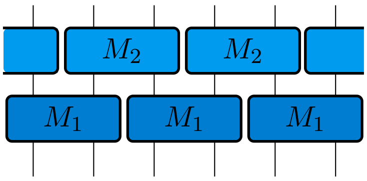
To such family of operators J we can naturally associate a family of $\mathcal{V}^{\dagger}, \mathcal{V}$ as in eq.(21). Each of those choices give rise to renormalised automata that differ by a local change of basis, namely, considering J and J' as above the respective unitaries that implement the induced maps are:

$$\begin{aligned}&(U_{\mathcal{T}^N})_r^J = J U_{\mathcal{T}^N} J^{\dagger} \\ &(U_{\mathcal{T}^N})_r^{J'} = \left(\bigotimes_{x \in \mathcal{L}'} U_x\right)(U_{\mathcal{T}^N})_r \left(\bigotimes_{y \in \mathcal{L}'} U_y\right)^{\dagger}.\end{aligned}$$

In other words, the choice of Π fixes the preserved degrees of freedom in the coarse-graining, while the choice of the basis in which we represent the renormalised evolution is still arbitrary.

4 Renormalisation of one-dimensional quantum circuits

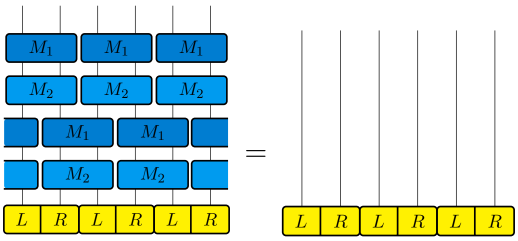
In this section we specify our analysis to QCAs on a one dimensional lattice that can be realised as FDQCs. In particular, we will consider size-2 coarse-graining, i.e. two cells and two steps are coarse-grained into one cell and one step (the projections Π_x act on two cells). Up to grouping neighbouring cells together, we can consider the case in which the QCA $\mathcal{T}$ is nearest-neighbour. From the index theory of QCAs [41] we have that $\mathcal{T}$ is realisable as a quantum circuit if and only if $\operatorname{ind}(\mathcal{T}) = 1$. Then $\mathcal{T}$ can be implemented as a Margolous partitioning scheme (see Proposition (2)) as follows:

$\mathcal{T} =$ 

, (47)

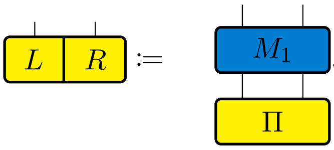
Let us now consider a size-2 renormalisation of a QCA that can be implemented as in Equation (47). Thanks to Proposition (3) we can consider the case in which $\mathcal{T}$ is defined on a finite lattice. We start with the following lemma.

Lemma 9. *Let $\mathcal{T}$ be QCA that admits a realisation as a finite depth quantum circuit as in Equation (47). Then $\mathcal{T}$ admits a size-2 renormalisation if and only if*



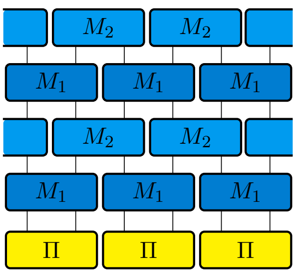
(48)

where we define

$P_{\Lambda_x} =$ 

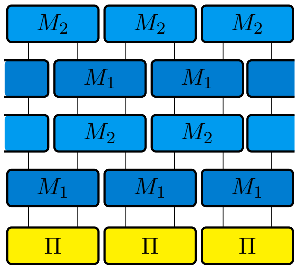
. (49)

Proof. We can write

$\mathcal{T}^2(\bigotimes_{x\in\mathcal{L}'} \Pi_x) =$ 

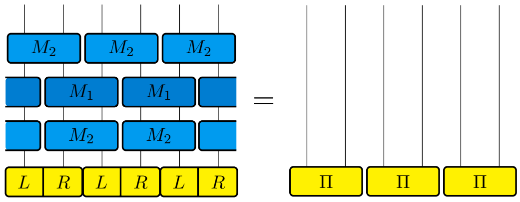
(50)

Since each step is translationally invariant we can shift the second evolution step in order to have

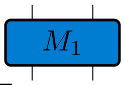
$\mathcal{T}^2(\bigotimes_{x\in\mathcal{L}'} \Pi_x) =$ 

. (51)

By introducing the projection P_{Λ_x}, we have

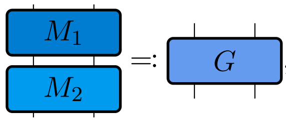


(52)

and by applying 

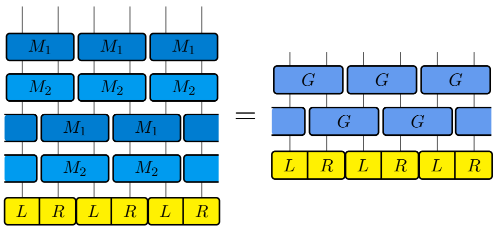
on both sides we obtain the thesis. ■

It is now convenient to define the operator

$G :=$ 

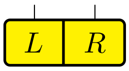
, (53)

so that we have



. (54)

It is also convenient to consider the Schmidt decomposition of the projection P_{Λ_x}:

$P_{\Lambda_x} =$ 

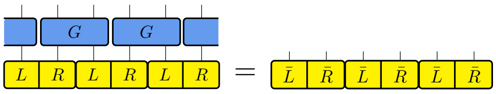
$= \sum_{\mu_x} \lambda_{\mu_x} \otimes \rho_{\mu_x}$. (55)

In terms of G and of the operators $\{\lambda_{\mu_x}, \rho_{\mu_x}\}$ we have the following necessary and sufficient condition for $\mathcal{T}$ to admit a size-2 renormalisation.

Proposition 4. *A one dimensional QCA $\mathcal{T}$ that has a Margolus partitioning scheme as in Equation (47) admits a size-2 renormalisation iff for every Schmidt decomposition $\sum_\mu \lambda_\mu \otimes \rho_\mu$ of P, there exists a Schmidt Decomposition $\sum_\mu \tilde{\lambda}_\mu \otimes \tilde{\rho}_\mu$ of $GPG^\dagger$ such that:*

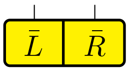
$$G^\dagger(\rho_\mu \otimes \lambda_\nu)G = \tilde{\rho}_\mu \otimes \tilde{\lambda}_\nu, \quad \forall \mu, \nu, \tag{56}$$

Proof. Inverting the second layer of G's on both sides of (48) we have the equivalent condition



$$ \tag{57} $$

where

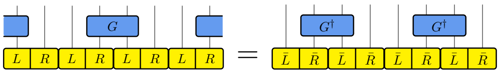


$$:= G\left(\sum_{\mu} \lambda_\mu \otimes \rho_\mu\right) G^\dagger. \tag{58} $$

In formulas,

$$ \begin{aligned} &\sum_{\boldsymbol{\mu}} \bigotimes_{x\in\mathcal{L}'} G^\dagger(\rho_{\mu_{x-1}} \otimes \lambda_{\mu_x})G = \\ &= \bigotimes_{y\in\mathcal{L}'} \sum_{\mu_y} G(\lambda_{\mu_y} \otimes \rho_{\mu_y})G^\dagger \\ &= \bigotimes_{y\in\mathcal{L}'} \sum_{\mu_y} \bar{\lambda}_{\mu_y} \otimes \bar{\rho}_{\mu_y}, \end{aligned} \tag{59} $$

where the sum over $\boldsymbol{\mu}$ is shorthand for the sum over all the indices μ_x, for $x \in \mathcal{L}'$, and $\sum_\mu \bar{\lambda}_\mu \otimes \bar{\rho}_\mu$ is some Schmidt decomposition of $GPG^\dagger$. Inverting half of the G's we obtain another equivalent condition:



$$ \tag{60} $$

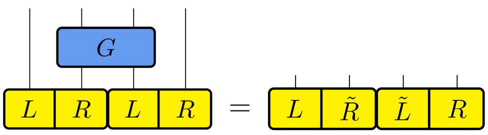
which in formulas reads

$$ \begin{aligned} &\sum_{\boldsymbol{\mu}} \bigotimes_{x\in\mathcal{L}'} \lambda_{\mu_{2x}} \otimes G^\dagger(\rho_{\mu_{2x}} \otimes \lambda_{\mu_{2x+1}})G \otimes \rho_{\mu_{2x+1}} = \\ &= \sum_{\boldsymbol{\nu}} \bigotimes_{x\in\mathcal{L}'} \bar{\lambda}_{\nu_{2x-1}} \otimes G\left(\bar{\rho}_{\nu_{2x-1}} \otimes \bar{\lambda}_{\nu_{2x}}\right) G^\dagger \otimes \bar{\rho}_{\nu_{2x}}. \end{aligned} \tag{61} $$

Comparing the factorisations on the left and right hand side, and considering the cells Λ_x, Λ_{x+1}, we have:

$$ \begin{aligned} &\sum_{\mu,\nu} \lambda_\mu \otimes G^\dagger(\rho_\mu \otimes \lambda_\nu)G \otimes \rho_\nu = \\ &\qquad = \sum_{\mu,\nu} \lambda_\mu \otimes \tilde{\rho}_\mu \otimes \tilde{\lambda}_\nu \otimes \rho_\nu, \end{aligned} \tag{62} $$

for suitable $\tilde{\lambda}_\mu, \tilde{\rho}_\mu$, or diagrammatically



$$ \tag{63} $$

Using the linear independence of all factors in the terms of the Schmidt decomposition λ_ν, ρ_μ we get to

$$ G^\dagger(\rho_\nu \otimes \lambda_\mu)G = \tilde{\rho}_\nu \otimes \tilde{\lambda}_\mu, \tag{64} $$

for all possible combinations of μ, ν. Finally, inserting this result in Eq. (59) we get to:

$$ \begin{aligned} &\sum_{\boldsymbol{\mu}} \bigotimes_{x\in\mathcal{L}'} \bar{\lambda}_\mu \otimes \bar{\rho}_\mu = \\ &= \sum_{\boldsymbol{\mu}} \bigotimes_{x\in\mathcal{L}'} \tilde{\lambda}_\mu \otimes \tilde{\rho}_\mu. \end{aligned} \tag{65} $$

Thus, invoking (64), one can prove that $\sum_\mu \tilde{\lambda}_\mu \otimes \tilde{\rho}_\mu$ is another Schmidt decomposition of $GPG^\dagger$. The proof of the converse statement is now straightforward, and this concludes the proof. ■

It is easy to see that the renormalisability conditions (56) are equivalent to

$$ G^\dagger(\rho^{(n)} \otimes \lambda^{(n)})G = \tilde{\rho}^{(n)} \otimes \tilde{\lambda}^{(n)} \tag{66} $$

where $\rho^{(n)}, \lambda^{(n)}, \tilde{\rho}^{(n)}$ and $\tilde{\lambda}^{(n)}$ denote the monomials of order n in $\rho_\mu, \lambda_\mu, \tilde{\rho}_\mu$ and $\tilde{\lambda}_\mu$ respectively, i.e.

$$ \begin{aligned} \lambda^{(n)} &= \lambda_{\nu_1} \ldots \lambda_{\nu_n}, \quad & \rho^{(n)} &= \rho_{\mu_1} \ldots \rho_{\mu_n} \\ \tilde{\lambda}^{(n)} &= \lambda_{\nu_1} \ldots \lambda_{\nu_n}, \quad & \tilde{\rho}^{(n)} &= \rho_{\mu_1} \ldots \rho_{\mu_n} \end{aligned} $$

We have then the following result.

