Squashed quantum non-Markovianity: a measure of genuine quantum non-Markovianity in states

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Quantum non-Markovianity in tripartite quantum states ρ_{ABC} represents a correlation between systems A and C when conditioned on the system B and is known to have both classical and quantum contributions. However, a systematic characterization of the latter is missing. To address this, we propose a faithful measure for non-Markovianity of genuine quantum origin called squashed quantum non-Markovianity (sQNM). It is based on the quantum conditional mutual information and is defined by the left-over non-Markovianity after squashing out all non-quantum contributions. It is lower bounded by the squashed entanglement between non-conditioning systems in the reduced state and is delimited by the extendibility of either of the non-conditioning systems. We show that the sQNM is monogamous, asymptotically continuous, convex, additive on tensor-product states, and generally super-additive. We characterize genuine quantum non-Markovianity as a resource via a convex resource theory after identifying free states with vanishing sQNM and free operations that do not increase sQNM in states. We use our resource-theoretic framework to bound the rate of state transformations under free operations and to study state transformation under non-free operations; in particular, we find the quantum communication cost from Bob (B) to Alice (A) or Charlie (C) is lower bounded by the change in sQNM in the states. The sQNM finds operational meaning; in particular, the optimal rate of private communication in a variant of conditional one-time pad protocol is twice the sQNM. Also, the minimum deconstruction cost for a variant of quantum deconstruction protocol is given twice the sQNM of the state.

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1 Introduction

Understanding different kinds of quantum correlations has been a long-standing research quest in the field of quantum information and computation from both fundamental and technological aspects [1, 2, 3, 4, 5, 6]. Quantum correlations are essential or useful to perform many of the desired quantum information and computation tasks that otherwise would not be feasible using only classical resources, such as quantum teleportation [7, 8], quantum key distribution [9, 6, 10], quantum computing [11, 12], true random number generation [13, 14, 15], etc. In the past couple of decades, resource-theoretic frameworks [16, 17, 18] for several quantum correlations and properties, e.g., nonlocality [2, 19, 20], steering [21, 22], entanglement [23, 24], unextendibility [25, 26, 27, 28], etc. have been introduced providing deeper insights into these and their viable applications. Another important correlation in composite quantum systems is quantum non-Markovianity: a correlation between two parties conditioned by a third one. Although it is widely studied, a consistent resource-theoretic framework to characterize genuine quantum non-Markovianity as a quantum resource is still lacking.

An arbitrary tripartite quantum system is in a quantum non-Markovian state if it is not a quantum Markov state [29, 30], a quantum analog of a Markov chain in classical systems. A quantum state ρ_{ABC} is a quantum Markov state ($A-B-C$) if and only if the quantum conditional mutual information (QCMI) $I(A;C|B)_\rho$ vanishes [29], i.e.,

$$I(A;C|B)_\rho \coloneqq S(AB)_\rho + S(BC)_\rho - S(ABC)_\rho - S(B)_\rho = 0,$$

where $S(B)_\rho \coloneqq -\operatorname{Tr}[\rho_B \log_2 \rho_B]$ is the von Neumann entropy of a reduced state $\rho_B = \operatorname{Tr}_{AC}[\rho_{ABC}]$. It is always non-negative for quantum states. This property is equivalent [31] to the strong subadditivity property of the von Neumann entropy [32, 33] and the monotonicity of quantum relative entropy between states under the action of quantum channels [34, 35], the quantum data-processing inequality. The QCMI is shown to have operational meaning in the quantum communication tasks of quantum state redistribution [36, 37], quantum state deconstruction and conditional erasing [38, 39], and quantum one-time pad [40]. It is one of the widely used entropic quantities in different areas of quantum physics such as characterization of quantum correlations [41, 42, 43, 44], quantum cryptography [45, 46, 47, 48], quantum error correction [49, 29, 50], characterization of memory effects in quantum dynamics [51, 52, 53, 54], analyzing properties of states and dynamics in many-body quantum systems [44, 55, 56].

Here we primarily focus on the resource theory of genuine quantum non-Markovianity for quantum states where the free states are (an extended class of) quantum Markov chain states [29, 57]. This is different from the notion of non-Markovian and Markovian memory effects in quantum dynamics [58, 59, 60], which we discuss only towards the end of the paper in Section 5. The connections between quantum Markov chain states and quantum Markovian dynamics are discussed in [51, 52]. In a recent work [60], the notion of genuine non-Markovianity in quantum states developed in this work is applied to characterize memory effects in the revival of information in quantum dynamics.

The condition $I(A;C|B)_\rho = 0$ signifies that the state ρ_{ABC} can be perfectly recovered from ρ_{AB} (or ρ_{BC}) via a quantum channel acting on B. The set of states with vanishing QCMI is not convex, i.e., a quantum non-Markovian state can be produced by taking a classical mixture of some quantum Markov states. That implies that non-Markovian behavior in many quantum non-Markovian states is not of quantum origin. The structure of quantum non-Markovian states is complicated. There exist quantum states with small QCMI that can be far, in terms of trace distance, from all quantum Markov chain states in the form (11) [30]. It is now well-understood that an approximate quantum Markov state (with very small QCMI) ρ_{ABC} is approximately recoverable from its bipartite reduced states ρ_{AB} (or ρ_{BC}) [30]. There are recent attempts to devise a resource-theoretic framework of quantum non-Markovianity in states [61, 62]. Still, they do not have all the desirable properties (see Table 1) and are unable to quantify *genuine* quantum non-Markovianity, i.e., quantum non-Markovian correlations in ρ_{ABC} that are not of classical origin.

To tackle the issues above, we introduce a measure of genuine quantum non-Markovianity, which we call squashed quantum non-Markovianity (sQNM). It is based on quantum conditional mutual information and quantifies non-Markovinaity after squashing out all possible non-Markovian correlations of a non-genuinely quantum origin. This measure is in a similar spirit of squashed entanglement [63], a faithful entanglement monotone. The genuine quantum non-Markovianity can also be quantified in terms of generalized divergence of recovery. With these in hand, we can formulate a resource-theoretic framework to characterize genuine quantum non-Markovianity in quantum states as a resource and discuss its role or connection in some of the operational quantum information processing tasks.

We start with the properties of sQNM. The states with vanishing sQNM form a convex set. These states are considered free states. The measure satisfies various desired information-theoretic properties, such as convexity, super-additivity, monogamy, asymptotic continuity, and faithfulness. We also characterize the quantum operations that do not increase or create sQNM in states. These operations also form a convex set and are considered free operations. With well-defined free states and operations, we study the conditions for state transformations under free operations and derive various bounds on the rate of transformations.

We explore the interrelation between quantum communication from the conditioning system to the non-conditioning system and sQNM and demonstrate that the quantum communication cost is lower bounded by the change in sQNM in the states. We also compare the sQNM with the squashed entanglement and observe that the sQNM of a state ρ_{ABC} is lower bounded by the squashed entanglement of the reduced state ρ_{AC} on a non-conditioning system. Furthermore, genuine quantum non-Markovianity is delimited by the extendibility in non-conditioning systems. We study the possible implications of our results in two quantum information-theoretic tasks. For conditional quantum one-time pad [40], we show that the amount of fully secured quantum key that can be shared using a tripartite state is given by the sQNM of the state. For quantum deconstruction of a state [38, 64], we find that, on average, the minimum number of unitaries required is given by the sQNM of the state.

