Interferometric binary phase estimations

Simone Roncallo [1,2], Xi Lu [3], and Lorenzo Maccone [1,2]

[1]Dipartimento di Fisica, Università degli Studi di Pavia, Via Agostino Bassi 6, I-27100, Pavia, Italy
[2]INFN Sezione di Pavia, Via Agostino Bassi 6, I-27100, Pavia, Italy
[3]School of Mathematical Science, Zhejiang University, Hangzhou, 310027, China

We propose an interferometric scheme where each photon returns one bit of the binary expansion of an unknown phase. It sets up a method for estimating the phase value at arbitrary uncertainty. This strategy is global, since it requires no prior information, and it achieves the Heisenberg bound independently of the output statistics. We provide simulations and a characterization of this architecture.

1 Introduction

Quantum metrology protocols [1–3] are typically described in terms of an interferometric phase shift. Examples include displacement interferometry [4], clock synchronization [5], key distribution [6, 7] and imaging [8, 9]. Several strategies can estimate a quantum phase, including computational approaches [10–14], interferometry [15–20] and estimation theory [21]. The solution is optimal when it reaches the Heisenberg limit, i.e. when it undergoes the precision scaling set by the quantum bounds [22–25].

In this paper, we introduce an optical protocol for the estimation of an unknown phase value. We design a two-mode interferometric setup, whose response function approximates a square wave. In the ideal case, the mode where the photon exits identifies which among the available intervals the phase value falls into. By iteratively changing the response period, i.e. by controlling the number of phase shifts applied to the photon, we combine all the binary outputs and provide a full phase estimation at arbitrary uncertainty. This step can be performed sequentially, or in parallel without feedforward propagation. Our method can be rephrased as a binary search, which undergoes the Heisenberg scaling in the number of phase shifts experienced by each photon. Such scaling is attained immediately, and not asymptotically, since statistics is not required to complete the search. The protocol requires no prior information, i.e. it is a global estimation method.

Simone Roncallo: simone.roncallo01@ateneopv.it
Xi Lu: luxi@zju.edu.cn
Lorenzo Maccone: lorenzo.maccone@unipv.it

In contrast to learning-based and optimization-based approaches, our strategy does not require training. The shape of the response function is determined by a fixed sequence of parameters. This ensures modularity in controlling the square wave approximation, giving the possibility to extend or shorten the interferometer without further optimization.

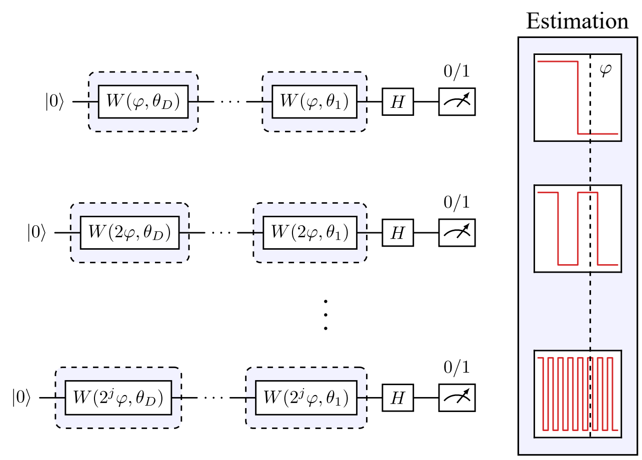

Figure 1: Binary phase estimation. Each step represents one photon going through the interferometer and providing a single bit of the binary expansion of the unknown phase φ. At the jth step, the output consists of a square-wave response with period $[0, 2\pi/2^j)$, with Fourier truncation order determined by the number D of subsequent applications of W (see Fig. 2). The true phase is sketched by the dashed vertical line. Combining all the outputs completes the estimation. The depth of the interferometer and the number of steps, i.e. the length of the binary expansion, determines the resource cost of the protocol.