Lemma 10. *Let us denote with* $\mathsf{M} = \langle\{\rho_\mu\}\rangle, \mathsf{N} = \langle\{\lambda_\mu\}\rangle, \tilde{\mathsf{M}} = \langle\{\tilde{\rho}_\mu\}\rangle$ *and* $\tilde{\mathsf{N}} = \langle\{\tilde{\lambda}_\mu\}\rangle$ *the algebras generated by* $\rho_\mu, \lambda_\mu, \tilde{\rho}_\mu$ *and* $\tilde{\lambda}_\mu$*. Then, for any* $A \in \mathsf{N}$*,* $B \in \mathsf{M}$ *there exist* $\tilde{A} \in \tilde{\mathsf{N}}$ *and* $\tilde{B} \in \tilde{\mathsf{M}}$ *such that*

$$ G^\dagger(A \otimes B)G = \tilde{A} \otimes \tilde{B} \tag{67} $$

Proof. Since $P^2 = P$, we have

$$ \lambda_\mu = \sum_{j,k} c_\mu^{j,k} \lambda_j \lambda_k \quad \rho_\nu = \sum_{j,k} d_\nu^{j,k} \rho_j \rho_k. \tag{68} $$

for suitable coefficients $c_\mu^{j,k}$ and $d_\nu^{j,k}$. By iteratively using Equation (68) we can express any homogeneous polynomial of degree n in the variables λ_μ as a homogeneous polynomial of degree $n+k$ for any k, and the same holds for homogeneous polynomial in the variables ρ_μ. Then, any arbitrary $A \in \mathsf{M}$ and $B \in \mathsf{N}$ can be written as homogeneous polynomials of the same degree n for sufficiently large n. The thesis now follows from Eq. (66)). ■

Corollary 2. *If G commutes with the swap then* $[G^2, P] = 0$ *and*

$$ \begin{aligned} &\forall A \otimes B \in \mathsf{K}, \exists \tilde{A} \otimes \tilde{B} \in \tilde{\mathsf{K}} \textit{ such that} \\ &G^\dagger(A \otimes B)G = \tilde{A} \otimes \tilde{B}, \end{aligned} \tag{69} $$

where we defined

$$\begin{aligned} \mathsf{K} &:= (\mathsf{M}\otimes\mathsf{N}\cup\mathsf{N}\otimes\mathsf{M})'' \\ \tilde{\mathsf{K}} &:= (\tilde{\mathsf{M}}\otimes\tilde{\mathsf{N}}\cup\tilde{\mathsf{N}}\otimes\tilde{\mathsf{M}})'' \end{aligned} \tag{70}$$

Proof. If G commutes with the swap, Equation (56)) implies that

$$\lambda_\mu\otimes\rho_\nu = G(\tilde{\lambda}_\mu\otimes\tilde{\rho}_\nu)G^\dagger. \tag{71}$$

Then we have: $G^2PG^{\dagger 2} = G(GPG^\dagger)G^\dagger = G(\sum_\mu\tilde{\lambda}_\mu\otimes\tilde{\rho}_\mu)G^\dagger = \sum_\mu\lambda_\mu\otimes\rho_\mu = P$. From Equation (71) we have that Equation (67) holds for every operator generated by $\{\lambda_\mu\otimes\rho_\nu\}\cup\{\rho_\nu\otimes\lambda_\mu\}$. ■

4.1 Renormalisation of a quantum circuit to a QCA with a different index

In this section we we will consider the case in which a one dimensional QCA with index equal to 1 admits a size-2 renormalisation to a QCA which is given by a left or right shift followed by local unitaries.

Thus, we need to be able to identify a full matrix algebra of $d\times d$ matrices in the macro-cell Λ_x, that after two steps is totally moved to the left macro-cell. The following proposition specialises the renormalisability condition to this case.

Proposition 5. *The condition* (26) *in the case of renormalisation of a unit index QCA to a QCA with different index reduces to*

$$\sum_{\mu}\bigotimes_{x\in\mathcal{L}}\tilde{\lambda}_{\mu_{x+1}}\otimes\tilde{\rho}_{\mu_{x-1}} = \sum_{\mu}\bigotimes_{x\in\mathcal{L}}\lambda_{\mu_x}\otimes\rho_{\mu_x}, \tag{72}$$

where $\tilde{\lambda}_{\mu_{x+1}} = V\lambda_{\mu_{x+1}}V^\dagger$ *and* $\tilde{\rho}_{\mu_{x-1}} = V\rho_{\mu_{x-1}}V^\dagger$ *for some unitary operator* V*.*

Proof. The support algebra of a macro-cell Λ_x in Λ_{x-1} after two steps is a subalgebra L' of $\mathsf{L}_{2x-1}\cap\mathcal{T}(\mathsf{L}_{2x})\simeq\mathcal{T}^{-1}(L_{2x-1})\cap\mathsf{L}_{2x}$, and that of Λ_x in Λ_{x+1} after two steps is a subalgebra R' of $\mathsf{R}_{2x+1}\cap\mathcal{T}(\mathsf{R}_{2x})\simeq\mathcal{T}^{-1}(R_{2x+1})\cap\mathsf{R}_{2x}$. Starting from an index 1-QCA $\mathcal{T}$, we have $\dim\mathsf{L} = \dim\mathsf{R} = d^2$. If we now consider a projection Π with rank d, and a general subalgebra S of $\mathsf{A}_{\Lambda_{x\pm1}}$, we have

$$\dim(\Pi\mathsf{S}\Pi)\le\dim(\mathsf{S}),$$

and thus, if we want to have

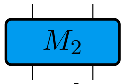
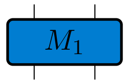
$$\dim(\Pi\mathsf{L}'\Pi) = d^2 \qquad (\text{or }\ \dim(\Pi\mathsf{R}'\Pi) = d^2),$$

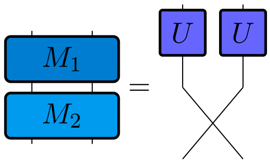
we need to have

$$\begin{aligned} &\mathsf{L}' = \mathcal{T}^{-1}(\mathsf{L}_{2x-1})\cap\mathsf{L}_{2x}\equiv\mathcal{T}^{-1}(\mathsf{L}_{2x-1})\equiv\mathsf{L}_{2x} \\ &(\mathsf{R}' = \mathcal{T}^{-1}(\mathsf{R}_{2x+1})\cap\mathsf{R}_{2x}\equiv\mathcal{T}^{-1}(\mathsf{R}_{2x+1})\equiv\mathsf{R}_{2x}) \end{aligned}$$

Finally, this implies that, in order to have a renormalisation to a left shift or right shift, we need to have $\mathcal{T}(\mathsf{L})\equiv\mathsf{L}$ or $\mathcal{T}(\mathsf{R})\equiv\mathsf{R}$, respectively. Notice however that $\mathcal{T}(\mathsf{L})\equiv\mathsf{L}$ if and only if $\mathcal{T}(\mathsf{R})\equiv\mathsf{R}$.

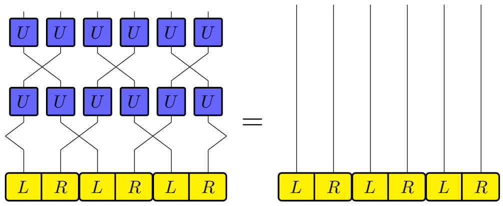
This implies that M_2 must map the algebra in its right input cell to $\mathcal{T}^{-1}(\mathsf{L})$ and that in its left input in $\mathcal{T}^{-1}(\mathsf{R})$, i.e. it must be equal to a swap followed by local isomorphisms followed by M_1^{-1}. Finally, this implies that

the composition of  and  results in a swap up to a local unitary, or—diagrammatically—



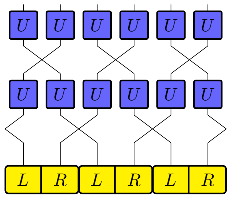
(73)

In this way eq.(48) becomes



, (74)

Using the decomposition in eq.(55)) and computing the left side of the previous equation we get



$$= \sum_{\mu}\bigotimes_{x\in\mathcal{L}}\tilde{\lambda}^{(x)}_{\mu_{x+1}}\otimes\tilde{\rho}^{(x)}_{\mu_{x-1}}, \tag{75}$$

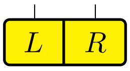
where $\tilde{\lambda}^{(x)}_{\mu_{x+1}} = U^2\lambda^{(x)}_{\mu_{x+1}}U^{\dagger 2}$ and $\tilde{\rho}^{(x)}_{\mu_{x-1}} = U^2\rho^{(x)}_{\mu_{x-1}}U^{\dagger 2}$ Here we introduced a superscript to keep track of the the macro-cell on which an operator acts *after* the evolution, while the subscript is a summation index that is reminiscent of the macro cell on which the operator acted *before* the evolution. In this way eq.(74)) becomes

$$\sum_{\mu}\bigotimes_{x\in\mathcal{L}}\tilde{\lambda}^{(x)}_{\mu_{x+1}}\otimes\tilde{\rho}^{(x)}_{\mu_{x-1}} = \sum_{\mu}\bigotimes_{x\in\mathcal{L}}\lambda^{(x)}_{\mu_x}\otimes\rho^{(x)}_{\mu_x}. \tag{76}$$

■

In the following we prove that, if eq.(76) holds, then no renormalisation other than the one that changes the index is allowed.

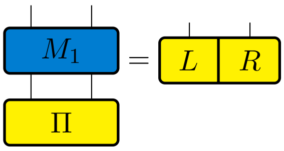
Proposition 6. *If eq.(76) holds for a given unitary index QCA, that QCA can only be renormalised into a QCA with non-unitary index. Moreover, the rank-2 projections P can only be:*

$P =$  $= \begin{cases} |\phi_0\rangle\langle\phi_0| \otimes I \\ I \otimes |\psi_0\rangle\langle\psi_0| \end{cases}$ (77)

Proof. On the left we have that the summation index μ is the same for $\tilde{\lambda}_\mu^{(x)}$ and $\tilde{\rho}_\mu^{(x+2)}$. So if we multiply both sides by a fixed $(\tilde{\rho}_\nu^{(x+2)})^\dagger$ in the macro-cell (Λ_{x+2}) and $(\tilde{\lambda}_\gamma^{(x-2)})^\dagger$ in the macro-cell (Λ_{x-2}), and take the partial trace on the corresponding spaces, the expression on both sides is left such that the operator on the macro-cell Λ_x is factorized from the rest, and we have

$$\tilde{\lambda}_\nu^{(x)} \otimes \tilde{\rho}_\gamma^{(x)} = \sum_{\mu_x} \lambda_{\mu_x}^{(x)} \otimes \rho_{\mu_x}^{(x)}. \tag{78}$$

In order to have the equality between the two members we need to have only one term in the expression of eq.(55)), i.e.

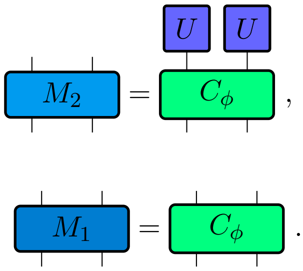
 $= \begin{cases} |\phi_0\rangle\langle\phi_0| \otimes I \\ I \otimes |\psi_0\rangle\langle\psi_0| \end{cases}$ (79)

Moreover, in order for eq.(74)) to be satisfied, either $|\phi_0\rangle$ or $|\psi_0\rangle$ should be eigenstates of the local unitary U in eq.(73)). But this means that our projection will select only the left or the right algebra. Thus the renormalised evolution will consist in an algebra moving to the left or to the right, depending on the choice of the projection. ■

5 Renormalisation of one dimensional Qubit QCA

In this section we will apply the tools developed so far to the renormalisation of one dimensional qubit QCAs with index 1, i.e. QCA on $\mathbb{Z}$ such that each cell is a two dimensional quantum system and such that they can be realized as a finite depth quantum circuit.

Comparing Equation (11) with Equation (47) we have

 (80)

and the operator G of Equation (53) becomes

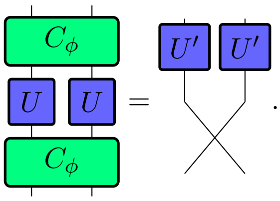
$$G = C_\phi(U \otimes U)C_\phi \tag{81}$$

Lemma 11. *A qubit QCA that is renormalisable only admits unit index renormalisations.*

Proof. By the argument in the previous section, in order to have a renormalised QCA with index different from unit we should have

$$G = (U' \otimes U')S, \tag{82}$$

where S is the swap operator, $S|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$. Diagrammatically this reads

 (83)

This is impossible, since we need at least three controlled gates in order to have a swap, and only two controlled phases appear in the expression of G. Thus the renormalised evolution will have index 1. ■

Lemma 12. *Let $\mathcal{T}$ be a one dimensional qubit QCA with index 1 which admits a size-2 renormalisation and let us denote with $\mathcal{M}$ the algebra of 2×2 complex matrices. Then either we have*

$$\begin{aligned} &\forall A, B \in \mathcal{M}, \exists \tilde{A}, \tilde{B} \in \mathcal{M} \textit{ such that} \\ &G(A \otimes B)G^\dagger = \tilde{A} \otimes \tilde{B} \end{aligned} \tag{84}$$

or there exist a unital abelian algebra $\mathcal{A}$ and a unital abelian algebra $\tilde{\mathcal{A}}$ of commuting 2×2 matrices such that

$$\begin{aligned} &\forall A, B \in \mathcal{A}, \exists \tilde{A}, \tilde{B} \in \tilde{\mathcal{A}} \textit{ such that} \\ &G(A \otimes B)G^\dagger = \tilde{A} \otimes \tilde{B} \end{aligned} \tag{85}$$

Renormalised evolutions			
$\mathcal{T}$	$P = Q_1$	$P = Q_2$	$P = I \otimes \lvert c\rangle\langle c\rvert$
$\begin{cases} \phi \notin \{0,\pi\} \\ n_x = n_y = 0 \end{cases}$	$\begin{cases} \phi' = 2\phi \\ \theta' = \phi - 4\theta \end{cases}$ $\begin{cases} \phi' = 2\phi \\ \theta' = 4\theta - 3\phi \end{cases}$	$\begin{cases} \phi' = -2\phi \\ \theta' = \phi \end{cases}$	$\begin{cases} \phi' = 0 \\ \theta' = \pm 2(\theta + \delta_{c,0}\phi) \end{cases}$
$\begin{cases} \phi \notin \{0,\pi\} \\ n_z = 0 \\ \theta = \frac{\pi}{2} \end{cases}$	$\begin{cases} \phi' = 2\phi \\ \theta' = -\phi \end{cases}$	$\begin{cases} \phi' = -2\phi \\ \theta' = \phi \end{cases}$	$\begin{cases} \phi' = 0 \\ \theta' = \mp(2c-1)\phi \end{cases}$

Table 1: The table sums up the renormalisation flow of QCAs with $\phi \neq 0$: for each choice of the parameters in $\mathcal{T}$ the renormalised evolutions associated to a particular choice of the projection P are shown: $Q_1 := |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$ and $Q_2 := |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|$.