The paper is organized as follows. Section 2 introduces the measure of genuine quantum non-Markovianity in states, outlines the properties of squashed quantum non-Markovianity (sQNM), and classifies the set of operations that do not create or increase sQNM in states. Section 3 deals with state transformations under free operations and explores the interrelation between sQNM and quantum communications costs in state transformations. Two quantum information theoretic tasks are reconsidered in Section 4 to provide an operational meaning to sQNM. Finally, we discuss the possible implications of our results and conclude in Section 5.

	$I(A;C|B)_\rho$ [61]	$M_f(A;C|B)_\rho$ [62]	$N_{\text{sq}}(A;C|B)_\rho$
Convexity	$\times$	$\times/\checkmark$ [a]	$\checkmark$
Super-additivity	$\times$	$\times$	$\checkmark$
Monogamy	$\times$	$\times$	$\checkmark$
Continuity	$\checkmark$	$\checkmark$	$\checkmark$
Faithfulness	$\checkmark$	$\checkmark$	$\checkmark$

[a]Convexity is satisfied only when the conditional system extends classically. This is termed as conditional convexity [62].

Table 1: The table provides a comparison between basic properties of the measures of quantum non-Markovianity in quantum states: QCMI $I(A;C|B)_\rho$ [61], non-Markovianity of formation $M_f(A;C|B)_\rho$ [62], and the sQNM $N_{\text{sq}}(A;C|B)_\rho$ (25). The checkmark ($\checkmark$) denotes that the corresponding property holds, and the cross ($\times$) denotes that the corresponding property does not hold. Due to differences in the properties of the measures used to define the set of free states, the corresponding resource theories are also different. For instance, the frameworks proposed in [61] and [62] are non-convex and conditionally convex, respectively, while the one we formulate here is convex.

2 Characterizing genuine quantum non-Markovianity

The QCMI for an arbitrary state ρ_{ABC} can also be written as

$$I(A;C|B)_\rho \coloneqq I(AB;C)_\rho - I(B;C)_\rho \geq 0, \quad (1)$$

where $I(B;C)_\rho \coloneqq S(B)_\rho + S(C)_\rho - S(BC)_\rho$ is the quantum mutual information between B, C in the state ρ_{BC}, and it also always holds that $I(A;C|B)_\rho = I(C;A|B)_\rho$. For any quantum state ρ_{ABC}, there exists a universal recovery map $\mathcal{R}_{B\to BC}$ such that [57, 65]

$$I(A;C|B)_\rho \geq -\log_2 F(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})), \quad (2)$$

where $F(\rho,\sigma) \coloneqq \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2$ is the fidelity with $\|M\|_1 \coloneqq \text{Tr}\left[\sqrt{M^\dagger M}\right]$ denoting the trace-norm of an operator M. A universal recovery map $\mathcal{R}_{B\to BC}$ in inequality (2) is a quantum channel, i.e., a completely positive, trace-preserving (CPTP) map, depending only on ρ_{BC} (not on ρ_{ABC}) and $\mathcal{R}_{B\to BC}(\rho_B) = \rho_{BC}$. From inequality (2), we see that for any state ρ_{ABC} there exists a universal recovery map $\mathcal{R}_{B\to BC}$ such that [30, 57, 65]

$$F(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})) \geq 2^{-I(A;C|B)_\rho}. \quad (3)$$

A state ρ_{ABC} is a quantum Markov state ($A - B - C$), i.e., $I(A;C|B)_\rho = 0$, if and only if there exists a universal map $\mathcal{R}_{B\to BC}$ such that $\rho_{ABC} = \mathcal{R}_{B\to BC}(\rho_{AB})$. An approximate Markov state ρ_{ABC} is approximately recoverable from ρ_{AB} (or ρ_{BC}) via an universal recovery map $\mathcal{R}_{B\to BC}$ (or $\mathcal{R}_{B\to AB}$) as they have small QCMI, i.e., $I(A;C|B)_\rho \leq \varepsilon$ for some appropriately small ε.

A quantum state ρ_{ABC} for which the QCMI is non-zero, $I(A;C|B)_\rho \neq 0$, is a non-Markov state. However, the Markov states do not form a convex set. There are many examples of quantum Markov states $\sigma^x_{ABC\,x\in\mathcal{X}}$, i.e., $I(A;C|B)_{\sigma^x} = 0$ for all $x \in \mathcal{X}$, whose non-trivial probabilistic mixtures $\sigma_{ABC} = \sum_{x\in\mathcal{X}} p_X(x)\sigma^x_{ABC}$ for $0 \leq p_x \leq 1$ and $\sum_x p_X(x) = 1$ yield non-zero QCMI, $I(A;C|B)_\sigma > 0$. It can be easily checked by assuming ABC as a three-qubit system with $\sigma_{ABC} = |000\rangle\langle 000|_{ABC}$ and $\tau_{ABC} = |101\rangle\langle 101|_{ABC}$. Mixing a set of states following a probability distribution is a classical operation. Thus, it is clear that the non-Markovian nature of a quantum state may have a classical origin. Beyond this, some states have non-Markov properties of genuine quantum origin and cannot be created by classical processing like the one mentioned above. On the other hand, as we show later, a probabilistic mixture of genuine quantum non-Markov states may result in a Markov state (see Observation 1). Hence, non-Markovianity in a quantum state may have both classical and quantum contributions.

2.1 Squashed quantum non-Markovianity

We aim to characterize states having non-Markovianity of genuine quantum origin. Here, we introduce a faithful measure quantifying genuine quantum non-Markovianity in a state. We call this measure as *squashed quantum non-Markovianity (sQNM)*.

Definition 1 (Squashed quantum non-Markovianity). *The squashed quantum non-Markovianity (sQNM) N_{sq} of an arbitrary tripartite quantum state ρ_{ABC} is defined by a genuine quantum correlation between two subsystems A and C when conditioned on the subsystem B, as*

$$N_{\text{sq}}(A;C|B)_\rho \coloneqq \frac{1}{2}\inf_{\rho_{ABCE}} I(A;C|BE)_\rho, \quad (4)$$

where the infimum is taken over the set of all state extensions ρ_{ABCE} of the state ρ_{ABC}, i.e., $\rho_{ABC} = \text{Tr}_E[\rho_{ABCE}]$.