2 Method

We consider the problem of estimating an unknown phase $\varphi \in [0, 2\pi)$, encoded by a unitary phase shift $P(\varphi)$. We combine a set of beam splitters and phase transformations into a two-mode interferometer, which takes single-photon states as input (or coherent states, with single-photon component post-selected by a photon-number resolving detector). This is achieved by appending a sequence of unitary blocks W with behaviour controlled by the phase parameters $\{\theta_i\}$. This provides a modular, i.e. extendible, structure. Ideally, the interferometric response function reproduces a square wave. When a photon goes through it, the probability of exiting the first mode yields one bit of the binary expansion of φ. The square wave period, controlled by the number of subsequent applications of $P(\varphi)$, determines which of the bits is returned. In practice, the interferometric response approximates the truncated Fourier series of the square wave. In this case, the truncation order is determined by the number of subsequent applications of W (with parameters changed at each composition). When φ cannot be encoded at different spatial positions, the sequential strategy is substituted by a multiround implementation, in which the photon passes multiple times through the same unknown phase shift [16, 26]. In this case, the truncation order is fixed by the number of rounds a single photon makes on the same module, with phase parameters coherently changed at each passage. Figs. 1 and 2 illustrate the estimation protocol and the interferometric module design, respectively.

Consider a binary estimation made of $n \in \mathbb{N}^+$ iterations, each associated to an interferometer, whose response function approximates a square wave with period $[0, 2\pi/k_j)$. Here, $j \in \{0, \dots, n-1\}$ and $k_j = 2^j$. We adopt the dual-rail encoding, where a photon can propagate in two orthogonal modes, represented by the quantum states $|0\rangle$ and $|1\rangle$. The photon is initially injected in mode 0. Let $\theta = \{\theta_i\}_{i\in\mathbb{N}}$ be a fixed set of parameters, labelled by $i \in \{1, \dots, D\}$. As building block of our setup, we define the operator

$$W(k_j\varphi, \theta_i) = P(k_j\varphi)HP(\theta_i)HP(k_j\varphi) \ , \qquad (1)$$

where H is the Hadamard gate (implemented by a 50 : 50 beam splitter), and P a phase shift transformation. Optically, this operator is equivalent to applying three phase shifts to mode 1, each interspersed by a beam splitter [27]. Consider D subsequent applications of W (each with a different parameter θ_i), followed by a final beam splitter (Hadamard). We measure the expectation value of the projector $\Pi_0 = |0\rangle\langle 0|$, i.e. the ratio of photons p_{jD} in mode 0, yielding

$$\begin{gathered} p_{jD}(\varphi) = \langle 0| U^\dagger(k_j\varphi, \theta)\Pi_0 U(k_j\varphi, \theta) |0\rangle \ , \\ \text{with } U(k_j\varphi, \theta) = H \prod_{i=1}^{D} W(k_j\varphi, \theta_i) \ . \end{gathered} \qquad (2)$$

Since $P(k_j\varphi)|0\rangle = |0\rangle$, it follows that

$$U(k_j\varphi, \theta)|0\rangle = HP(k_j\varphi)\widetilde{U}(k_j\varphi, \theta)|0\rangle \ , \qquad (3)$$

$$\widetilde{U}(k_j\varphi, \theta) = L(\theta_1)P(2k_j\varphi)L(\theta_2)\cdots P(2k_j\varphi)L(\theta_D) \ , \qquad (4)$$

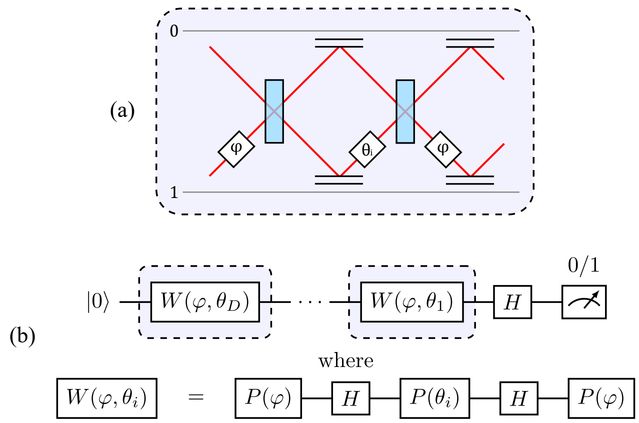

Figure 2: Interferometric module. (a) Optical implementation of $W(k_j\varphi, \theta_i)$, for $k_j = 1$. In mode 1, the photon undergoes three phase transformations, each separated by a beam splitter. Subsequent applications of W (with different θ_i) approximate the truncated Fourier series of the square wave. (b) First iteration ($k_j = 1$) after D subsequent applications of W, represented as a single-qubit circuit. The circuit depth determines the Fourier truncation order.