Proof. From Equation (81) we have that $[G, S] = 0$, where S is the swap operator. Then Corollary (2) holds. If X is a C^*-subalgebra of $\mathcal{M}$, then one of the following must be the case $i)$ $\mathsf{X} = \mathcal{M}$, $ii)$ $\mathsf{X} = \mathcal{A}$ where $\mathcal{A}$ is 2-dimensional unital abelian algebra, $iii)$ $\mathsf{X} = \mathsf{I} := \{zI, z \in \mathbb{C}\}$, or $iv)$ $\mathsf{X} = \mathsf{O} := \{z|\psi\rangle\langle\psi| \mid z \in \mathbb{C}\})$. Let M, N, $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{N}}$ be the algebras defined in Lemma (10) and let K and $\tilde{\mathsf{K}}$ be the algebras defined in Corollary (2). If $\mathsf{M} = \mathsf{O}$ and $\mathsf{N} = \mathsf{O}'$ then $\operatorname{rank}(P) = 1$, and since we require that $\operatorname{rank}(P) = 2$, this case is ruled out. Then at least one between M and N must be a unital algebra. Without loss of generality, let us assume that M is unital. Let us now examine the several cases that can occur. If either $\mathsf{M} = \mathcal{M}$ or $[\mathsf{M}, \mathsf{N}] \neq 0$ then it must be $\mathsf{K} = \mathcal{M} \otimes \mathcal{M}$. If $\mathsf{M} = \mathcal{A}$ and $[\mathsf{M}, \mathsf{N}] = 0$ then $\mathsf{K} = \mathcal{A} \otimes \mathcal{A}$. If $\mathsf{M} = \mathsf{I}$ then $\mathsf{N} \neq \mathsf{I}$, since otherwise we would have rank $P = 4$. If $\mathsf{M} = \mathsf{I}$ and $\mathsf{N} = \mathcal{M}$ then $\mathsf{K} = \mathcal{M} \otimes \mathcal{M}$. If $\mathsf{M} = \mathsf{I}$ and $\mathsf{N} = \mathcal{A}$ then $\mathsf{K} = \mathcal{A} \otimes \mathcal{A}$. Finally, If $\mathsf{M} = \mathsf{I}$ and $\mathsf{N} = \mathsf{O}$ then $\mathsf{K} = \mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A}$ is the abelian algebra generated by I and $|\psi\rangle\langle\psi|$. Then either $\mathsf{K} = \mathcal{A} \otimes \mathcal{A}$ or $\mathsf{K} = \mathcal{M} \otimes \mathcal{M}$. In the same way, we can prove that either $\tilde{\mathsf{K}} = \mathcal{M} \otimes \mathcal{M}$ or $\tilde{\mathsf{K}} = \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}$. Since G is unitary, $\mathsf{K} = \mathcal{M} \otimes \mathcal{M}$ implies $\tilde{\mathsf{K}} = \mathcal{M} \otimes \mathcal{M}$ and $\mathsf{K} = \mathcal{A} \otimes \mathcal{A}$ implies $\tilde{\mathsf{K}} = \mathcal{A} \otimes \mathcal{A}$. ■

Proposition 7. *Let $\mathcal{T}$ be a one dimensional qubit QCA with index 1, and let us express the corresponding operator G as follows: $G = C_\phi(U_n \otimes U_n)C_\phi$ where*

$$\begin{aligned} &U_n := \exp(-i\theta\mathbf{n} \cdot \sigma) \\ &C_\phi|a\rangle|b\rangle = e^{i\phi ab}|a\rangle|b\rangle, \quad a, b \in \{0,1\} \\ &\mathbf{n} = (n_x, n_y, n_z)^T, |\mathbf{n}| = 1 \quad \sigma_z|a\rangle = (-1)^a|a\rangle. \end{aligned} \tag{86}$$

Then $\mathcal{T}$ admits a size-2 renormalisation iff

$$(\phi = 0) \vee (n_x = n_y = 0) \vee (n_z = 0, \theta = \tfrac{\pi}{2}). \tag{87}$$

Proof. Let us suppose that $\mathcal{T}$ admits a size-2 renormalisation. From Lemma (12) we know that either Equation (84) or Equation (85) must hold. If Equation (84) holds then G is a unitary operator that maps factorised states into factorised states, which implies [47] that either $G = V \otimes W$ or $G = (V \otimes W)S$ where V, W are some unitary operators and S is the swap gate. Since $[G, S] = 0$ we must have $G = V \otimes V$ or $G = (V \otimes V)S$. However, the case $G = (V \otimes V)S$ was ruled out in the proof of Lemma (11) and and we are left with $G = V \otimes V$. Then we must have the identity $V \otimes V = C_\phi(U_n \otimes U_n)C_\phi$, which in the basis of the eigenstates of σ_z reads as follows

$$\begin{aligned}
&\begin{pmatrix} v_{00}V & v_{01}V \\ v_{10}V & v_{11}V \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} u_{00}U & u_{01}U \\ u_{10}U & u_{11}U \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \Phi \end{pmatrix} \\
&= \begin{pmatrix} u_{00}U & u_{01}U\Phi \\ u_{10}\Phi U & u_{11}\Phi U\Phi \end{pmatrix} \\
&V = \begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{pmatrix}, \ U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}, \\
&\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}
\end{aligned}$$

If $v_{00} \neq 0$ then $v_{00}V = u_{00}U$ implies that $u_{00} \neq 0$ and $V \propto U$. Then the condition $v_{01}V = u_{01}U\Phi$ implies that either $\Phi \propto I$, i.e. $\phi = 0$, or $v_{01} = u_{01} = 0$, i.e. $n_y = n_x = 0$. On the other hand if $v_{00} = 0$ then we must have that $u_{00} = u_{11} = 0$, i.e. $n_z = 0, \theta = \frac{\pi}{2}$. The case in which Equation (85) holds is more elaborate and it is carried out in Appendix (C). Conversely, suppose that $(\phi = 0) \vee (n_x = n_y = 0) \vee (n_z = 0, \theta = \pi/2)$. If $\phi = 0$ then $\mathcal{T}$ is a QCA made of local unitaries and it clearly admits a size-2 renormalisation (see Fig. (4)). We then focus on the remaining cases. Let us consider a wrapping of $\mathcal{T}$ on $\mathbb{Z}_n$ where n is an even number greater than 8 [5]. One can see that the regular neighborhood condition is satisfied for $\mathcal{T}^2$ and that $\mathcal{T}^2$ is diagonal in the computational basis, i.e. $\mathcal{T}^2 = \sum_{i_1=0}^{1} \cdots \sum_{i_n=0}^{1} \lambda(i_1,\ldots,i_n) \bigotimes_{x=1}^{n} |i_x\rangle\langle i_x|$ for suitable phases $\lambda(i_1,\ldots,i_n)$. Then $[U_{\mathcal{T}^2}, \Pi] = 0$ for $\Pi := \otimes_x \Pi_x$ and $\Pi_x := |a\rangle\langle a| \otimes |b\rangle\langle b| + |a'\rangle\langle a'| \otimes |b'\rangle\langle b'|$ for any choices of $a, b, a', b' \in \{0,1\}$ such that Π_x has rank 2. ■

We now want to classify all the renormaisable QCAs, along with their respective renormalised evolution. The explicit calculations are straightforward but tedious and are carried out in Appendix (D). There are three main classes of such QCA:

1. **QCA that trivialise after two steps**: This class of QCA corresponds to $\mathcal{T}^2 = \mathcal{I}$ and is renormalised in a local unitary. Since two steps of the evolution correspond to the identity, any choice of the projection P constitutes a legitimate choice for the coarse-graining.

2. **QCA that apply a local unitary transformation**: This class of QCA corresponds to an evolution made of local unitaries. Imposing the commutation of two steps with the projection, it is easy to see that the only suitable projections are the abelian one over a factorised basis of U^2 (see Figure (4)).

3. **QCA that after two steps are diagonal on a factorised basis**: This class of QCA can be renormalised through a projection diagonal in the factorised computational basis. Those are the only QCAs that admit a size-2 renormalisation without reducing to a local unitary, i.e. they have $\phi \neq 0$. There are two main subclasses:

   (a) QCA with a single step diagonal over the computational basis

   (b) QCA with a single step anti-diagonal over the computational basis

The first two cases are trivial, and essentially boil down to the example in Fig.(4)r . Table (1) summarises the results for the non trivial case where $\phi \neq 0$. We can observe that all the size-2 renormalisable QCAs do not propagate information, since either they are factorised, or their Margolus partitioning scheme involves commuting layers, which implies that all the odd layers can be grouped into one, and the same for the even layers, thus collapsing in a transformation with a neighbourhood contained in that of a single step, independently of how many steps are performed (see Fig.(5)). r

By direct inspection of the solutions in Table (1), one can straightforwardly see that the only fixed point with $\phi \neq 0$ is given by case 3.(a). This is given by the evolution:

$$\begin{cases} \phi = \frac{2n\pi}{3}, \\ \theta = \frac{2n\pi}{3}, \end{cases} \tag{88}$$

which is mapped into itself by means of the projection $P = |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|$. This

[5] Actually, it is shown in the Appendix that it is sufficient to consider a lattice $\mathbb{Z}_n$ with n even and greater than 6.

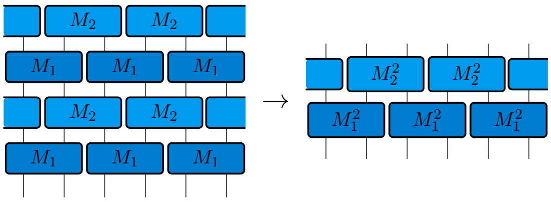


Figure 5: Two steps of the evolutions that can be renormalised collapse in a single one: no information can be propagated after the first step.

case corresponds to the local evolution given by:

$$\begin{aligned} \mathcal{T}_0(A_0) &= X^\dagger (I \otimes e^{-i\frac{2n\pi}{3}\sigma_z} A_0 e^{i\frac{2n\pi}{3}\sigma_z} \otimes I) X, \\ X &= (C_{\frac{2n\pi}{3}} \otimes I_3)(I_1 \otimes C_{\frac{2n\pi}{3}})\,. \end{aligned} \tag{89}$$

6 Conclusion

In conclusion, we analysed the problem of renormalising cellular automata on $\mathbb{Z}^s$ via a Kadanoff-like coarse-graining procedure. An analysis similar in its conceptual structure is at the core of the recently proposed *operator algebraic* approach to Wilson's renormalisation group [48, 49, 50, 51]. We provide necessary and sufficient conditions for renormalisability that are amenable to further analysis. We then specialised to the case where $s = 1$ and the cells are qubits, with a coarse-graining where two cells and two time steps are mapped into a single one. We completely solved this case. The solution that we found tells two important facts: for our special case $s = 1$ and qubit cells, we cannot change the index upon renormalisation, and most QCAs are not renormalisable, with the exception of QCAs where information does not actually propagate. We conjecture that this very restrictive result is due to the extremely limited choice of degrees of freedom that one can "erase" in the coarse-graining process in this simple case. In order to prove that more interesting cases exist, the present analysis should be developed either for coarse-grainings where more than two cells and more than two steps are renormalised into a single one, or move beyond the qubit-cell case. One can easily prove that the second choice includes the first one as a special case. Further developments can then be achieved provided a generalisation of the classification of Margolus partitioning schemes.