It trivially follows from the above definition that $N_{\text{sq}}(A;C|B)_\rho = N_{\text{sq}}(C;A|B)_\rho$ for any state ρ_{ABC}. A priori $|E|$ is unbounded (cf. [63, 66]). As a direct consequence of the above definition, we see that for an arbitrary pure state ψ_{ABC} of arbitrary dimensional systems ABC,

$$N_{\text{sq}}(A;C|B)_\psi = \frac{1}{2}I(A;C|B)_\psi = \frac{1}{2}I(A;C)_\psi. \quad (5)$$

For example, let us consider two important classes of pure entangled states for three-qubit systems: Greenberger-Horne-Zeilinger (GHZ) [67, 68] and W states [69]. The three-qubit GHZ state $|\text{GHZ}\rangle\langle\text{GHZ}|$

and W state $|W\rangle\langle W|$ are given by

$$|\mathrm{GHZ}\rangle_{ABC} = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)_{ABC},$$
$$|\mathrm{W}\rangle_{ABC} = \frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle\Big)_{ABC}$$
$$= \sqrt{\frac{2}{3}}|0\rangle_A|\psi^+\rangle_{BC} + \sqrt{\frac{1}{3}}|1\rangle_A|00\rangle_{BC}, \quad (6)$$

where $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ corresponds to an EPR-Bell state. The sQNM for three-qubit GHZ and W states are $N_{\mathrm{sq}}(A;C|B)_{\mathrm{GHZ}} = 0.5$ and $N_{\mathrm{sq}}(A;C|B)_{\mathrm{W}} = 0.4591$, respectively. More generally, for the following families $\{\mathrm{GHZ}(p)\}_{0\leq p\leq 1}$ and $\{\mathrm{W}(p)\}_{0\leq p\leq 1}$ of three-qubit GHZ and W states, where $\mathrm{GHZ}(p) = |\mathrm{GHZ}(p)\rangle\langle\mathrm{GHZ}(p)|$ and $\mathrm{W}(p) = |\mathrm{W}(p)\rangle\langle\mathrm{W}(p)|$ such that

$$|\mathrm{GHZ}(p)\rangle_{ABC} := \sqrt{p}|000\rangle_{ABC} + \sqrt{1-p}|111\rangle_{ABC}, \quad (7)$$

$$|\mathrm{W}(p)\rangle_{ABC} := \sqrt{p}|0\rangle_A|\psi^+\rangle_{BC} + \sqrt{1-p}|1\rangle_A|00\rangle_{BC}, \quad (8)$$

we have $N_{\mathrm{sq}}(A;C|B)_{\mathrm{GHZ}(p)} = h_2(p)/2 = N_{\mathrm{sq}}(A;C|B)_{\mathrm{W}(p)}$ with $h_2(p) := -(1-p)\log_2(1-p) - p\log_2 p$ being the binary Shannon entropy.

In contrast, if the systems B and A are swapped in W_{ABC} (8) above to get $\widetilde{\mathrm{W}}_{ABC}$, we have another family $\{\widetilde{\mathrm{W}}(p)\}_{0\leq p\leq 1}$ of states for which $N_{\mathrm{sq}}(A;C|B)_{\widetilde{\mathrm{W}}(p)} = h_2(p/2) - h_2(p)/2$, a monotonically increasing function in p. In the limiting case of $p = 1$, we have $|\widetilde{\mathrm{W}}(1)\rangle = |0\rangle_B|\psi^+\rangle_{AC}$, and $N_{\mathrm{sq}}(A;C|B)_{\widetilde{\mathrm{W}}(p)} = 1$. This is a special case of the following observation. If the pure state ψ_{ABC} is a product state between AC and B, i.e., of the form $\psi_{ABC} = \varphi_{AC} \otimes \phi_B$, then

$$N_{\mathrm{sq}}(A;C|B)_{\psi} = \frac{1}{2}I(A;C)_{\varphi} = S(A)_{\varphi} = S(C)_{\varphi}. \quad (9)$$

The definition of N_{sq} shares the same spirit of squashed entanglement (E_{sq}), where the latter is defined as [63]

$$E_{\mathrm{sq}}(A;C)_{\sigma} := \frac{1}{2}\inf_{\sigma_{ACF}} I(A;C|F)_{\sigma}, \quad (10)$$

for an arbitrary bipartite state σ_{AC}, where the infimum is carried out over the set of all state extensions σ_{ACF}, i.e., $\sigma_{AC} = \mathrm{Tr}_F[\sigma_{ACF}]$. It is a faithful measure of quantum entanglement and vanishes if and only if σ_{AC} is a separable state in the partition $A|C$ [41].

Any finite-dimensional quantum state ρ_{ACF} with $I(A;C|F)_{\rho} = 0$ satisfies a Markov chain structure [29],

$$\rho_{ACF} = \oplus_i p_i\ \rho_{AF_i^L} \otimes \rho_{F_i^R C}, \quad (11)$$

where the Hilbert space of system F has a decomposition $F = \oplus_i F_i^L \otimes F_i^R$. It can be easily seen that the reduced state ρ_{AC} is separable. While the measures N_{sq} and E_{sq} share several interesting properties, they represent qualitatively different aspects of a quantum state. The E_{sq} signifies genuine bipartite quantum correlation, that is, quantum entanglement. In contrast, for a tripartite system, N_{sq} represents a genuine bipartite quantum correlation with the condition of having complete knowledge of the third system. The E_{sq} and N_{sq} satisfy the relationship mentioned in the lemma below. We refer to Appendix B for the proof.

Lemma 1. *For any finite-dimensional tripartite system ABC in an arbitrary state ρ_{ABC} and any isometry $V : B \to B_L \otimes B_R$ transforming $\rho_{ABC} \to \sigma_{AB_LB_RC}$, i.e., $V(\rho_{ABC})V^{\dagger} = \sigma_{AB_LB_RC}$,*

$$E_{\mathrm{sq}}(A;C)_{\rho} \leq N_{\mathrm{sq}}(A;C|B)_{\rho} \leq$$
$$\min\left\{E_{\mathrm{sq}}(AB_L;B_RC)_{\sigma}, E_{\mathrm{sq}}(AB;C)_{\rho}, E_{\mathrm{sq}}(A;BC)_{\rho}\right\}. \quad (12)$$

Clearly, if a state ρ_{ABC} is separable in any of the partitions $A|BC$, $AB|C$, or $\sigma_{AB_LB_RC}$ is separable in the partition $AB_L|B_RC$, then the sQNM vanishes. Separability between A and C does not necessarily imply that the state has vanishing sQNM, e.g., a tripartite GHZ state (7) for which $0 = E_{\mathrm{sq}}(A;C)_{\mathrm{GHZ}(\frac{1}{2})} < N_{\mathrm{sq}}(A;C|B)_{\mathrm{GHZ}(\frac{1}{2})} = 1$. For a tripartite state $\psi_{AC} \otimes \omega_B$, $E_{\mathrm{sq}}(A;C) = N_{\mathrm{sq}}(A;C|B)$ if ψ_{AC} is pure. The measure N_{sq} is a genuine tripartite quantum correlation that can be considered in-between bipartite and tripartite entanglement.

Now, we dwell on various important properties satisfied by the measure sQNM and state them in the following lemma. See Appendix C for the proof of (P1)-(P5).