where $L(\theta_i) = HP(\theta_i)H$. By Stone's theorem [28], we write $P(2k_j\varphi)$ in terms of its generator $2k_j|1\rangle\langle 1|$, which has spectrum $\{0, k_j\}$. Using the spectral decomposition [29], the response reads

$$p_{jD}(\varphi) = \sum_{q\in I} c_q(\theta)e^{ik_j q\varphi} \ , \qquad (5)$$

with $I \subseteq \{-2D+1, -2D+2, \dots, 2D-1\}$. This is a truncated Fourier series, with period $[0, 2\pi/k_j)$ and coefficients fixed by θ. Notice that the depth of the interferometer determines the truncation point of the series, i.e. the higher is D the more accurate is the approximation of the square wave. We summarize the whole setup in Fig. 2, which represents the first iteration of the binary search estimation. The subsequent iterations are obtained by progressively doubling k_j.

As a generalization of our method, it is possible to train the parameters θ to approximate an arbitrary response function, with Fourier expressivity controlled by the circuit architecture [30]. Here, we limit to a fixed set of parameters that simulates the square wave. This choice brings several advantages. On the one hand, it bypasses the training procedure, i.e. the additional cost of repeating the protocol for a sufficient amount of optimization epochs. On the other hand, it provides modularity, i.e. it makes possible to extend the interferometer, improving the truncation accuracy without having to reoptimize the former parameters. We evaluate the interferometer response by simulating the circuit of Fig. 2 when $\theta_i = \alpha/(2i-1)$, with $\alpha \in \mathbb{R}^+$. The results are reported in Fig. 3. Under this choice of parameters,