Another interesting development would be to carry a similar analysis in the case of *Fermionic* cellular automata. Also this case requires a generalisation of the classification of Margolus partitioning schemes, which is then a required intermediate step.

Finally, our necessary and sufficient condition might be exploited to study the renormalisation problem for QCAs on higher-dimensional graphs, such as $\mathbb{Z}^2$ or $\mathbb{Z}^3$. We expect that in these cases more interesting situations can arise, with fixed points that might represent phase transitions for the statistical mechanics of QCAs, opening interesting questions about the universality classes of QCAs and their relation to other models, both in statistical mechanics and in quantum field theory. It is worth noting that our necessary and sufficient condition for a QCA to admit a coarse-graining coincides with the one identified in [52] for a Quantum Walk to be Lorentz covariant. Additionally, we have focused on the problem of exact coarse-graining. This approach can serve as a foundation for developing more flexible notions of coarse-graining, where some of the current framework's constraints are relaxed. For instance, in [53], an approximate coarse-graining method is proposed for quantum walks, in which a given evolution is replaced by the unitary dynamics that best approximates it—after tracing out part of the system's degrees of freedom.

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A The quasi-local algebra

Here we define rigorously the quasi-local algebra. In the following Ω and Ω_i denote finite subsets of $\mathbb{Z}^s$. Consider now the algebras of operators acting on a finite set of sites $\Omega \subset \mathbb{Z}^s$

$$\mathsf{A}(\Omega) = \bigotimes_{x \in \Omega} \mathsf{A}_x, \tag{90}$$

where A_x is the C*-algebra of operators acting on cell x. For $\Omega_1 \subset \Omega_2$ we can consider $\mathsf{A}(\Omega_1)$ as a subalgebra of $\mathsf{A}(\Omega_2)$ by tensoring with the identity over the region $\Omega_2 \setminus \Omega_1$ the operators in $\mathsf{A}(\Omega_1)$, that is:

$$O \otimes I_{\Omega_2 \setminus \Omega_1} \in \mathsf{A}(\Omega_2) \;\; \forall O \in \mathsf{A}(\Omega_1). \tag{91}$$

In this way the product $O_1 O_2$ is well defined over $\Omega_1 \cup \Omega_2$ for $O_1 \in \mathsf{A}(\Omega_1)$ and $O_2 \in \mathsf{A}(\Omega_2)$. We can now define the equivalence relation $\sim$ by saying that $O_1 \sim O_2$ with $O_i \in \mathsf{A}(\Omega_i)$ if

$$O_1 \otimes I_{(\Omega_1 \cup \Omega_2) \setminus \Omega_1} = O_2 \otimes I_{(\Omega_1 \cup \Omega_2) \setminus \Omega_2}. \tag{92}$$

In other words two operators over Ω_1 and Ω_2 are equivalent under $\sim$ if they are are the same operator when seen as elements of $\Omega_1 \cup \Omega_2$. The local algebra is then the algebra of the equivalence classes under $\sim$ of all operators algebras over finite set of sites, or

Definition 9 (Local Algebra). *The Local Algebra* $\mathsf{A}^{\mathrm{loc}}$ *is defined as*

$$\mathsf{A}^{\mathrm{loc}} = \bigcup_{\Omega} \mathsf{A}(\Omega) \; / \sim \tag{93}$$

Moreover, since $\mathsf{A}(\Omega)$ for every finite set $\Omega \subseteq \mathbb{Z}^s$ is a normed space with the uniform operator norm $\|\cdot\|_\infty$, and $\|O \otimes I\|_\infty = \|O\|_\infty$, the norm of $O \in \mathsf{A}_{\Omega_1}$ when embedded in $\mathsf{A}(\Omega)$ as in Eq. (91) is the same as that of O seen as an element of A_{Ω_1}. We can then make also $\mathsf{A}^{\mathrm{loc}}$ into a normed space by defining the norm of an equivalence class as the norm of any of its representatives which is indeed independent of the chosen representative.

We can then define an equivalence relation for Cauchy sequences, setting $\{\mathsf{A}_n\}_{n=0}^\infty \simeq \{\mathsf{B}_n\}_{n=0}^\infty$ if

$$\forall \epsilon > 0 \;\; \exists n_0 \text{ s.t. } \|\mathsf{A}_k - \mathsf{B}_k\|_\infty < \epsilon \quad \forall k \geq n_0. \tag{94}$$

We can then complete the local algebra by the standard procedure of defining the algebra of equivalence classes of Cauchy sequences. As a result of this procedure, we obtain the quasi-local algebra $\mathsf{A}(\mathbb{Z}^s)$. The detailed definition follows.

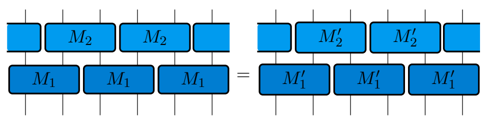
Definition 10 (quasi-local algebra). *Let* $\mathcal{C}(\mathsf{A}^{\mathrm{loc}})$ *denote the space of Cauchy sequences over* $\mathsf{A}^{\mathrm{loc}}$. *The quasi-local algebra* $\mathsf{A}(\mathbb{Z}^s)$ *is defined as*

$$\mathsf{A}(\mathbb{Z}^s) = \mathcal{C}(\mathsf{A}^{\mathrm{loc}}) / \simeq \tag{95}$$

Intuitively speaking, this algebra contains all the operators that can be arbitrarily well approximated by local operators in $\mathsf{A}^{\mathrm{loc}}$. We remark that $\mathsf{A}^{\mathrm{loc}} \subset \mathsf{A}(\mathbb{Z}^s)$.

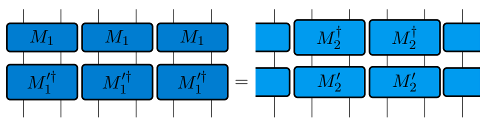
B Uniqueness of the Margolus partitioning scheme

Let M_1' and M_2' be a pair of unitary operators such that eq. (47) is satisfied. Then we must have



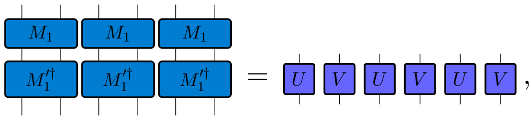
, (96)

which implies that



, (97)

The two different factorisations imply that both diagrams must be equal to a factorised unitary of the form



and this shows that

$$M_1' = (U \otimes V)M_1, \quad M_2' = M_2(V^\dagger \otimes U^\dagger). \tag{98}$$

C Proof of Proposition (7): K $= \mathcal{A} \otimes \mathcal{A}$ case

To prove the proposition, we start showing that if Eq.(85) holds then $(\phi = 0) \vee (n_x = n_y = 0) \vee (n_z = 0)$, which means that either $C_\phi = I$, or U is either diagonal or antidiagonal. Let then $\{|\mu_i\rangle\}_{i=0,1}$ be an orthonormal basis such that $\mathcal{A} := \langle |\mu_0\rangle\langle\mu_0| \cup |\mu_1\rangle\langle\mu_1|\rangle$. If Equation (85) holds, we have

$$P = \sum_{i,j} c_{i,j} |\mu_i\rangle\langle\mu_i| \otimes |\mu_j\rangle\langle\mu_j|$$
$$G^\dagger(|\mu_i\rangle\langle\mu_i| \otimes |\mu_j\rangle\langle\mu_j|)G = |\tilde{\mu}_i\rangle\langle\tilde{\mu}_i| \otimes |\tilde{\mu}_j\rangle\langle\tilde{\mu}_j|, \tag{99}$$

where $c_{i,j} \in \{0,1\}$ and $\{|\tilde{\mu}_i\rangle\}_{i=0,1}$ is another orthonormal basis. Clearly, we have $\tilde{\mathcal{A}} := \langle |\tilde{\mu}_0\rangle\langle\tilde{\mu}_0| \cup |\tilde{\mu}_1\rangle\langle\tilde{\mu}_1|\rangle$. We now find a convenient form to express G that acts as above.

Lemma 13. *If eq.(99) holds, then we can write G as:*

$$G = (V \otimes V)F \tag{100}$$

$$V = \begin{pmatrix} |a| & b \\ -b^* & |a| \end{pmatrix} \quad F = \begin{pmatrix} e^{i\theta_{00}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta_{11}} \end{pmatrix} \tag{101}$$

Proof. Let us now define a unitary operator V such that

$$|\tilde{\mu}_i\rangle := V|\mu_i\rangle, \quad i = 0,1. \tag{102}$$

Then, from Equation (99) we have

$$G = \sum_{i,j} e^{i\theta(i,j)} |\tilde{\mu}_i\rangle\langle\mu_i| \otimes |\tilde{\mu}_j\rangle\langle\mu_j| = (V \otimes V)F,$$
$$F := \sum_{i,j} e^{i\theta(i,j)} |\mu_i\rangle\langle\mu_i| \otimes |\mu_j\rangle\langle\mu_j|$$

We remark that invariance of G under conjugation by the swap implies that also $EFE = F$, and thus $\theta_{01} = \theta_{10}$, i.e. F has a degenerate eigenspace with dimension $d \geq 2$, with a 2-dimensional subspace spanned by $\{|0\rangle|1\rangle, |1\rangle|0\rangle\}$. In particular, since we are not interested in global phases, we can set F to be:

$$F = \begin{pmatrix} e^{i\theta_{00}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta_{11}} \end{pmatrix} \tag{103}$$

Moreover, if we write V as:

$$V = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$
$$a = |a|e^{i\phi}, b \in \mathbb{C},$$

we have

$$V = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$
$$= \begin{pmatrix} |a| & \tilde{b} \\ -\tilde{b}^* & |a| \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = \tilde{V} R_\phi,$$

with

$$\tilde{V} := \begin{pmatrix} |a| & \tilde{b} \\ -\tilde{b}^* & |a| \end{pmatrix}, \qquad R_\phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Thus we have:

$$G = (V \otimes V)F = (\tilde{V} \otimes \tilde{V})\tilde{F},$$

where $\tilde{F} := (R_\phi \otimes R_\phi)F$ is:

$$\tilde{F} = \begin{pmatrix} e^{i\tilde{\theta}_{00}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\tilde{\theta}_{11}} \end{pmatrix},$$

with

$$\tilde{\theta}_{00} := \theta_{00} + \phi, \quad \tilde{\theta}_{11} := \theta_{11} - \phi.$$

We then have the thesis. ■

As a consequence of the above result, we clearly have

$$[F, P] = 0. \tag{104}$$

Inserting the above decomposition of G in $[G^2, P] = 0$, we have

$$[(V \otimes V)F(V \otimes V)F, P] = 0, \tag{105}$$

and considering eq.(104)) we have

$$[(V\otimes V)F(V\otimes V),P]=0 \tag{106}$$

or, in other words,

$$(V\otimes V)F(V\otimes V)P=P(V\otimes V)F(V\otimes V) \tag{107}$$

Notice that eq. (107) also implies

$$(V\otimes V)F(V\otimes V)\bar{P}=\bar{P}(V\otimes V)F(V\otimes V), \tag{108}$$

with $\bar{P}:=I-P$, thus the operator $(V\otimes V)F(V\otimes V)$ is block-diagonal in the basis $\{|\mu\rangle|\nu\rangle\}$, i.e.

$$R:=(V\otimes V)F(V\otimes V)=\begin{pmatrix}R_{00} & 0\\ 0 & R_{11}\end{pmatrix},$$

where R_{00} and R_{11} are unitary.

Lemma 14. *If we consider the projection*

$$P=I\otimes|\mu\rangle\langle\mu|,$$

then either $G=U\otimes U$ or $V=U$ is diagonal or antidiagonal.