Lemma 2. *The squashed quantum non-Markovianity N_{sq} of arbitrary finite-dimensional systems satisfies the following properties:*
(P1) Convexity: For any state $\rho_{ABC} = \sum_m p_m\ \rho^m_{ABC}$, where $0 \leq p_m \leq 1$ for all m and $\sum_m p_m = 1$,

$$N_{\mathrm{sq}}(A;C|B)_{\rho} \leq \sum_m p_m\ N_{\mathrm{sq}}(A;C|B)_{\rho^m}. \quad (13)$$

(P2) Super-additivity: For every quantum state $\rho_{AA'BB'CC'}$, the sQNM is super additive,

$$N_{\mathrm{sq}}(AA';CC'|BB')_{\rho}$$
$$\geq N_{\mathrm{sq}}(A;C|B)_{\rho} + N_{\mathrm{sq}}(A';C'|B')_{\rho}, \quad (14)$$

and, for a state with tensor product structure $\rho_{AA'BB'CC'} = \rho_{ABC} \otimes \rho_{A'B'C'}$, it is additive,

$$N_{\mathrm{sq}}(AA';CC'|BB')_{\rho}$$
$$= N_{\mathrm{sq}}(A;C|B)_{\rho} + N_{\mathrm{sq}}(A';C'|B')_{\rho}. \quad (15)$$

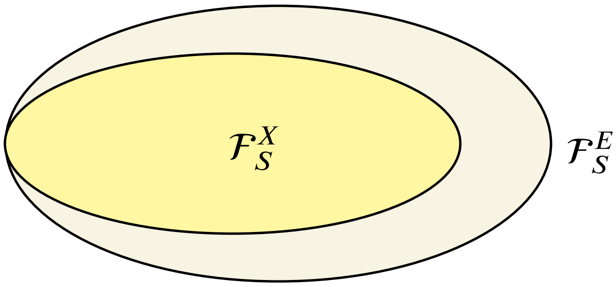

Figure 1: *Set of states with vanishing sQNM.* Here, $\mathcal{F}_S^X$ refers to the convex set of free states with the optimal extension for $N_{\rm sq}$=0 achieved through a classical extension, see Eq. (19). Outside of $\mathcal{F}_S^X$, there exist free states for which $N_{\rm sq}$=0 is achieved via extensions beyond classical registers, see Observation 1. In general, we represent the convex set of free states with the most general extension by $\mathcal{F}_S^E$, with the set $\mathcal{F}_S^X$ contained within it, i.e., $\mathcal{F}_S^X \subsetneq \mathcal{F}_S^E$ holds.

(P3) Monogamy: For every state $\rho_{AA'BC}$ the sQNM satisfies a monogamy relation

$$N_{\rm sq}(AA';C|B)_\rho \geq N_{\rm sq}(A;C|B)_\rho + N_{\rm sq}(A';C|B)_\rho. \tag{16}$$

(P4) Asymptotic continuity: For any two states ρ_{ABC} and σ_{ABC} with trace-distance $\frac{1}{2}\|\rho_{ABC}-\sigma_{ABC}\|_1 \leq \varepsilon$ with $0 \leq \varepsilon \leq 1$,

$$|N_{\rm sq}(A;C|B)_\rho - N_{\rm sq}(A;C|B)_\sigma| \leq \frac{f(\varepsilon)}{2}, \tag{17}$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.
(P5) Faithfulness: For any state ρ_{ABC}, squashed quantum non-Markovianity is zero, i.e., $N_{\rm sq}(A;C|B)_\rho = 0$, if and only if the state ρ_{ABC} is not genuinely quantum non-Markovian. Meaning that, $N_{\rm sq}(A;C|B)_\rho = 0$ if and only if there exists a universal recovery channel $\mathcal{R}_{BE\to BCE}$ such that

$$\mathcal{R}_{BE\to BCE}(\rho_{ABE}) = \rho_{ABCE}, \tag{18}$$

there exits state extension ρ_{ABCE} of ρ_{ABC}, such that Eq. (18) is true. Also see Remark 1.

Any quantum property, such as sQNM, quantified after squashing out all the non-quantum contributions should satisfy the convexity property (P1). This, in other words, implies that one cannot increase sQNM on average by classically mixing any set of states. As consequence, all the states with $N_{\rm sq} = 0$ (i.e., *free states*), form a convex set $\mathcal{F}_S^E$ (see Fig. 1 for details). Note that a convex mixture of the states in this set may result in a state with non-zero QCMI. However, a classical extension of the state and considering the extended system as part of the condition leads to a vanishing QCMI. For example, consider a set $\{\rho^x_{ABC}\}_{x\in\mathcal{X}}$ of quantum states such that $I(A;C|B)_{\rho^x} = 0$ for each $x \in \mathcal{X}$. In general, for nontrivial convex combination $\rho^X_{ABC} := \sum_x p_X(x)\rho^x_{ABC}$, $I(A;C|B)_{\rho^X} \neq 0$, unless states ρ^x_{ABC} have special structure. If we take state extension

$$\rho_{ABCX} = \sum_x p_X(x)\rho^x_{ABC} \otimes |x\rangle\langle x|_X\,, \tag{19}$$

given that $\mathrm{Tr}_X\{\rho_{ABCX}\} = \sum_x p(x)\rho^x_{ABC}$, then $I(A;C|BX)_\rho = 0$ and hence there exists a universal map $\mathcal{R}_{BX\to BCX}$ where $\mathcal{R}_{BX\to BCX}(\rho_{ABX}) = \rho_{ABCX}$. This is how the classical contribution to the non-vanishing QCMI is squashed out. Nevertheless, these states form a convex set, which we denote as $\mathcal{F}_S^X$. The convex set $\mathcal{F}_S^E$ of states with vanishing sQNM is proper super-set of $\mathcal{F}_S^X$, that is $\mathcal{F}_S^X \subsetneq \mathcal{F}_S^E$ (see Fig. 1).

In a more general setting, there are states ρ_{ABC} with respective state extension ρ_{ABCE}, i.e., $\mathrm{Tr}_E[\rho_{ABCE}] = \rho_{ABC}$, such that $I(A;C|B)_\rho \neq 0$ but do have $I(A;C|BE)_\rho = 0$, where the system E need not be a classical register. For each such states ρ_{ABC}, there exists a universal recovery map $\mathcal{R}_{BE\to BCE}$ so that $\mathcal{R}_{BE\to BEC}(\rho_{ABE}) = \rho_{ABCE}$. In this regard, we highlight the following observation.

Observation 1. *A probabilistic mixture of genuine quantum non-Markov states may result in a state with vanishing sQNM. One such example is a set of states $\{|i\rangle\langle i|_B \otimes |\phi_i\rangle\langle\phi_i|_{AC}\}_i$ where $\{|\phi_i\rangle\langle\phi_i|_{AC}\}_i$ are the four Bell states shared between A and C. While each state $|i\rangle\langle i|_B \otimes |\phi_i\rangle\langle\phi_i|_{AC}$ has $N_{\rm sq}(A;C|B) = 1$, a probabilistic mixture $\sigma_{ABC} = \frac{1}{4}\sum_i |i\rangle\langle i|_B \otimes |\phi_i\rangle\langle\phi_i|_{AC}$ has vanishing sQNM. More importantly, the extension of σ_{ABC} for which we have $I(A;C|BE)_\sigma = 0$, is a four-partite pure state $\sigma_{ABCE} = |\psi\rangle\langle\psi|_{ABCE}$ with $|\psi\rangle_{ABCE} = \frac{1}{2}\sum_i |ii\rangle_{BE} \otimes |\phi_i\rangle_{AC}$. Here, the state extension is genuinely quantum, and having access to that extension as part of the condition leads to vanishing QCMI.*

Generally, for any form of classical correlation in a multipartite state, the global amount of that property cannot exceed the sum of local ones. However, on the contrary, this is not true for quantum correlations such as (squashed) entanglement. The sQNM satisfies super-additivity (P2) and quantifies a correlation of genuine quantum nature. The sQNM also satisfies monogamy property (P3), signifying that, for a multipartite system, the more a quantum system shares a genuine quantum non-Markovian correlation with another, then the less genuine quantum non-Markovian correlation it shares with the others. The asymptotic continuity (P4) is another desirable property satisfied by the sQNM measure of quantum non-Markovianity. As expected, any two states, asymptotically close in trace distance, are also asymptotically close in their sQNM. These properties, along with faithfulness (P5), indicate that the sQNM may be considered a quantum resource, and a systematic characterization would prepare the base for a resource theory of genuine quantum non-Markovianity. It is worth mentioning that there have been attempts to study quantum non-Markovianity as a resource. For instance, the QCMI is used to quantify quantum non-Markovianity in [61]. In [62], non-Markovianity of formation is introduced from a resource theoretic approach. However, these quantifiers do not satisfy all the properties satisfied by sQNM. See Table 1 for a comparison. Apart from that, the set $\mathcal{F}_S^W$ of free states considered in [61] and [62] are the Markov states, and, thus, it is a subset of $\mathcal{F}_S^E$, i.e., $\mathcal{F}_S^W \subsetneq \mathcal{F}_S^E$.