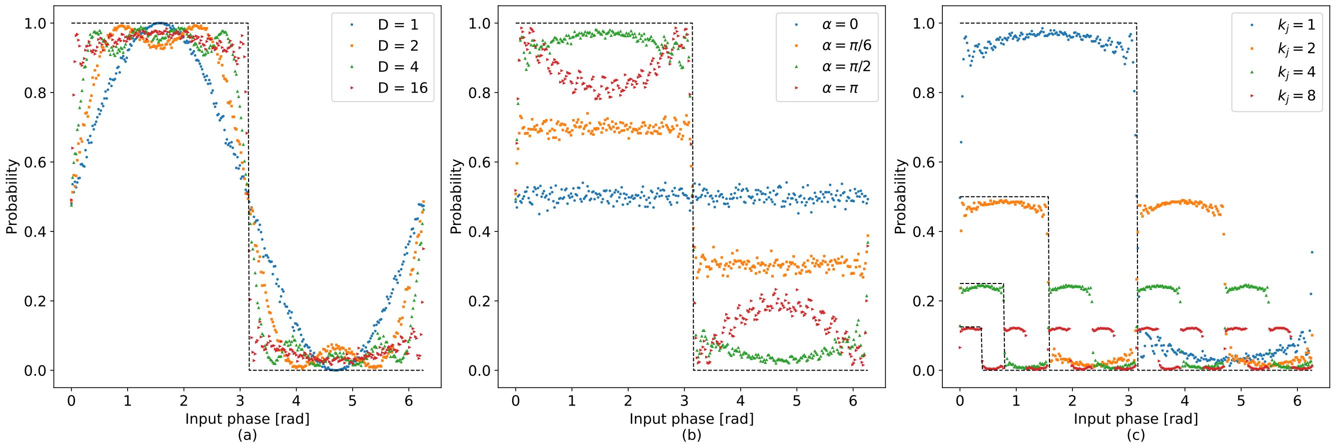

Figure 3: Comparison between the square wave and the optical response p_{jD}, obtained as the ratio of photons observed in mode 0. Simulation performed with Qiskit Aer, sampled for 300 evenly-spaced phases $\varphi \in [0, 2\pi)$ and 1000 shots, for different choices of depth D, α and k_j. (a) Simulation performed with $\alpha = \pi/2$, $k_j = 1$, and different depths D. The response function shows a binary behaviour, with approximation accuracy that increases with the interferometer depth. (b) Simulation with $D = 16$ and $k_j = 1$, plotted for different values of α. As α increases, the extrema of p_{jD} approximates 0 and 1, up to a critical value, after which the output departs from the expected behaviour. (c) Response function simulated with $D = 16$ and $\alpha = \pi/2$, plotted for different values of k_j. As k_j increases, the response period is progressively halved, representing different iterations of the binary search. To enhance visualization, each response is renormalized to k_j: all the outputs have true range in $[0, 1]$.

the output reproduces a square wave. However, this identification holds only approximately. The truncated Fourier series of the square wave admits both negative and greater-than-one values. Such values are strictly forbidden at the interferometer output, which is probabilistic, i.e. $p_{jD}(\varphi) \in [0, 1]\ \forall \varphi \in [0, 2\pi)$. This constraint is counterbalanced by α, which can be set to saturate the infimum and supremum of $p_{jD}(\varphi)$ to 0 and 1, respectively. See Appendix A for a discussion. Alternatively, α can be optimized to mitigate uncertainties or optical imperfections, e.g. using the mean squared error between the ideal and the experimental responses or information metrics on the unknown phase.

We outline the binary estimation of the unknown phase φ, at arbitrary uncertainty ε. Consider a sequence of square waves $\{f_j\}_{j \in \mathbb{N}}$, obtained through the periodic extension of the step function

$$f_j(\varphi) = \begin{cases} 1 \text{ for } \varphi \in \left[0, \frac{\pi}{k_j}\right) \\ 0 \text{ for } \varphi \in \left[\frac{\pi}{k_j}, \frac{2\pi}{k_j}\right) \end{cases} , \qquad (6)$$

simulated by $p_{jD}(\varphi)$ at the output of the interferometer. At each iteration, f_j partitions the interval $[0, 2\pi)$ in 2^{j+1} subsets of size π/k_j, namely

$$\left\{ \left[\frac{m\pi}{k_j}, \frac{(m+1)\pi}{k_j} \right) \right\}_m \text{ with } m \in \{0, 1, \ldots, k_{j+1} - 1\} . \quad (7)$$

These subsets are associated to an alternating sequence of 0s and 1s, which correspond to the values of f_j. Combining them identifies which subset includes the true value of the unknown phase. In practice, we do this by measuring the ratio of photons in mode 0 at the output of the interferometer. The result is rounded to the nearest binary digit, with threshold at 1/2. The process is repeated, each time doubling the value of k_j, which subsequently halves the response period as $f_{j+1}(\varphi) = f_j(2\varphi)$. The estimation halts as soon as the partition size becomes double the uncertainty, i.e. $2\varepsilon = \pi/k_{n-1}$ and the number of iterations is $n = \log_2(\pi/\varepsilon)$. If $n \notin \mathbb{N}^+$ we round it down to the first integer, such that $2\varepsilon = \inf_j \pi/k_j$. Denoting $\widetilde{m}$ the index of the decision outcome, we estimate the unknown phase φ as

$$\widetilde{\varphi} = \frac{(2\widetilde{m} + 1)\pi}{2^n} \pm \frac{\pi}{2^n} . \qquad (8)$$

The full method is outlined in Protocol 1. The above procedure can also be parallelized using n interferometers, each set to a different value of k_j. Although the parallel strategy brings no advantage in terms of resource counting, as we discuss below, it does not require the feedforward propagation of the sequential one.