Proof. From the above considerations, we can write :

$$F=(V^\dagger\otimes V^\dagger)R(V^\dagger\otimes V^\dagger)=\begin{pmatrix}F_{00} & F_{01}\\ F_{10} & F_{11}\end{pmatrix}, \tag{109}$$

with blocks

$$\begin{aligned}
F_{00}&=a^2V^\dagger R_{00}V^\dagger-|b|^2V^\dagger R_{11}V^\dagger,\\
F_{01}&=-ab(V^\dagger R_{00}V^\dagger+V^\dagger R_{11}V^\dagger),\\
F_{10}&=ab^*(V^\dagger R_{00}V^\dagger+V^\dagger R_{11}V^\dagger),\\
F_{11}&=-|b|^2V^\dagger R_{00}V^\dagger+a^2V^\dagger R_{11}V^\dagger.
\end{aligned}$$

Since F is block-diagonal, we must have $F_{01}=F_{10}=0$, that is, (upon multiplying by V on both sides)

$$\begin{cases}ab(R_{00}+R_{11})=0,\\ ab^*(R_{00}+R_{11})=0.\end{cases} \tag{110}$$

We have now two possibilities: i) $R_{00}\propto R_{11}$, which however implies $F=e^{i\theta Z}\otimes e^{i\theta Z}$ and $G=U\otimes U$; ii) $ab=0$ that implies that V is either diagonal or off-diagonal, and this concludes the proof. ■

We want now to consider the case where

$$P=|\mu\rangle\langle\mu|\otimes|\nu\rangle\langle\nu|+|\mu\oplus1\rangle\langle\mu\oplus1|\otimes|\nu\oplus1\rangle\langle\nu\oplus1|.$$

It is convenient to define

$$\begin{aligned}
\delta&:=\frac{\theta_{00}-\theta_{11}}{2}, &(111)\\
U&=:e^{i\alpha_1\sigma_z}e^{i\gamma\sigma_y}e^{i\alpha_2\sigma_z}, &(112)\\
\alpha&:=\alpha_1+\alpha_2, &(113)\\
\chi&:=2\alpha-\phi, &(114)
\end{aligned}$$

where in (112) we used the representation of $\mathbb{SU}(2)$ matrices in terms of Euler angles. We can impose preliminary constraints on G by computing the determinant and the trace of $G=(V\otimes V)F$ and $G=C_\phi(U\otimes U)C_\phi$ and imposing the equality between the two forms. The constraints are summarised in the following Lemma.

Lemma 15. *Considering the form of G in eq.(100) and the one in eq.(86) it should hold:*

$$\begin{cases}a^2(1+\cos\phi\cos\delta)=\cos^2\gamma(1+(-1)^n\cos\phi\cos\chi),\\ a^2\sin\phi\cos\delta=(-1)^n\cos^2\gamma\sin\phi\cos\chi.\end{cases} \tag{115}$$

Proof. Computing

$$\det(G)=\det(F)=\det(C_\phi)^2,$$

and remembering that $\det(U)=\det(V)=1$ we get to:

$$e^{i(\theta_{00}+\theta_{11})}=e^{i2\phi}. \tag{116}$$

Moreover, considering $G=(V\otimes V)F$,

$$\mathrm{Tr}[G]=a^2[2+2e^{i\frac{\theta_{00}+\theta_{11}}{2}}\cos\delta],$$

while, for $G=C_\phi(U\otimes U)C_\phi$, we have

$$\mathrm{Tr}[G]=\cos^2\gamma[2+2e^{i\phi}\cos\chi],$$

Taking $(\theta_{00}+\theta_{11})/2=\phi+n\pi$, with $n\in\{0,1\}$, as from Eq. (116), we obtain

$$a^2[1+e^{i\phi}\cos\delta]=\cos^2\gamma[1+(-1)^n e^{i\phi}\cos\chi], \tag{117}$$

and equating the real and imaginary parts of both sides gives the thesis:

$$\begin{cases}a^2(1+\cos\phi\cos\delta)=\cos^2\gamma(1+(-1)^n\cos\phi\cos\chi),\\ a^2\sin\phi\cos\delta=(-1)^n\cos^2\gamma\sin\phi\cos\chi.\end{cases}$$

■

Now, the cases are the following

1. $\phi = 0$, which implies that $G = U \otimes U$.
2. $\phi = \pi$, that we will analyse as the last.
3. $\phi \neq n\pi$ for $n \in \mathbb{N}$, in which case

$$a^2 \cos\delta = (-1)^n \cos^2\gamma \cos\chi. \qquad (118)$$

In case (3), condition (118) can be substituted in (117) to obtain

$$a^2 + a^2 e^{i\phi}\cos\delta = \cos^2\gamma + a^2 e^{i\phi}\cos\delta,$$

leading to $a^2 = \cos^2\gamma$. Since the sign of a is irrelevant, we can take $a = \cos\gamma$. Moreover, either $a = 0$, which leads to the case where $U = i\mathbf{n}\cdot\boldsymbol{\sigma}$, or, from (118), $\cos\delta = (-1)^n \cos\chi$, i.e.

$$\begin{cases} \frac{\theta_{00}-\theta_{11}}{2} = s(2\alpha - \phi) + n\pi, \\ \frac{\theta_{00}+\theta_{11}}{2} = \phi + n\pi, \end{cases}$$

with $s \in \{+1, -1\}$. Finally, we get to

$$\theta_{00} = 2s\alpha + (1-s)\phi + 2n\pi,$$
$$\theta_{11} = -2s\alpha + (1+s)\phi.$$

If $s = +1$, then

$$\theta_{00} = 2\alpha,$$
$$\theta_{11} = -2\alpha + 2\phi,$$

while if $s = -1$, then

$$\theta_{00} = -2\alpha + 2\phi,$$
$$\theta_{11} = 2\alpha.$$

Now, since $a = \cos\gamma$, we can write

$$V = e^{-i\beta Z} e^{i\gamma Y} e^{i\beta Z},$$

where where X, Y, Z represent the Pauli matrices in the basis where F is diagonal, and $b = |b|e^{-2i\beta}$. Moreover, since $\alpha = \alpha_1 + \alpha_2$, if $s = 1$ we can write

$$F = (e^{i\alpha Z} \otimes e^{i\alpha Z})\tilde{C}_{2\phi},$$

where $\tilde{C}_{2\phi}$ represents a controlled-phase diagonal in the same basis as F. Now, considering that

$$\begin{aligned} G &= (e^{-i\beta Z} e^{i\gamma Y} e^{i(\alpha+\beta)Z})^{\otimes 2}\tilde{C}_{2\phi} \\ &= (e^{-i(\beta+\alpha_1)Z} e^{i\alpha_1 Z} e^{i\gamma Y} e^{i\alpha_2 Z} e^{i(\beta+\alpha_1)Z})^{\otimes 2}\tilde{C}_{2\phi} \\ &= (e^{-i(\beta+\alpha_1)Z})^{\otimes 2}(e^{i\alpha_1 Z} e^{i\gamma Y} e^{i\alpha_2 Z})^{\otimes 2} \\ &\quad \times \tilde{C}_{2\phi}(e^{i(\beta+\alpha_1)Z})^{\otimes 2}, \end{aligned}$$

we can choose

$$V = e^{-i(\beta+\alpha_1)Z}(e^{i\alpha_1 Z} e^{i\gamma Y} e^{i\alpha_2 Z})e^{i(\beta+\alpha_1)Z},$$
$$F = (e^{-i(\beta+\alpha_1)Z})^{\otimes 2}\tilde{C}_{2\phi}(e^{i(\beta+\alpha_1)Z})^{\otimes 2}.$$

On the other hand, if $s = -1$ we can write

$$F = (Xe^{i\alpha Z} \otimes Xe^{i\alpha Z})\tilde{C}_{2\phi}(X \otimes X).$$

In this case, considering that

$$\begin{aligned} G &= (e^{-i\beta Z} e^{i\gamma Y} e^{i\beta Z} X e^{i\alpha Z})^{\otimes 2}\tilde{C}_{2\phi} X^{\otimes 2} \\ &= X^{\otimes 2}(e^{i\beta Z} e^{-i\gamma Y} e^{-i\beta Z} e^{i\alpha Z})^{\otimes 2} \\ &\quad \times \tilde{C}_{2\phi} X^{\otimes 2} \\ &= X^{\otimes 2}(e^{i(\beta+\frac{\pi}{2})Z} e^{i\gamma Y} e^{-i(\beta+\frac{\pi}{2})Z} e^{i\alpha Z})^{\otimes 2}\tilde{C}_{2\phi} X^{\otimes 2} \\ &= X^{\otimes 2}(e^{i\tilde{\beta} Z} e^{i\alpha_1 Z} e^{i\gamma Y} e^{i\alpha_2 Z} e^{-i\tilde{\beta} Z})^{\otimes 2}\tilde{C}_{2\phi} X^{\otimes 2} \\ &= (Xe^{i\tilde{\beta} Z})^{\otimes 2}(e^{i\alpha_1 Z} e^{i\gamma Y} e^{i\alpha_2 Z})^{\otimes 2} \\ &\quad \times \tilde{C}_{2\phi}(e^{-i\tilde{\beta} Z} X)^{\otimes 2}, \end{aligned}$$

with $\tilde{\beta} := (\beta + \frac{\pi}{2} - \alpha_1)$, we can choose

$$V = Xe^{i\tilde{\beta} Z}(e^{i\alpha_1 Z} e^{i\gamma Y} e^{i\alpha_2 Z})e^{-i\tilde{\beta} Z} X,$$
$$F = (Xe^{i\tilde{\beta} Z})^{\otimes 2}\tilde{C}_{2\phi}(e^{-i\tilde{\beta} Z} X)^{\otimes 2}.$$

Thus, in both cases, for a suitable special unitary W,

$$V = WUW^\dagger, \quad F = (W \otimes W)C^2_\phi(W^\dagger \otimes W^\dagger).$$

Eq. (107) and (108) imply that the operator R has the form

$$R = \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix},$$

with R_{00} and R_{11} diagonal, and R_{01} and R_{10} anti-diagonal. To impose this form, it is convenient to consider the basis given by triplet and singlet states:

$$|I\rangle\!\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle), \qquad (119)$$
$$|\sigma_z\rangle\!\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle), \qquad (120)$$
$$|\sigma_x\rangle\!\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle), \qquad (121)$$
$$|\sigma_y\rangle\!\rangle = \frac{-i}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle). \qquad (122)$$

It is easy to see that:

$$|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| = |I\rangle\!\rangle\langle\!\langle I| + |\sigma_z\rangle\!\rangle\langle\!\langle \sigma_z|,$$
$$|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| = |\sigma_x\rangle\!\rangle\langle\!\langle \sigma_x| + |\sigma_y\rangle\!\rangle\langle\!\langle \sigma_y|.$$

Thus, R must be block diagonal in the above basis.

Moreover, we have:

$$E|I\rangle\rangle = |I\rangle\rangle, \quad E|\sigma_z\rangle\rangle = |\sigma_z\rangle\rangle, \quad E|\sigma_x\rangle\rangle = |\sigma_x\rangle\rangle,$$
$$E|\sigma_y\rangle\rangle = -|\sigma_y\rangle\rangle.$$

Reminding that, for V with unit determinant,

$$(V \otimes V)|\sigma_y\rangle\rangle = |\sigma_y\rangle\rangle,$$

and by direct inspection of the matrix form of F

$$F|\sigma_y\rangle\rangle = |\sigma_y\rangle\rangle,$$

we have

$$R|\sigma_y\rangle\rangle = |\sigma_y\rangle\rangle.$$

Thus, being R block diagonal, and being $|\sigma_y\rangle\rangle$ necessarily an eigenvector, it must hold that

$$R|\sigma_x\rangle\rangle = \omega|\sigma_x\rangle\rangle, \quad |\omega| = 1. \tag{123}$$

Moreover, since

$$(W \otimes W)(U \otimes U)C_\phi^2(W^\dagger \otimes W^\dagger)|\mu\rangle|\nu\rangle$$
$$= e^{i\theta_{\mu\nu}}|\mu\rangle|\nu\rangle,$$

and since the unique tensorised basis that is mapped to a tensorised basis by C_γ for $\gamma \neq k\pi$, $k \in \mathbb{Z}$, is the computational basis, we must have

$$(W^\dagger \otimes W^\dagger)|\sigma_\mu\rangle\rangle = (e^{iZ\chi} \otimes e^{iZ\chi})|M\rangle\rangle,$$

where $M = I$ for $\mu = 0$, and $M = X, Y, Z$ for $\mu = x, y, z$, respectively, and $|M\rangle\rangle$ represents the four Pauli operators in the computational basis (as opposed to $|\sigma_\mu\rangle\rangle$ that represent them in the eigenbasis of P). Hence

$$(U \otimes U)C_\phi^2(U \otimes U)|X\rangle\rangle = \omega|X\rangle\rangle.$$

Equivalently, we can write

$$C_\phi^2|UXU^T\rangle\rangle = \omega|U^\dagger XU^*\rangle\rangle. \tag{124}$$