The extendibility of quantum states in non-conditioning systems limits the quantum non-Markovianity, and it depicts the monogamous behavior of quantum non-Markovianity, as shown below. A state ρ_{AC} is said to be k-extendible in C for a $k \geq 2, k \in \mathbb{N}$, if there exists a state extension $\sigma_{AC_1C_2\ldots C_k}$, where $C_1 \simeq C_2 \simeq \cdots \simeq C_k \simeq C$, such that $\sigma_{AC_i} \coloneqq \mathrm{Tr}_{C_1\ldots C_k\setminus C_i}[\sigma_{C_1\ldots C_k}] = \rho_{AC}$ for all $i \in \{1, 2, \ldots, k\}$ [25].

Theorem 1. *For an arbitrary quantum state ρ_{ABC} that is k-extendible in C, the squashed quantum non-Markovianity is upper bounded by*

$$N_{\mathrm{sq}}(A; C|B)_\rho \leq \frac{\log_2 |A|}{k}. \tag{20}$$

Proof. Let ρ_{ABC} be k-extendible in C. Using properties of QCMI and monogamy property of squashed quantum non-Markovianity, we have

$$\begin{aligned}\log_2 |A| &\geq N_{\mathrm{sq}}(A; C_1C_2\ldots C_k|B)_\rho \\ &\geq N_{\mathrm{sq}}(A; C_1|B)_\rho + \cdots + N_{\mathrm{sq}}(A; C_k|B)_\rho \\ &= k\ N_{\mathrm{sq}}(A; C|B)_\rho, \end{aligned} \tag{21}$$

which yields bound in Eq. (20). □

It was shown in [70] that for any state ρ_{AC} that is k-extendibile in C has $E_{\mathrm{sq}}(A; C)_\rho \leq \frac{\log_2 |A|}{k}$, and we have $E_{\mathrm{sq}}(A; C)_\rho \leq N_{\mathrm{sq}}(A; C|B)_\rho$ from Lemma 1. Based on the theorem above, we can see that the bound (20) provides a tighter relation for E_{sq} in the sense that for all states ρ_{ABC} that is k-extendible in C, we have

$$E_{\mathrm{sq}}(A; C)_\rho \leq N_{\mathrm{sq}}(A; C|B)_\rho \leq \frac{\log_2 |A|}{k}. \tag{22}$$

Now, we turn to the allowed operations that do not increase sQNM. These operations can also be considered the *free operations* for a resource theory characterizing quantum non-Markovianity.

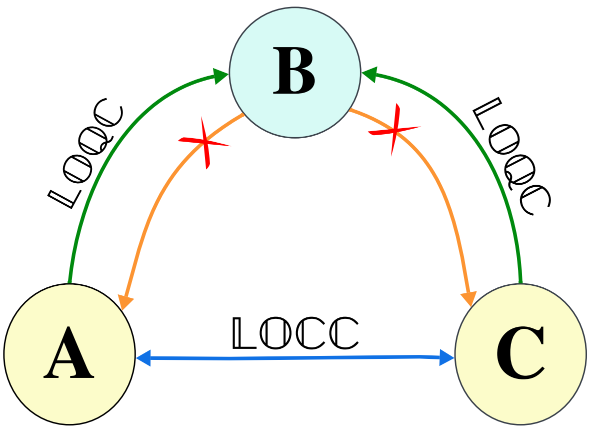

Figure 2: The figure depicts the free operations under which the sQNM is monotonically non increasing. Here, we assume Alice, Bob, and Charlie possess the systems **A**, **B**, and **C**, respectively, and Bob is used as the conditioning system. The arrows represent the direction of communication while implementing the operations. The free operations are local operation and classical communication between Alice and Charlie $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$, involving both-way classical communications; and local operation and quantum communication $\mathbb{LOQC}_{\mathbf{A}\to\mathbf{B}}$ ($\mathbb{LOQC}_{\mathbf{C}\to\mathbf{B}}$) involving one-way quantum communication from Alice (Charlie) to Bob, while no classical communication is allowed from Bob to Alice (Charlie). See Lemma 3 for more details.

For notational convenience, whenever there is no ambiguity, we denote Alice, Bob, and Charlie's laboratories with **A**, **B**, and **C**, respectively.

Lemma 3 (Free operations)**.** *The measure $N_{\mathrm{sq}}(A; C|B)_\rho$ of any state ρ_{ABC} is a monotone, i.e., non-increasing, under the following operations (see Fig. 2):*
(O1) $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$ — Local quantum operations and classical communication (LOCC) channels [71] between Alice and Charlie.
(O2) $\mathbb{LOQC}_{\mathbf{A}\to\mathbf{B}}$ and $\mathbb{LOQC}_{\mathbf{C}\to\mathbf{B}}$ — Local quantum operation and (one-way) quantum communication (LOQC) [72] channel, involving quantum communication from Alice to Bob and from Charlie to Bob respectively. It includes local quantum operations (channels) $\mathbb{LO}_{\mathbf{B}}$ on Bob.

The lemma is proved in Appendix D. The sQNM is a monotone under local quantum channels on Alice and Charlie ($\mathbb{LO}_{\mathbf{A}}$ and $\mathbb{LO}_{\mathbf{C}}$). Moreover, it is a strong monotone under selective operations (or quantum instruments) on Alice and Charlie. For a set of instruments $\{\mathcal{E}_{m,A}\}_m$, where each $\mathcal{E}_{m,A}$ is a completely positive map and their sum is a trace-preserving map $\mathcal{E}_A$, applied on a state ρ_{ABC} with $\sigma^m_{ABC} = \mathcal{E}_{m,A} \otimes \mathrm{id}_{BC}(\rho_{ABC})/p_m$ and $p_m = \mathrm{Tr}[\mathcal{E}_{m,A} \otimes \mathrm{id}_{BC}(\rho_{ABC})]$, the strong monotonicity implies

$$N_{\mathrm{sq}}(A; C|B)_\rho \geq \sum_m p_m\ N_{\mathrm{sq}}(A; C|B)_{\sigma^m}. \tag{23}$$

For the resultant state $\sigma_{ABC} = \mathcal{E}_A \otimes \mathrm{id}_{BC}(\rho_{ABC})$, we have $\sum_m p_m N_{\rm sq}(A;C|B)_{\sigma^m} \geq N_{\rm sq}(A;C|B)_\sigma$ as the consequence of convexity of sQNM. Hence, we find the monotonicity relation

$$N_{\rm sq}(A;C|B)_\rho \geq N_{\rm sq}(A;C|B)_\sigma. \tag{24}$$

A similar relation also holds for selective operations applied to Charlie. Together with convexity, the strong monotonicity of operations on Alice and Charlie renders that the sQNM is monotonically non-increasing under $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$, similar to what we have seen for squashed entanglement [73, 63]; the sQNM $N_{sq}(A;C|B)_\rho$ could be deemed as a measure of the conditional entanglement in ρ_{ABC} quantifying entanglement between A and C when conditioned on B. The $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$ may involve implementing one-way or multiple rounds of classical communication between Alice and Charlie.