We complete our analysis by discussing the resource cost of our protocol. In resource counting, the metrological figure of merit is represented by the number N_p of unknown phase shifts $P(\varphi)$, underwent by each photon [16, 31]. For $k_j = 1$, a single application of W costs 2 resources. Since $P(k_j\varphi)$ corresponds to k_j compositions of single phase shifts, $W(k_j\varphi, \theta_i)$ requires $2k_j$ resources. After D subsequent applications of W, the cost of approximating a binary iteration f_j is $2Dk_j$. The binary

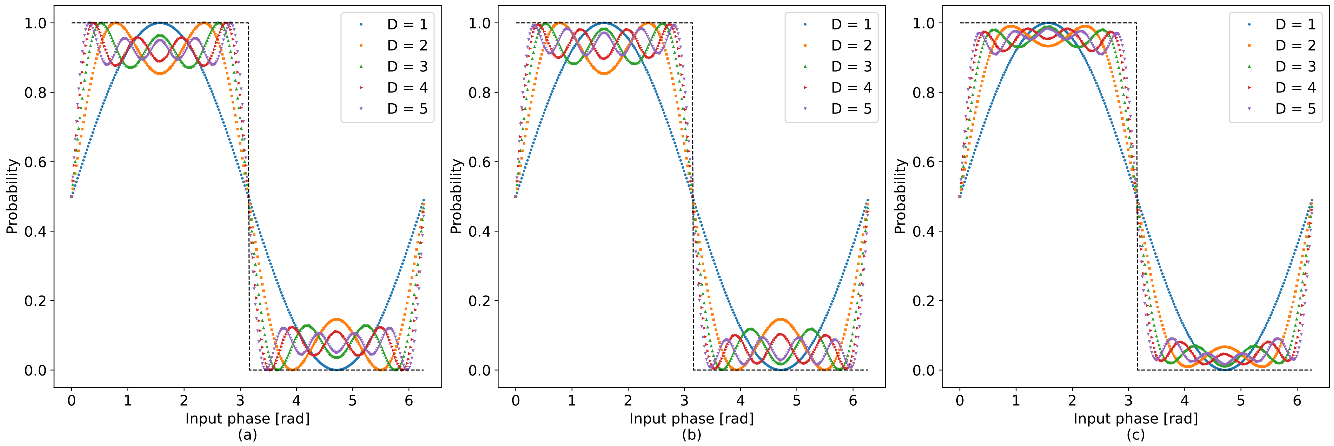

Figure 4: Comparison between the square wave f_j and the optical response function p_{jD}. Results obtained by direct numerical computation of Eq. (10) and Eq. (2), evaluated for 300 evenly-spaced phases $\varphi \in [0, 2\pi)$, and different truncation orders D. The interferometric response is computed exactly, with no statistical sampling method involved. (a) Truncated Fourier series of f_j, renormalized to the range $[0, 1]$. (b) Interferometer response p_{jD}, evaluated for $\beta = 1$ and $\alpha = \pi/2$. (c) Interferometer response p_{jD}, evaluated for $\beta = 2$ and $\alpha = \pi/2$.

search method consists in repeating the above protocol, doubling the value of k_j until condition $\varepsilon = \pi/2^n$ is met. The total cost becomes

$$N_p = 2D \sum_{j=0}^{n-1} k_j = 2D(\pi\varepsilon^{-1} - 1) \ , \tag{9}$$

namely $\mathcal{O}(D\pi\varepsilon^{-1})$ resources, with uncertainty $\mathcal{O}(N_p^{-1})$. Our strategy reaches optimality with respect to the Heisenberg bound, with uncertainty obtained by averaging the mean squared error over a single interval of the partition. Notice that protocols that performs below the Heisenberg bound are usually not attainable, since they would require more prior information, making the whole estimation strategy ineffective [32].