We can now explicitly calculate

$$UXU^T = \begin{pmatrix} 2abe^{i\alpha} & a^2 - |b|^2 \\ a^2 - |b|^2 & -2ab^*e^{-i\alpha} \end{pmatrix}$$
$$U^\dagger XU^* = \begin{pmatrix} -2abe^{-i\alpha} & a^2 - |b|^2 \\ a^2 - |b|^2 & 2ab^*e^{i\alpha} \end{pmatrix}.$$

This allows us to write the vectors in eq. (124), upon suitable redefinition of b by multiplication by a phase factor as

$$\begin{aligned} |UXU^T\rangle\rangle =& 2abe^{i\alpha}|00\rangle - 2ab^*e^{-i\alpha}|11\rangle \\ &+ (a^2 - |b|^2)|X\rangle\rangle, \\ |U^\dagger XU^*\rangle\rangle =& 2ab^*e^{-i\alpha}|11\rangle - 2abe^{i\alpha}|00\rangle \\ &+ (a^2 - |b|^2)|X\rangle\rangle, \end{aligned}$$

and after applying $C_\phi^2 = C_{2\phi}$ to the first one we can rewrite eq. (124) as

$$2abe^{i\alpha}|00\rangle - 2ab^*e^{i(2\phi-\alpha)}|11\rangle + (a^2 - |b|^2)|X\rangle\rangle$$
$$= \omega(-2abe^{-i\alpha}|00\rangle + 2ab^*e^{i\alpha}|11\rangle + (a^2 - |b|^2)|X\rangle\rangle)$$

If $a - |b|^2 \neq 0$, we must have $\omega = 1$, and thus

$$abe^{2i\alpha} = -ab,$$
$$ab^*e^{2i(\phi-\alpha)} = -ab^*,$$

which admits a solution for $ab = 0$—we then have either $U = e^{i\alpha\sigma_z}$ or $U = i\mathbf{n} \cdot \mathbf{\Sigma}$ with $\mathbf{n} = (n_x, n_y, 0)$ and $\mathbf{\Sigma} = (X, Y, Z)$—or for $\alpha = (2k+1)\pi/2$ and $\phi = n\pi$, which falls under case (2) that we will treat separately. If $a = |b| = 1/\sqrt{2}$, one has

$$e^{2i\alpha} = e^{2i(\phi-\alpha)} = -\omega,$$

which implies $2\alpha - \phi = n\pi$. In this case, we have

$$(U \otimes U)C_\phi^2 = (V_y \otimes V_y)F_\phi^s,$$

where, up to conjugation by $e^{-i\alpha_1\sigma_z}$ on every factor, we have

$$V_y = \frac{1}{\sqrt{2}}(I + i\sigma_y),$$
$$F_\phi^s = \begin{pmatrix} se^{i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & se^{i\phi} \end{pmatrix} = (e^{i\alpha\sigma_z} \otimes e^{i\alpha\sigma_z})C_{2\phi}.$$

Where $s = e^{in\pi} \in \{-1, 1\}$. We thus end up having:

$$(V_y \otimes V_y)F_\phi^s = (W^\dagger \otimes W^\dagger)C_\phi(U \otimes U)C_\phi(W \otimes W),$$
$$W \otimes W|M\rangle\rangle = |\sigma_\mu\rangle\rangle.$$

We can decompose $C_\phi(U\otimes U)C_\phi$ as

$$\begin{aligned}&C_\phi(e^{i\alpha_1\sigma_z}V_y e^{i\alpha_2\sigma_z}\otimes e^{i\alpha_1\sigma_z}V_y e^{i\alpha_2\sigma_z})C_\phi =\\ &=(e^{i\alpha_1\sigma_z}\otimes e^{i\alpha_1\sigma_z})C_\phi(V_y\otimes V_y)\\ &\;\times \tilde{C}^s_\phi(e^{-i\alpha_1\sigma_z}\otimes e^{-i\alpha_1\sigma_z})\\ &\tilde{C}^s_\phi = \begin{pmatrix} se^{i\phi} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & s\end{pmatrix}.\end{aligned}$$

Then, writing:

$$\begin{aligned}C_\phi &= (e^{-i\frac{\phi}{4}\sigma_z}\otimes e^{-i\frac{\phi}{4}\sigma_z})F^+_{\frac{\phi}{2}}\\ \tilde{C}^s_\phi &= F^s_{\frac{\phi}{2}}(e^{i\frac{\phi}{4}\sigma_z}\otimes e^{i\frac{\phi}{4}\sigma_z}),\end{aligned}$$

we get to the equality:

$$\begin{aligned}(V_y\otimes V_y)F^s_\phi =&(\tilde{W}^\dagger\otimes\tilde{W}^\dagger)F^+_{\frac{\phi}{2}}(V_y\otimes V_y)\\ &\times F^s_{\frac{\phi}{2}}(\tilde{W}\otimes\tilde{W}),\end{aligned}$$

where $\tilde{W}=e^{i\frac{\phi}{4}\sigma_z}e^{-i\alpha_1\sigma_z}W$. In order to find the values of ϕ that satisfy this equality, let us compute eigenvalues and eigenvectors of the two matrices. Computing $(V_y\otimes V_y)F^s_\phi$ in the basis $|\sigma_\mu\rangle\rangle$ ordered as $\{|\sigma_0\rangle\rangle,|\sigma_y\rangle\rangle,|\sigma_x\rangle\rangle,|\sigma_z\rangle\rangle\}$ we get to:

$$(V_y\otimes V_y)F^s_\phi = \begin{pmatrix} se^{i\phi} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -se^{i\phi}\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

This has eigenvalues $\{se^{i\phi},1,(\frac{s+1}{2}i+\frac{s-1}{2})e^{i\frac{\phi}{2}},-(\frac{s+1}{2}i+\frac{s-1}{2})e^{i\frac{\phi}{2}}\}$. To ease the notation we introduce the function $f:\{-1,1\}\to\{i,1\}$ defined as follows:

$$f(1)=i \qquad\qquad f(-1)=1$$

with the property that $f(s)^2=-s$. With this notation the spectrum of the above matrix reads: $\{se^{i\phi},1,f(s)e^{i\frac{\phi}{2}},-f(s)e^{i\frac{\phi}{2}}\}$. On the other hand, computing $F^+_{\frac{\phi}{2}}(V_y\otimes V_y)F^s_{\frac{\phi}{2}}$ in the computational basis $|M\rangle\rangle$ ordered as $\{|I\rangle\rangle,|Y\rangle\rangle,|X\rangle\rangle,|Z\rangle\rangle\}$ we have:

$$F^+_{\frac{\phi}{2}}(V_y\otimes V_y)F^s_{\frac{\phi}{2}} = \begin{pmatrix} se^{i\phi} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -se^{i\frac{\phi}{2}}\\ 0 & 0 & e^{i\frac{\phi}{2}} & 0\end{pmatrix}.$$

This has the same spectrum $\{se^{i\phi},1,f(s)e^{i\frac{\phi}{2}},-f(s)e^{i\frac{\phi}{2}}\}$. Let us now compute the eigenvectors. If $\phi\neq n\pi$, there is no degeneracy and it must be $|\sigma_0\rangle\rangle=e^{i\gamma_0}|I\rangle\rangle$, thus:

$$\tilde{W}\otimes\tilde{W}|I\rangle\rangle = e^{i\gamma_0}|I\rangle\rangle,$$

or, in other words

$$\tilde{W}\tilde{W}^T = e^{i\gamma_0}I.$$

This means that $e^{-i\gamma_0/2}\tilde{W}\in O(2)$. However, since $\det(\tilde{W})=1$, it must be $e^{-i\gamma_0}=1$, i.e. $\gamma_0=0$. Moreover, the singlet state is invariant under local transformations with unit determinant, so $|Y\rangle\rangle=|\sigma_y\rangle\rangle$. Then the matrix form of $\tilde{W}$ is

$$c_0I+ic_1\sigma_y = c_0I+ic_1Y, \quad c_0,c_1\in\mathbb{R},\ c_0^2+c_1^2=1.$$

Since $\tilde{W}$ is special, the remaining eigenvalues of $\tilde{W}\otimes\tilde{W}$ must be $\{e^{i\gamma},e^{-i\gamma}\}$. The other two eigenvectors of $(V_y\otimes V_y)F^2_\phi$ are given by linear combinations of $|\sigma_x\rangle\rangle$ and $|\sigma_z\rangle\rangle$. In particular, restricting to this two dimensional subspace, we have the two equations:

$$\begin{aligned}\begin{pmatrix}0 & -se^{i\phi}\\ 1 & 0\end{pmatrix}|U\rangle\rangle &= f(s)e^{i\frac{\phi}{2}}|U\rangle\rangle,\\ \begin{pmatrix}0 & -se^{i\phi}\\ 1 & 0\end{pmatrix}|V\rangle\rangle &= -f(s)e^{i\frac{\phi}{2}}|V\rangle\rangle\end{aligned}$$

This gives:

$$\begin{aligned}|U\rangle\rangle &= \frac{1}{\sqrt{2}}(e^{-i\frac{\phi}{4}}|\sigma_z\rangle\rangle+f(s)e^{i\frac{\phi}{4}}|\sigma_x\rangle\rangle),\\ |V\rangle\rangle &= \frac{1}{\sqrt{2}}(e^{-i\frac{\phi}{4}}|\sigma_z\rangle\rangle-f(s)e^{i\frac{\phi}{4}}|\sigma_x\rangle\rangle).\end{aligned}$$

In the same way, we can compute the eigenvectors $|\tilde{U}\rangle\rangle,|\tilde{V}\rangle\rangle$ in the $|X\rangle\rangle,|Z\rangle\rangle$ basis. We have:

$$\begin{aligned}|\tilde{U}\rangle\rangle &= \frac{1}{\sqrt{2}}(|Z\rangle\rangle+f(s)|X\rangle\rangle)\\ |\tilde{V}\rangle\rangle &= \frac{1}{\sqrt{2}}(|Z\rangle\rangle-f(s)|X\rangle\rangle)\end{aligned}$$

Finally, we have to impose:

$$\begin{aligned}\tilde{W}\otimes\tilde{W}|\tilde{U}\rangle\rangle &= |\tilde{W}\tilde{U}\tilde{W}^T\rangle\rangle = e^{i\gamma}|\tilde{U}\rangle\rangle,\\ \tilde{W}\otimes\tilde{W}|\tilde{V}\rangle\rangle &= |\tilde{W}\tilde{V}\tilde{W}^T\rangle\rangle = e^{-i\gamma}|\tilde{V}\rangle\rangle.\end{aligned}$$

We have:

$$
\begin{aligned}
&|\tilde{W}\tilde{U}\tilde{W}^T\rangle\rangle = \\
&= \frac{1}{\sqrt{2}}(|\tilde{W}Z\tilde{W}^\dagger\rangle\rangle - f(s)|\tilde{W}X\tilde{W}^\dagger\rangle\rangle) = \\
&= \frac{1}{\sqrt{2}}(|\tilde{W}^2 Z\rangle\rangle - f(s)|\tilde{W}^2 X\rangle\rangle), \\
&|\tilde{W}\tilde{V}\tilde{W}^T\rangle\rangle = \\
&= \frac{1}{\sqrt{2}}(|\tilde{W}Z\tilde{W}^\dagger\rangle\rangle + f(s)|\tilde{W}X\tilde{W}^\dagger\rangle\rangle) = \\
&= \frac{1}{\sqrt{2}}(|\tilde{W}^2 Z\rangle\rangle + f(s)|\tilde{W}^2 X\rangle\rangle).
\end{aligned}
$$

Considering that $\tilde{W}^2 = (c_0^2 - c_1^2)I + 2ic_0c_1Y$, we finally have

$$
\begin{aligned}
&|\tilde{W}\tilde{U}\tilde{W}^T\rangle\rangle \\
&= \frac{1}{\sqrt{2}}[(c_0^2 - c_1^2 - 2f(s)c_0c_1)|Z\rangle\rangle \\
&- f(s)(c_0^2 - c_1^2 - 2sf(s)c_0c_1)|X\rangle\rangle], \\
&|\tilde{W}\tilde{V}\tilde{W}^T\rangle\rangle \\
&= \frac{1}{\sqrt{2}}[(c_0^2 - c_1^2 + 2f(s)c_0c_1)|Z\rangle\rangle \\
&+ f(s)(c_0^2 - c_1^2 + 2sf(s)c_0c_1)|X\rangle\rangle]
\end{aligned}
$$

This yields

$$
\begin{aligned}
&(c_0^2 - c_1^2 - 2f(s)c_0c_1) = e^{i\gamma}, \\
&(c_0^2 - c_1^2 - 2sf(s)c_0c_1) = e^{i\gamma}, \\
&(c_0^2 - c_1^2 + 2f(s)c_0c_1) = e^{-i\gamma}, \\
&(c_0^2 - c_1^2 + 2sf(s)c_0c_1) = e^{-i\gamma}.
\end{aligned}
$$

Clearly, l.h.s. of the above equations is a real number, thus $e^{i\gamma} = e^{-i\gamma} = \pm 1$. This implies that

$$
\begin{aligned}
&\tilde{W} \otimes \tilde{W}|X\rangle\rangle = \pm|X\rangle\rangle, \\
&\tilde{W} \otimes \tilde{W}|Z\rangle\rangle = \pm|Z\rangle\rangle,
\end{aligned}
$$

and consequently

$$
\begin{cases}
\tilde{W} = (iY) & e^{i\gamma} = -1, \\
\tilde{W} = I & e^{i\gamma} = 1.
\end{cases}
$$

Thus,

$$
\begin{aligned}
&f(s)e^{i\frac{\phi}{4}}\sigma_x = \frac{1}{\sqrt{2}}(U - V) = \pm X, \\
&e^{-i\frac{\phi}{4}}\sigma_z = \frac{1}{\sqrt{2}}(U + V) = \pm Z.
\end{aligned}
$$

If $s = -1$ (and $f(s) = 1$), it must be $\phi = 0$ and we end up with a completely factorised QCA, that has been already treated. If $s = 1$ (and $f(s = 1) = i$), on the other hand, it must be

$$
-e^{-i\frac{\phi}{2}} = e^{i\frac{\phi}{2}} = 1,
$$

which is impossible.