The sQNM, $N_{\rm sq}(A;C|B)_\rho$ for a state ρ_{ABC}, remains invariant under a local unitary (or isometry) operation by Bob. In general, it is monotonically non-increasing under local quantum channel $\mathbb{LO}_{\mathbf{B}}$ on Bob's system. Any non-unitary operation introduces noise in Bob's system, which should not increase a genuine quantum non-Markovianity as expected. It is, however, not a strong monotone under selective operation (or instruments) on Bob. As a consequence, we see that the local operations implemented via one-way classical communication from Alice to Bob (and from Charlie to Bob), i.e., $\mathbb{LOCC}_{\mathbf{A}\rightarrow\mathbf{B}}$ (and $\mathbb{LOCC}_{\mathbf{C}\rightarrow\mathbf{B}}$), are the operations that do not increase sQNM. Any operation that involves local operations and classical communications from Bob to Alice or Charlie may increase the sQNM. For instance, with two initial Bell states shared between Alice and Bob and Bob and Charlie, local Bell measurements by Bob and classical communication to Charlie followed by a local unitary operation by Charlie can implement entanglement swapping, leading to the maximally entangled state between Alice and Charlie. Thus, $\mathbb{LOCC}_{\mathbf{B}\rightarrow\mathbf{A}}$ and $\mathbb{LOCC}_{\mathbf{B}\rightarrow\mathbf{C}}$ are not allowed, and this is a consequence of the fact that sQNM is not a strong monotone under local operations on Bob.

It can be seen from the additivity property of sQNM that attaching an additional bipartite system in an arbitrary state and sharing it with Alice and Bob (or with Charlie and Bob) does not increase sQNM. These states could be maximally entangled states, and, utilizing these, the operations $\mathbb{LOCC}_{\mathbf{A}\rightarrow\mathbf{B}}$ (and $\mathbb{LOCC}_{\mathbf{C}\rightarrow\mathbf{B}}$) enable perfect quantum communications $\mathbb{QC}_{\mathbf{A}\rightarrow\mathbf{B}}$ (and $\mathbb{QC}_{\mathbf{C}\rightarrow\mathbf{B}}$) from Alice (and Charlie) to Bob without increasing sQNM. This can also seen from the weak chain rule of sQNM. That is, for any four partite state ρ_{AMBC},

$$\begin{aligned} N_{\rm sq}(AM;C|B)_\rho &\geq N_{\rm sq}(A;C|BM)_\rho + N_{\rm sq}(M;C|B)_\rho \\ &\geq N_{\rm sq}(A;C|BM)_\rho, \end{aligned}$$

the first inequality directly follows from the chain rule of QCMI and the definition of sQNM, and the second inequality is because $N_{\rm sq}(M;C|B)_\rho \geq 0$. The strong monotonicity under selective operations by Alice and Charlie and monotonicity under $\mathbb{QC}_{\mathbf{A}\rightarrow\mathbf{B}}$ and $\mathbb{QC}_{\mathbf{C}\rightarrow\mathbf{B}}$ together imply that the sQNM is monotonically non-increasing under (one-way) $\mathbb{LOQC}_{\mathbf{A}\rightarrow\mathbf{B}}$ and $\mathbb{LOQC}_{\mathbf{C}\rightarrow\mathbf{B}}$.

Note that the set of free operations, denoted by $\mathcal{F}_O^E$, characterized above form a convex set. The set of free operations $(\mathcal{F}_O^{W_1})$ considered in [61] is a subset of the free operations $(\mathcal{F}_O^{W_2})$ considered in [62]. As expected, both these sets are subsets of the set of free operations we characterize, and $\mathcal{F}_O^{W_1} \cup \mathcal{F}_O^{W_2} \subsetneq \mathcal{F}_O^E$. It is worth mentioning that intrinsic information [74, 75], a correlation in tripartite systems, is different from sQNM. For instance, the former is not convex and is not monotonically non-increasing under $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$.

With the measure of sQNM and the free states and operations, we have the necessary ingredients to formulate a resource-theoretic framework to characterize genuine quantum non-Markovianity as a resource. One aim of this framework would be to study transformations between arbitrary states when restricted to free operations. This we shall consider in Section 3. Before that, we provide other measures of genuine quantum non-Markovianity based on the generalized divergence and the recoverability of quantum states. We also discuss the applicability of some of the measures of genuine QNM for infinite-dimensional systems.

2.2 Generalized squashed quantum non-Markovianity and fidelity of recovery

We now introduce other measures to quantify squashed quantum non-Markovianity for arbitrary quantum systems applicable to both discrete and continuous variable quantum systems. Different distinguishability quantifiers have different operational meanings in terms of discrimination tasks, so it is pertinent to have measures based on different distinguishability measures. Depending on physical or analytical requirements, one can choose from appropriate quantifiers or measures accordingly.

A generalized divergence $\mathbf{D}(\rho\|\sigma)$ between states ρ, σ is a monotone functional for all states under the action of quantum channels [76], i.e., $\mathbf{D}(\rho_{A'}\|\sigma_{A'}) \geq \mathbf{D}(\mathcal{N}_{A'\rightarrow A}(\rho_{A'})\|\mathcal{N}_{A'\rightarrow A}(\sigma_{A'}))$, where $\mathcal{N}$ is an arbitrary quantum channel and ρ, σ are arbitrary states. Consequently, generalized divergences remain invariant under the action of isometric operations. Some examples of generalized divergences are the trace distance, sandwiched Rényi relative entropy [77, 78], hypothesis testing relative entropy [79, 80], relative entropy [34], negative of fidelity, etc. Based on the generalized divergence of recovery, in the same spirit as the fidelity of recovery [81] (see also [82]), we now define generalized divergence of squashed quantum Markovianity and discuss specific properties with an example of the fidelity of squashed quantum non-Markovianity.

Definition 2. *Generalized divergence* $\mathbf{D}_{\mathrm{sqnm}}$ *of squashed quantum non-Markovianity of an arbitrary state* ρ_{ABC} *is defined as*

$$\mathbf{D}_{\mathrm{sqnm}}(A;C|B)_\rho \coloneqq \inf_{\rho_{ABCE}} \inf_{\mathcal{R}_{BE\to BCE}} \mathbf{D}(\rho_{ABCE}\|\mathcal{R}_{BE\to BCE}(\rho_{ABE})), \tag{25}$$

where ρ_{ABCE} *is an arbitrary state extension of* ρ_{ABC} *and* $\mathcal{R}_{BE\to BCE}$ *is a universal recovery map. Generally, this definition holds for arbitrary dimensional quantum systems, finite or infinite.*

We now consider an example of $\mathbf{D}_{\mathrm{sqnm}}$ which is obtained by taking generalized divergence $\mathbf{D}(\cdot\|\cdot)$ to be the negative fidelity $-F(\cdot,\cdot)$. Fidelity between arbitrary quantum states is non-decreasing under the action of quantum channels.