Protocol 1 Binary phase estimation

Input: Phase φ, uncertainty ε
Output: Estimation $\widetilde{\varphi}$

initialize $b \leftarrow (0, \ldots, 0)$
initialize $j \leftarrow 0$
while $\pi/2^j \geq 2\varepsilon$ **do**
  measure $p_{jD}(\varphi)$ ▷ See Eq. (2)
  if $p_{jD}(\varphi) \geq 1/2$ **then** set $b_j \leftarrow 1$
  else set $b_j \leftarrow 0$
  increment $j \leftarrow j + 1$
get $b \leftarrow \mathrm{NOT}(b)$
get $b \leftarrow \mathrm{Reverse}(b)$
convert $\widetilde{m} \leftarrow b_0 2^0 + b_1 2^1 + \ldots + b_{n-1} 2^{n-1}$
estimate $\widetilde{\varphi}$ ▷ See Eq. (8)

3 Conclusions

We introduced an interferometric setup for the estimation of an unknown phase value. In principle, it can approximate any response function, with Fourier expressivity determined by the depth of the interferometer. In practice, we considered an ad hoc architecture, whose response is a square wave, bypassing the optimization of the interferometer parameters. This choice rephrases the estimation task as a binary search problem, which can estimate the unknown phase at arbitrary uncertainty. Our protocol is compatible with parallel, sequential and multiround strategies. Moreover, it undergoes the Heisenberg scaling in the number of phase transformations, with no prior information required.

Acknowledgements

S.R. acknowledges the PRIN MUR Project 2022RATBS4. X.L. thanks National Natural Science Foundation of China under Grant Nos. 62272406, 61932018 and Zhejiang University for funding, and the University of Pavia for hospitality. L.M. acknowledges the U.S. DOE, Office of Science, National Quantum Information Science Research Centers, SQMS Center under Contract No. DE-AC02-07CH11359, and thanks Alexander Sergienko for fruitful discussions.

Data Availability

The underlying code that generated the data for this study is openly available in GitHub [33].

A Comparison

In this section, we investigate the relationship between the square wave f_j and the interferometer output p_{jD}. We discuss the role of the parameter α in saturating the response extrema to 0 and 1, as shown in Fig. 3. Consider the square wave f_j with period $[0, 2\pi/k_j)$ and range $[0, 1]$. By direct computation [34], its Fourier series and first-term truncation read

$$f_j(\varphi) = \frac{1}{2} + \sum_q \frac{2}{\pi q} \sin(qk_j\varphi) \ , \tag{10}$$

$$f_j(\varphi) \simeq \frac{1}{2} + \frac{2}{\pi} \sin(k_j\varphi) \ , \tag{11}$$

with $q \in 2\mathbb{N} + 1$. Compare this expression against the interferometer response, computed for $D = 1$. Consider the set of parameters $\theta_i = \alpha/(\beta i + 1 - \beta)$, with $\alpha, \beta \in \mathbb{R}^+$ and $i \in \{1, \ldots, D\}$. Then

$$\begin{aligned} p_{j1}(\varphi) &= \frac{1}{2} + \frac{1}{8} \operatorname{Re}\left[e^{ik_j\varphi}\left(e^{-i\alpha} - e^{i\alpha}\right)\right] \\ &= \frac{1}{2} + \frac{1}{2} \sin\alpha \sin(k_j\varphi) \ . \end{aligned} \tag{12}$$

Independently of β, we guarantee that $\max p_{j1}(\varphi) = 1$ and $\min p_{j1}(\varphi) = 0$, by choosing $\alpha = \pi/2 \bmod 2\pi$. We numerically evaluate f_j and p_{jD}, for different truncation orders D, and ensure a representative comparison by renormalizing the truncated Fourier series of f_j, so that its range becomes $[0, 1]$. The results are reported in Fig. 4, for different values of β. For $\beta = 1$, p_{jD} better approximates f_j. When $\beta = 2$, p_{jD} better saturates its extrema to 0 and 1, while mitigating the Gibbs phenomenon.

References

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photon. **5**, 222–229 (2011).

[2] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. **07**, 125 (2009).

[3] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. **89**, 035002 (2017).

[4] C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D **23**, 1693 (1981).

[5] P. Kómár, E. M. Kessler, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, A quantum network of clocks, Nat. Phys. **10**, 582–587 (2014).

[6] C. H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, Theor. Comput. Sci. **560**, 7 (2014).

[7] K. Tamaki, H.-K. Lo, C.-H. F. Fung, and B. Qi, Phase encoding schemes for measurement-device-independent quantum key distribution with basis-dependent flaw, Phys. Rev. A **85**, 042307 (2012).