Let us then finally analyse case (2), i.e. $\phi = \pi$. In this case, by eq. (100), we have

$$
C_\pi(U \otimes U)C_\pi = (V \otimes V)F,
$$

and now the determinant of r.h.s. is 1, so we must have $\theta_{11} = 2n\pi - \theta_{00}$, that leads to $F = (e^{i\theta\sigma_z})^{\otimes 2}$, that is $G = W \otimes W$. We only need to determine what are the U that allow for this condition. Now, the first case that we consider is $W^2 = I$, that is $U^2 = I$, and then either $U = I$ or $U = i\mathbf{n} \cdot \boldsymbol{\sigma}$. In the latter situation one has

$$
\begin{aligned}
C_\pi(U \otimes U)C_\pi &= \sum_{i,j=0,1} U_{ij}|i\rangle\langle j| \otimes \sigma_z^i U \sigma_z^j \\
&= U_{01}|0\rangle\langle 1| \otimes U\sigma_z + U_{10}|1\rangle\langle 0| \otimes \sigma_z U \\
&= (U_{01}|0\rangle\langle 1| - U_{10}|1\rangle\langle 0|) \otimes (n_x\sigma_y - n_y\sigma_x),
\end{aligned}
$$

thus $C_\pi(U \otimes U)C_\pi = W \otimes W$ if and only if $U = i\sigma_j$ with $j = x, y$. We remain with the case $W^2 \neq I$, which implies that $W \otimes W$ has the same eigenspaces as $W^2 \otimes W^2$, and then $[G, P] = 0$. In this case one has

$$
\begin{aligned}
&(U \otimes U)C_\pi|\mu\rangle|\nu\rangle = e^{i\theta_{\mu\nu}}C_\pi|\mu\rangle|\nu\rangle, \\
&\theta_{00} = -\theta_{11}, \quad \theta_{01} = \theta_{10} = 0,
\end{aligned}
$$

and then

$$
C_\pi|\mu\rangle|\nu\rangle = |\tilde{\mu}\rangle|\tilde{\nu}\rangle, \tag{125}
$$

where $\{|\tilde{0}\rangle, |\tilde{1}\rangle\}$ is the eigenbasis of U. However, this is possible if and only if $|\tilde{\mu}\rangle = |\mu\rangle$ for $\mu = 1, 0$ is the computational basis. Then eq. (123) becomes

$$
U \otimes U|\sigma_x\rangle\rangle = \omega|\sigma_x\rangle\rangle,
$$

or equivalently

$$
V \otimes V|\sigma_x\rangle\rangle = \omega|\sigma_x\rangle\rangle,
$$

that is

$$
\begin{aligned}
&2ab|00\rangle - 2ab^*|11\rangle + (a^2 - |b|^2)|\sigma\rangle\rangle_x \\
&= \omega[-2ab|00\rangle + 2ab^*|11\rangle + (a^2 - |b|^2)|\sigma_x\rangle\rangle].
\end{aligned}
$$

The solutions are i) $ab = 0$ and $\omega = 1$, ii) $a^2 = |b|^2 = 1/2$ and $\omega = -1$. In the case (i), we have $V = U = I$, or $V = i\sigma_x$ and $U = i\sigma_y$, or $V = i\sigma_y$ and $U = i\sigma_x$, which contradict the hypothesis. In case (ii) we have, without loss of generality, $V = H$, the Hadamard gate. However, $C_\pi(H \otimes H)C_\pi$ is not factorised, and thus it cannot be accepted as a solution.

D Renormalised evolution

Considering eq. (27)) it is easy to see that the renormalised evolution of the single cell algebra B_0 over the periodic lattice $\mathcal{L}'$ is:

$$\mathcal{S}(B) := (\mathcal{T}^2)_r(B) = \mathcal{V}^\dagger \circ \mathcal{T}^2_w \circ \mathcal{V}(B), \quad \forall B \in \mathsf{B}_0. \tag{126}$$

Since the evolution $\mathcal{S}$ consists in a single step of a nearest neighbours automaton, the minimum regular lattice $\mathcal{L}'$ is made of four cells, i.e. $\mathcal{L}' = \{-1, 0, 1, 2\}$, while for the evolution $\mathcal{T}^2_{\mathcal{L}}$ we need $|\mathcal{L}| = 8$. However, the following Lemma tells us that the renormalised local rule can be computed over a lattice with $|\mathcal{L}| = 6$.

Lemma 16. *The matrix implementing the renormalised local rule is*

$$V_{\mathcal{S}} = \left(\bigotimes_{x\in\mathcal{L}'} \tilde{J}^\dagger_{\Lambda_x}\right) \tilde{U}^2_{\mathcal{T}} \left(\bigotimes_{x\in\mathcal{L}'} \tilde{J}_{\Lambda_x}\right), \tag{127}$$

with:

$$\tilde{J}_{\Lambda_x} := M_1 J_{\Lambda_x}.$$

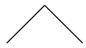
To compute this rule, we can use a lattice $\mathcal{L}$ with $|\mathcal{L}| = 6$.

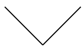
Proof. If the evolution $\mathcal{T}$ is implemented by the unitary $U_{\mathcal{T}}$, eq. (126) reads:

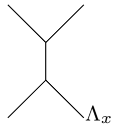
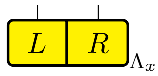
$$\mathcal{S}(B) = V_{\mathcal{S}} B_0 V^\dagger_{\mathcal{S}} \tag{128}$$

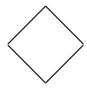
$$V_{\mathcal{S}} = \left(\bigotimes_{x\in\mathcal{L}'} J^\dagger_{\Lambda_x}\right) U^2_{\mathcal{T}} \left(\bigotimes_{x\in\mathcal{L}'} J_{\Lambda_x}\right) \tag{129}$$

Then repeating the same manipulations on the evolution as in eq.(52)), we can define the operator $\tilde{J}_{\Lambda_x}$ as:

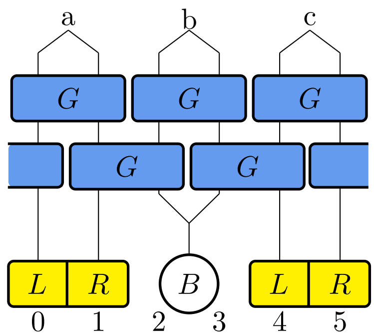
$\tilde{J}^\dagger_{\Lambda_x} := J^\dagger_{\Lambda_x} M^\dagger_1 =:$ 

$\tilde{J}_{\Lambda_x} := M_1 J_{\Lambda_x} =:$ 

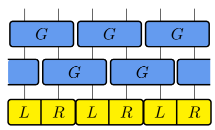
$\tilde{J}_{\Lambda_x}\tilde{J}^\dagger_{\Lambda_x} =$  $=$  $= P_{\Lambda_x}$ (130)

$\tilde{J}^\dagger_{\Lambda_x}\tilde{J}_{\Lambda_x} =$  $= I_x \in \mathsf{A}_r(\mathcal{L}')$

With those definitions we get to the diagrammatic equation:

$\mathcal{S}(B) =$  (131)

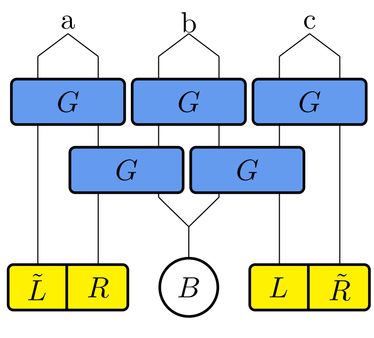
where the numbers label the cells in $\mathcal{L}$, the letters the cells of the coarse-grained lattice $\mathcal{L}'$, and the half G-transformations are meant to act on the $(-1, 0)$ and $(5, 6)$ cell, where possibly $-1 \equiv 7$. This allows us to express eq. (129) using the evolution with G operators as in eq. (54). Since our constraints are expressed in terms of G, this is amenable for a direct application of our results. Calling $\tilde{U}^2_{\mathcal{T}}$ the operator implementing the evolution (48),i.e.

$\tilde{U}^{\dagger 2}_{\mathcal{T}}\left(\bigotimes_{x\in\mathcal{L}'} P_x\right)(\tilde{U}^2_{\mathcal{T}}) =$  , (132)

eq. (129) reads

$$V_{\mathcal{S}} = \left(\bigotimes_{x\in\mathcal{L}'} \tilde{J}^\dagger_{\Lambda_x}\right) \tilde{U}^2_{\mathcal{T}} \left(\bigotimes_{x\in\mathcal{L}'} \tilde{J}_{\Lambda_x}\right). \tag{133}$$

Exploiting eq.(64)) we have

$\mathcal{S}(B) =$  (134)

Since there are no interaction between the cells $5-6$ nor $-1-0$, this equation reduces to a local expression for the evolution $\mathcal{S}(B)$ on a lattice $\mathcal{L} = \{0,1,2,3,4,5\}$. ■

Thus , in the following, we will compute the evolution using eq.(127) over only three projections.

D.1 Renormalised Qubit automata

We now explicitly calculate the renormalised evolution for the above case of study of qubit cellular automata. Let us start from the case where we can choose P to have factorised eigenvectors, i.e.

$$\begin{aligned} P_{\Lambda_x} &= |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| \\ |\psi_k\rangle &= |l(k)\rangle|j(k)\rangle \\ P_{\Lambda_x} &: \mathsf{A}(\Lambda_x) \to \mathsf{A}(\Lambda_x) \end{aligned} \tag{135}$$

where l, j are functions $\{0,1\} \to \{0,1\}$, and $\{|0\rangle, |1\rangle\}$ are a basis of the single cell algebra isomorphic to $\mathbb{C}^2$. In the same way as in Eq. (19) the choice of a basis for the space $(\mathsf{A}_r)_x \simeq \mathbb{C}^2$ allows us to define the operator J_{Λ_x} corresponding to Π_x, we can define the operator $\tilde{J}_{\Lambda_x}$ in Eq. (130) corresponding to P_{Λ_x} as well, i.e.

$$\tilde{J}_{\Lambda_x} = |0\rangle_r\langle\psi_0| + |1\rangle_r\langle\psi_1|, \tag{136}$$

that for the choice of the projection in eq. (135) becomes

$$\tilde{J}_{\Lambda_x} = |0\rangle_r\langle l(0)|\langle j(0)| + |1\rangle_r\langle l(1)|\langle j(1)|. \tag{137}$$

Here the subscript r simply indicates that

$$\tilde{J}_{\Lambda_x} : \mathsf{A}(\Lambda_x) \to (\mathsf{A}_r)_x. \tag{138}$$

Let us consider for the remainder a minimal coarse-grained lattice $\mathcal{L}'$, i.e. with four cells. Eq. (127) states that the evolution of the renormalised CA has eigenvalues equal to those of two layers of G, corresponding to eigenvectors of the form

$$|\Psi_{\mathbf{k}}\rangle := |\psi_{k_0}\rangle|\psi_{k_1}\rangle|\psi_{k_2}\rangle|\psi_{k_3}\rangle. \tag{139}$$

The corresponding eigenvector is then given by

$$|\mathbf{k}\rangle := |k_0\rangle_r|k_1\rangle_r|k_2\rangle_r|k_3\rangle_r. \tag{140}$$

Let us begin computing the case corresponding to G factorised, i.e. $G = U \otimes U$. In this case, in order to compute the renormalised evolution it is enough to specify the action over the single $|\psi_k\rangle$. In the basis identified by P, we have

$$U^2 = \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix}, \tag{141}$$

whose action on $|\psi_k\rangle = |l(k)\rangle|j(k)\rangle$ is

$$(U^2 \otimes U^2)|\psi_k\rangle = e^{i[(-1)^{l(k)} + (-1)^{j(k)}]\lambda}|\psi_k\rangle. \tag{142}$$

Thus the renormalised evolution will be a local unitary with eigenstates given by the basis chosen in (135), whose diagonal form is

$$V = \begin{pmatrix} e^{i\delta_{l(0),j(0)}(-1)^{l(0)}2\lambda} & 0 \\ 0 & e^{i\delta_{l(1),j(1)}(-1)^{l(1)}2\lambda} \end{pmatrix}. \tag{143}$$

Notice that, since the renormalised evolution have phases proportional to an integer multiple of λ, if λ is an irrational number, the iteration of the renormalisation will reach a fixed point (the identity CA) only if one chooses $l(k) = j(k) \oplus 1$ for both $k = 0, 1$.