Definition 3. *The fidelity of squashed quantum non-Markovianity* F_{sqnm} *for an arbitrary state* ρ_{ABC} *is defined as*

$$F_{\mathrm{sqnm}}(A;C|B)_\rho \coloneqq \sup_{\rho_{ABCE}} \sup_{\mathcal{R}_{BE\to BCE}} F(\rho_{ABCE}\|\mathcal{R}_{BE\to BCE}(\rho_{ABE})), \tag{26}$$

where ρ_{ABCE} *is an arbitrary state extension of* ρ_{ABC} *and* $\mathcal{R}_{BE\to BCE}$ *is a universal recovery map.*

As $0\le F(\rho,\sigma)\le 1$ between any two states defined on the same Hilbert space, $0\le F_{\mathrm{sqnm}}(A;C|B)_\rho\le 1$ holds for an arbitrary state ρ_{ABC}. We note that $F_{\mathrm{sqnm}}(A;C|B)_\rho$ is the supremum of the fidelity of recovery $F(A;C|BE)_\rho$ for all possible state extensions ρ_{ABCE}, with recovery map being a universal recovery map. It holds that $F_{\mathrm{sqnm}}(A;C|B)_\rho \le F^{\mathrm{sq}}(A;C)_\rho$, where $F^{\mathrm{sq}}(A;C)_\rho$ is defined in [81, Eq. (23)] with recovery map being a universal recovery map.

Remark 1. *Consider the generalized divergences* $\mathbf{D}$ *that are faithful, meaning that, the generalized divergence* $\mathbf{D}(\rho\|\sigma)$ *between states* ρ *and* σ *is vanishing,* $\mathbf{D}(\rho\|\sigma)=0$*, if and only the states are equal,* $\rho=\sigma$*. Then, the generalized divergence of squashed quantum non-Markovianity (Definition 2) is a faithful measure of genuine quantum non-Markovianity in states whenever the associated generalized divergence is faithful. That is, for any faithful generalized divergence* $\mathbf{D}$ *in Eq. (25),* $\mathbf{D}_{\mathrm{sqnm}}(A;C|B)_\rho=0$ *for a state* ρ_{ABC} *if and only if there exists a universal recovery map* $\mathcal{R}_{BE\to BCE}$ *such that* $\rho_{ABCE}=\mathcal{R}_{BE\to BCE}(\rho_{ABE})$ *and* $\mathrm{Tr}_E(\rho_{ABCE})=\rho_{ABC}$*. In this case,* $\mathbf{D}_{\mathrm{sqnm}}(A;C|B)_\rho=0$ *if and only if* $N_{\mathrm{sq}}(A;C|B)_\rho=0$*. For an instance,* $-\log_2 F_{\mathrm{sqnm}}(A;C|B)_\rho=0$ *if and only if* $N_{\mathrm{sq}}(A;C|B)_\rho=0$*, as* $-\log_2 F(\cdot,\cdot)$ *is a faithful generalized divergence.*

We note that the definitions for the quantifiers of squashed quantum non-Markovianity (Definitions 1 and 2) also apply to continuous-variable quantum systems as long as the associated measures (appearing on the right-hand sides of Eqs. (4) and (25)) are well-defined. For instance, the fidelity and the trace distance are well-defined and always finite for both finite and infinite-dimensional quantum systems. On the other hand, QCMI is always finite for finite-dimensional systems and is defined also for infinite-dimensional systems with some subtleties involved (cf. [83]). For the sake of simplicity, we have limited our discussions in the other parts of this paper to finite-dimensional systems.

3 Quantum non-Markovianity and state transformations

We now discuss the state transformations under the free operations and find out the conditions in terms of the squashed quantum non-Markovianity. As discussed earlier, all states with vanishing sQNM form a convex set. All these states are free states. On the other hand, the operations mentioned in Lemma 3 are free operations as these operations neither create nor increase sQNM.

As expected, all free states can be created using free operations. As we see below, free operations such as $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$, $\mathbb{QC}_{\mathbf{A}\to\mathbf{B}}$, $\mathbb{QC}_{\mathbf{C}\to\mathbf{B}}$, and partial tracing on Bob's system are sufficient for this. For instance, consider any free state σ_{ABC} with $N_{\mathrm{sq}}(A;C|B)_\sigma=0$. Further, consider the extension σ_{ABCE} for which $I(A;C|BE)_\sigma=0$, and the state assumes the structure $\sigma_{ABCE}=\oplus_i p_i\ \sigma_{A(BE)_i^L}\otimes\sigma_{(BE)_i^R C}$, up to a local isometry by Bob. This is a separable state with the partition $A(BE)^L|(BE)^R C$. Say, initially, Alice and Charlie perform $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{B}}$ between them and create a state $\oplus_i p_i\ \sigma_{A(BE)_i^L}\otimes\sigma_{(BE)_i^R C}$, where Alice and Charlie possess the systems $A(BE)^L$ and $(BE)^R C$ respectively. After that, both Alice and Charlie transfer the states corresponding to $(BE)^L$ and $(BE)^R$ to Bob via quantum communications $\mathbb{QC}_{\mathbf{A}\to\mathbf{B}}$ and $\mathbb{QC}_{\mathbf{C}\to\mathbf{B}}$ respectively followed by tracing out of $E^L E^R$ by Bob – resulting in the desired state σ_{ABC}. This implies that the preparation of free states does not require any resources.

A resource state, i.e., a state with genuine QNM, cannot be created by the action of free operations on the free states. However, a resource state can be generated by the action of resource (non-free) operations on the free states. For a tripartite system ABC, the maximal sQNM state is a maximally entangled state between Alice and Charlie $\phi^+_{AC}=\frac{1}{d}\sum_{n,n'}|nn\rangle\langle n'n'|$ of Schmidt-rank $d=\min\{|A|,|C|\}$

and $N_{\mathrm{sq}}(A;C|B)_{\phi^+} = \log_2 d$. With the help of states with maximal sQNM and free operations, we provide examples of preparing arbitrary state ρ_{ABC} shared between Alice, Bob, and Charlie, with $d = \min\{|A|,|C|\}$. Before the preparation starts, Alice and Charlie share a maximally entangled state $\phi^+_{A'C'}$. Without loss of generality, we assume $d = |C|$. The protocol for the preparation involves the following steps: (a) Alice locally prepares the state ρ_{ABC}; (b) Alice sends the local state of B to Bob using noiseless quantum communication $\mathbb{QC}_{\mathbf{A}\to\mathbf{B}}$; (c) finally, Alice transfers the state of C to Charlie via quantum teleportation exploiting the shared entangled state $\phi^+_{A'C'}$ and $\mathbb{LOCC}_{A\to C}$.

Thus, an arbitrary tripartite quantum state can be created using free operations upon consuming a state with maximal sQNM. However, arbitrary tripartite operations cannot generally be simulated by exploiting the states with maximal sQNM and free operations. We discuss one such example later in Section 3.2.