[8] A. N. Boto, P. Kok, D. S. Abrams, S. L. Braunstein, C. P. Williams, and J. P. Dowling, Quantum interferometric optical lithography: Exploiting entanglement to beat the diffraction limit, Phys. Rev. Lett. **85**, 2733 (2000).

[9] M. Gessner, N. Treps, and C. Fabre, Estimation of a parameter encoded in the modal structure of a light beam: a quantum theory, Optica **10**, 996 (2023).

[10] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2010).

[11] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, Quantum algorithms revisited, Proc. R. Soc. Lond. A. **454**, 339–354 (1998).

[12] M. Hassani, C. Macchiavello, and L. Maccone, Digital quantum estimation, Phys. Rev. Lett. **119**, 200502 (2017).

[13] C. Huerta Alderete, M. H. Gordon, F. Sauvage, A. Sone, A. T. Sornborger, P. J. Coles, and M. Cerezo, Inference-based quantum sensing, Phys. Rev. Lett. **129**, 190501 (2022).

[14] M.-A. Filip, D. M. Ramo, and N. Fitzpatrick, Variational phase estimation with variational fast forwarding, Quantum **8**, 1278 (2024).

[15] C. Gerry and P. Knight, *Introductory Quantum Optics* (Cambridge University Press, 2004).

[16] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Entanglement-free Heisenberg-limited phase estimation, Nature **450**, 393–396 (2007).

[17] L. Pezzé and A. Smerzi, Mach-Zehnder interferometry at the Heisenberg limit with coherent and squeezed-vacuum light, Phys. Rev. Lett. **100**, 073601 (2008).

[18] M. G. Genoni, S. Olivares, and M. G. A. Paris, Optical phase estimation in the presence of phase diffusion, Phys. Rev. Lett. **106**, 153603 (2011).

[19] C. R. Schwarze, D. S. Simon, and A. V. Sergienko, Enhanced-sensitivity interferometry with phase-sensitive unbiased multiports, Phys. Rev. A **107**, 052615 (2023).

[20] J. Sinanan-Singh, G. L. Mintzer, I. L. Chuang, and Y. Liu, Single-shot quantum signal processing interferometry, Quantum **8**, 1427 (2024).

[21] N. Wiebe and C. Granade, Efficient bayesian phase estimation, Phys. Rev. Lett. **117**, 010503 (2016).

[22] P. Busch, T. Heinonen, and P. Lahti, Heisenberg's uncertainty principle, Phys. Rep. **452**, 155 (2007).

[23] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science **306**, 1330 (2004).

[24] W. Górecki, R. Demkowicz-Dobrzański, H. M. Wiseman, and D. W. Berry, π-Corrected Heisenberg limit, Phys. Rev. Lett. **124**, 030501 (2020).

[25] F. Belliardo and V. Giovannetti, Achieving heisenberg scaling with maximally entangled states: An analytic upper bound for the attainable root-mean-square error, Phys. Rev. A **102**, 042613 (2020).

[26] L. Maccone and G. De Cillis, Robust strategies for lossy quantum interferometry, Phys. Rev. A **79**, 023812 (2009).

[27] H. Lee, P. Kok, and J. P. Dowling, A quantum Rosetta stone for interferometry, J. Mod. Opt. **49**, 2325–2338 (2002).

[28] M. H. Stone, On one-parameter unitary groups in hilbert space, Ann. Math. **33**, 643 (1932).

[29] M. Schuld, R. Sweke, and J. J. Meyer, Effect of data encoding on the expressive power of variational quantum-machine-learning models, Phys. Rev. A **103**, 032430 (2021).

[30] M. Schuld and F. Petruccione, *Machine Learning with Quantum Computers* (Springer, 2021).

[31] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, Phys. Rev. Lett. **96**, 010401 (2006).

[32] V. Giovannetti and L. Maccone, Sub-Heisenberg estimation strategies are ineffective, Phys. Rev. Lett. **108**, 210404 (2012).

[33] `https://github.com/simoneroncallo/binary-phase-estimation`.

[34] G. Arfken, G. Arfken, H. Weber, and F. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide* (Elsevier, 2012).