Let us now consider the case of G of the form

$$G = C_\phi(R_z(\theta) \otimes R_z(\theta))C_\phi = \\ = \begin{pmatrix} e^{2i\theta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2i\theta+2i\phi} \end{pmatrix}. \tag{144}$$

Every G in the first layer of $\tilde{U}_T^2$ will act on the second qubit of a cell (say α) and the first one of the next cell ($\alpha \oplus 1$).

It is clear that these operators represent commuting gates between neighbouring cells in the lattice $\mathcal{L}'$, thus we can artificially split them in two layers, e.g. $(\tilde{G}_{12} \otimes \tilde{G}_{30})(\tilde{G}_{01} \otimes \tilde{G}_{23})$ where

$$\tilde{G}|k_\alpha\rangle|k_{\alpha\oplus1}\rangle = \omega(k_\alpha, k_{\alpha\oplus1})|k_\alpha\rangle|k_{\alpha\oplus1}\rangle, \tag{145}$$

$$\omega(k_\alpha, k_{\alpha\oplus1}) := e^{i\{[(-1)^{j(k_\alpha)} + (-1)^{l(k_{\alpha\oplus1})}]\theta + l(k_{\alpha\oplus1})j(k_\alpha)2\phi\}}. \tag{146}$$

After the application of these gates, the vector $|\mathbf{k}\rangle$ acquires a phase

$$\omega_1(\mathbf{k}) = \prod_{\alpha=0}^{3} \omega(k_\alpha, k_{\alpha\oplus1}).$$

Now, the second layer of G operators will act on the two qubits in the same cell, and can be interpreted as a layer of local gates acting on the single cells α as

$$\tilde{R}|k_\alpha\rangle = \nu(k_\alpha)|k_\alpha\rangle, \tag{147}$$

$$\nu(k_\alpha) := e^{i\{[(-1)^{j(k_\alpha)}+(-1)^{l(k_\alpha)}]\theta+l(k_\alpha)j(k_\alpha)2\phi\}}. \tag{148}$$

After the application of these gates, the vector $|\mathbf{k}\rangle$ acquires a phase

$$\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} \nu(k_\alpha).$$

The final phase $\omega_1(\mathbf{k})\omega_2(\mathbf{k})$ can have different values depending on the functions l and j. Let us consider the various cases, using the following equalities holding for $l \in \{0,1\}$

$$(-1)^l = -(2l-1), \quad l \oplus 1 = 1 - l.$$

1. $l(k) = j(k) = k$. In this case, neglecting an overall phase factor $e^{i16\theta}$, we have

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{i[k_\alpha(2\phi-8\theta)+k_\alpha k_{\alpha\oplus 1}2\phi]} \tag{149}$$

This evolution corresponds to a QCA obtained with two layers of controled-phase gates $C_{\phi'}$ and one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\begin{cases} \phi' = 2\phi, \\ \theta' = \phi - 4\theta. \end{cases} \tag{150}$$

2. $l(k) = j(k) = k \oplus 1$. In this case, neglecting an overall phase $e^{i16(\phi-\theta)}$, we have

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{i[k_\alpha(8\theta-6\phi)+k_\alpha k_{\alpha\oplus 1}2\phi]}. \tag{151}$$

This evolution corresponds to a QCA obtained with two layers of controled-phase gates $C_{\phi'}$ and one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\begin{cases} \phi' = 2\phi, \\ \theta' = 4\theta - 3\phi. \end{cases} \tag{152}$$

3. $l(k) = j(k) \oplus 1$. In this case, we have

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{i[k_\alpha 2\phi - k_\alpha k_{\alpha\oplus 1}2\phi]}. \tag{153}$$

This evolution corresponds to a QCA obtained with two layers of controled-phase gates $C_{\phi'}$ and one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\begin{cases} \phi' = -2\phi, \\ \theta' = \phi. \end{cases} \tag{154}$$

4. $l(k) = c_1$ or $j(k) = c_1$ constant. In this case, the other function will be either k or $k \oplus 1$, that we can express as $1/2+s_2(1/2-k)$ with $s_2 = \pm 1$, and we have, up to an irrelevant phase factor,

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{i[s_2 k_\alpha 4(\theta+c_1\phi)]}. \tag{155}$$

This evolution corresponds to a QCA obtained one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\theta' = 2s_2(\theta + c_1\phi). \tag{156}$$

Finally, consider the case

$$G = C_\phi(R_z(\theta)\mathbf{n}\cdot\boldsymbol{\sigma}R_z(\epsilon)) \otimes (R_z(\theta)\mathbf{n}\cdot\boldsymbol{\sigma}R_z(\epsilon))C_\phi.$$

Remembering that C_ϕ is diagonal in the basis of eigenvectors of σ_z, we can always choose the σ_y and σ_x in such a way that $(R_z(\theta)\mathbf{n}\cdot\boldsymbol{\sigma}R_z(\epsilon)) = \sigma_y$ and:

$$G = C_\phi \sigma_y \otimes \sigma_y C_\phi.$$

Moreover, we can insert $\sigma_y\sigma_y = I$ on the left and get:

$$G = \sigma_y \otimes \sigma_y(\sigma_y \otimes \sigma_y C_\phi \sigma_y \otimes \sigma_y)C_\phi. \tag{157}$$

We then have:

$$\sigma_y \otimes \sigma_y C_\phi \sigma_y \otimes \sigma_y = \begin{pmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{C}_\phi := (\sigma_y \otimes \sigma_y C_\phi \sigma_y \otimes \sigma_y)C_\phi = \begin{pmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}.$$

Thus, we can write G as:

$$G = \sigma_y \otimes \sigma_y \tilde{C}_\phi = \\ = \begin{pmatrix} 0 & 0 & 0 & e^{i(\phi+\pi)} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ e^{i(\phi+\pi)} & 0 & 0 & 0 \end{pmatrix}. \tag{158}$$

Following the steps of the previous calculation and considering the gates $\tilde{G}$ in $\mathcal{L}'$, we can write the action of the single $\tilde{G}$ on $|k_\alpha\rangle|k_{\alpha\oplus 1}\rangle$ as:

$$\tilde{G}|k_\alpha\rangle|k_{\alpha\oplus 1}\rangle = \omega(k_\alpha, k_{\alpha\oplus 1})|k_\alpha \oplus 1\rangle|k_{\alpha\oplus 1} \oplus 1\rangle, \tag{159}$$

$$\omega(k_\alpha, k_{\alpha\oplus 1}) \coloneqq e^{i\{\phi+\pi\}\delta_{j(k_\alpha),l(k_{\alpha\oplus 1})}}. \tag{160}$$

Conversely to the previous case, we can see as $\tilde{G}$ changes the vector $|k_\alpha\rangle|k_{\alpha\oplus 1}\rangle$ to $|k_\alpha\oplus 1\rangle|k_{\alpha\oplus 1}\oplus 1\rangle$. Thus, we have to evaluate the action of the second layer of $\tilde{G}$ over the new vectors $|k_\alpha\oplus 1\rangle|k_{\alpha\oplus 1}\oplus 1\rangle$. Again, this second layer will act on the two qubits in the same cells, and thus we can interpret it as a layer of local gates. This gives:

$$\tilde{R}|k_\alpha \oplus 1\rangle = \nu(k_\alpha)|k_\alpha\rangle, \tag{161}$$

$$\nu(k_\alpha) \coloneqq e^{i\{\phi+\pi\}\delta_{j(k_\alpha)\oplus 1,l(k_\alpha)\oplus 1}} = e^{i\{\phi+\pi\}\delta_{j(k_\alpha),l(k_\alpha)}}. \tag{162}$$

We can then define the phases:

$$\omega_1(\mathbf{k}) \coloneqq \prod_{\alpha=0}^{3} \omega(k_\alpha, k_{\alpha\oplus 1}) \tag{163}$$

$$\omega_2(\mathbf{k}) \coloneqq \prod_{\alpha=0}^{3} \nu(k_\alpha). \tag{164}$$

This gives the final phase:

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{i\{\phi+\pi\}\delta_{j(k_\alpha),l(k_\alpha)}} e^{i\{\phi+\pi\}\delta_{j(k_\alpha),l(k_{\alpha\oplus 1})}}. \tag{165}$$

Notice that the cases $j(k) = l(k) = k$ and $j(k) = l(k) = k \oplus 1$ give the same results. Moreover, the term $\omega_2(\mathbf{k})$ contributes only if $j(k) = c$ (or equivalently if $l(k) = c$). Indeed, it is easy to see that we have:

$$\omega_2(\mathbf{k}) = 1 \qquad \text{if } j(k) \neq l(k) \tag{166}$$

$$\omega_2(\mathbf{k}) = e^{i4(\pi+\phi)} \qquad \text{if} j(k) = l(k). \tag{167}$$

We can now consider the different cases exploiting the equality:

$$\delta_{ij} = 1 + 2ij - i - j. \tag{168}$$

1. $l(k) = j(k)$. We can consider without loss of generality the case $l(k) = k$. In this case, neglecting an overall phase factor $e^{i8\phi} = e^{i8\{\phi+\pi\}}$, we have

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \omega_1(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{-2i\phi k_\alpha} e^{2i\phi k_\alpha k_{\alpha\oplus 1}} \tag{169}$$

This evolution corresponds to a QCA obtained with two layers of controled-phase gates $C_{\phi'}$ and one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\begin{cases} \phi' = 2\phi, \\ \theta' = -\phi. \end{cases} \tag{170}$$

2. $l(k) = j(k) \oplus 1$. In this case,neglecting an overall phase factor $e^{i4\phi} = e^{i4\{\phi+\pi\}}$, we have

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \omega_1(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{2i\phi k_\alpha} e^{-2i\phi k_\alpha k_{\alpha\oplus 1}}. \tag{171}$$

This evolution corresponds to a QCA obtained with two layers of controled-phase gates $C_{\phi'}$ and one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\begin{cases} \phi' = -2\phi, \\ \theta' = \phi. \end{cases} \tag{172}$$

3. $l(k) = c_1$ or $j(k) = c_1$ constant. Let us assume without loss of generality that $l(k) = c_1$. It is easy to see that eq.(165) reduces to :

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{2i\{\phi+\pi\}\delta_{c_1,l(k)}} =$$

$$= \prod_{\alpha=0}^{3} e^{2i\{\phi+\pi\}\{2c_1 l(k_\alpha) - c_1 - l(k_\alpha)+1\}}.$$

Again, we can express the two possible choices of $l(k)$ as $\frac{1}{2} + s_2(\frac{1}{2} - k)$ for $s_2 = \pm 1$. Up to an irrelevant phase factor, this reduces to:

$$\omega_1(\mathbf{k})\omega_2(\mathbf{k}) = \prod_{\alpha=0}^{3} e^{-2is_2(2c_1-1)k_\alpha\{\phi+\pi\}}.$$

This evolution corresponds to a QCA obtained one layer of local unitary gates given by $\exp(-i\sigma_z\theta')$, with

$$\theta' = -s_2(2c_1 - 1)\phi. \tag{173}$$