3.1 Transformation between arbitrary states

A significant problem in any quantum resource theory is to investigate the possibility of transforming one resource state into another using free operations defined by the theory. Here, we explore the criteria for conversion between quantum states in the context of the resource theory of quantum non-Markovianity. Particularly, we are exploring asymptotic convertibility, i.e., a scenario where we have large copies (n) of the initial state $(\rho_1)_{ABC}$, and we want to convert to m_n copies of the final state $(\rho_2)_{ABC}$ using a free operation Λ_{free}, such that

$$\frac{1}{2}\left\|\Lambda_{\mathrm{free}}(\rho_1^{\otimes n}) - \rho_2^{\otimes m_n}\right\|_1 \leq \varepsilon. \tag{27}$$

In problems of resource interconversion, a quintessential quantity of interest is the rate of transformation, $R{=}\lim_{\varepsilon\to 0}\lim_{n\to\infty}(m_n/n)$. We present two important upper bounds of the rate R: (1) the *converse bound*, R_C, which gives an upper bound allowed by the resource theory, and (2) the *achievable bound*, R_A, that we could achieve using a specific quantum protocol. Our protocol is generally not optimal, and the inequality $R_A{\leq}R_C$ holds. We will discuss the question of optimality later shortly. In the following, we present both bounds and defer the proof to Appendix E. Let us first present the converse bound.

Theorem 2 (Converse bound)**.** *The maximum rate R, at which a state $(\rho_1)_{ABC}$ can be asymptotically converted to the state $(\rho_2)_{ABC}$, respects the upper bound*

$$R \leq R_C = \frac{N_{\mathrm{sq}}(A;C|B)_{\rho_1}}{N_{\mathrm{sq}}(A;C|B)_{\rho_2}}, \tag{28}$$

where $N_{\mathrm{sq}}(A;C|B)_\rho$ is the squashed quantum non-Markovianity of the state ρ_{ABC}.

See Appendix E.1 for proof of the converse bound. Next, we present the achievable bound.

Theorem 3 (Achievable bound)**.** *There exists a free operation that allows us to asymptotically convert state $(\rho_1)_{ABC}$ to $(\rho_2)_{ABC}$ at a rate*

$$R \leq R_A = \frac{\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}}{\min\left\{S(A)_{\rho_2}, S(C)_{\rho_2}\right\}}. \tag{29}$$

where $I(A_1\rangle A_2)_\rho := S(A_2)_\rho - S(A_1A_2)_\rho$ is the coherent information of the state between the subsystems A_1 and A_2 of ρ.

See Appendix E.2 for detailed proof of the achievable bound and a detailed protocol description.

Here, we briefly describe the protocol. Bob discards his system, and Alice and Charlie perform $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$ to generate shared Bell states at a rate $\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}$. Subsequently, Alice/Charlie locally prepares the target state ρ_2, sends Bob's system using $\mathbb{QC}_{\mathbf{A}\to\mathbf{B}}$ and $\mathbb{QC}_{\mathbf{C}\to\mathbf{B}}$, and teleports (or performs state splitting of) Charlie's/Alice's systems using the previously generated Bell states. As a sanity check, we can ensure $R_A{\leq}R_C$ exploiting the following inequality:

Corollary 1. *For any state ρ_{ABC},*

$$\begin{aligned}\min\left\{S(A)_\rho, S(C)_\rho\right\} &\geq N_{\mathrm{sq}}(A;C|B)_\rho \\ &\geq \max\left\{I(A\rangle C)_\rho, I(C\rangle A)_\rho\right\}.\end{aligned} \tag{30}$$

We prove the above inequalities in Appendix E.3.

Next, we discuss the optimality of our achievability protocol with a few examples:

Example 1: If $(\rho_1)_{ABC}$ and $(\rho_2)_{ABC}$ are pure bipartite states in AC, i.e., of the form $(\rho_i)_{ABC}{=}|\psi_i\rangle\langle\psi_i|_{AC}\otimes(\gamma_i)_B$, we have $R_A{=}R_C$ and the achievability protocol reduces to interconversion between bipartite entangled states proposed by Nielsen [84] and a local state preparation by Bob. However, we generally have a persisting gap between the achievable and converse rates. The gap raises the question of whether a better protocol can yield a higher achievable rate.

Example 2: We answer this positively for interconversion between a specific class of states $|\psi_1\rangle\langle\psi_1|{\to}|\psi_2\rangle\langle\psi_2|$ where

$$\begin{aligned}|\psi_i\rangle_{ABC} = V^{(i)}_{1,\ A_1A_2\to A}\otimes V^{(i)}_{2,\ B_1B_2\to B}\otimes V^{(i)}_{3,\ C_1C_2\to C} \\ \left(|\alpha_i\rangle_{A_1C_1}\otimes|\beta_i\rangle_{A_2B_1}\otimes|\gamma_i\rangle_{C_2B_2}\right).\end{aligned} \tag{31}$$

For these states, we have $N_{\mathrm{sq}}(A;C|B)_{\psi_i}{=}S(A_1)_{\alpha_i}$, i.e., all the resources for quantum non-Markovianity are concentrated in the state $|\alpha_i\rangle\langle\alpha_i|$. Hence to convert $|\psi_1\rangle\langle\psi_1| \rightarrow |\psi_2\rangle\langle\psi_2|$, Alice and Charlie can first uncouple $|\alpha_1\rangle\langle\alpha_1|_{A_1C_1}$ by applying local isometries $V_1^{(1)\dagger}$ and $V_3^{(1)\dagger}$ respectively. The rest of the state is a free state and can be discarded. Alice and Charlie now share the pure bipartite state $|\alpha_1\rangle\langle\alpha_1|_{A_1C_1}$, which can be asymptotically converted to $|\alpha_2\rangle\langle\alpha_2|_{A_1C_1}$ at a rate

$$R = \frac{S(A_1)_{\alpha_1}}{S(A_1)_{\alpha_2}} = \frac{N_{\rm sq}(A;C|B)_{\psi_1}}{N_{\rm sq}(A;C|B)_{\psi_2}}. \tag{32}$$

Subsequently, Alice and Charlie and Bob prepare free states $|\beta_2\rangle\langle\beta_2|_{A_2B_1}$ and $|\gamma_2\rangle\langle\gamma_2|_{C_2B_2}$ and apply the local isometries $V_1^{(2)}, V_2^{(2)}$, and $V_3^{(2)}$ to get the state $|\psi_2\rangle\langle\psi_2|^{ABC}$. Note this approach of uncoupling $|\alpha_1\rangle\langle\alpha_1|$ first and then converting to $|\alpha_2\rangle\langle\alpha_2|$ is better than discarding Bob's system in $|\psi_1\rangle\langle\psi_1|_{ABC}$ and performing an entanglement distillation on Alice and Charlie's subsystems as suggested in our achievability protocol. In this case, we have a strict inequality: $N_{\rm sq}(A;C|B)_{\psi_i} > \max\{I(A\rangle C)_{\psi_i}, I(C\rangle A)_{\psi_i}\}$. This indicates, for this specific example, we can find a protocol to achieve the rate of transformation $R_{\psi_1\rightarrow\psi_2}$ satisfying $R_A{<}R_{\psi_1\rightarrow\psi_2}{=}R_C$.

However, in general, for a tripartite state, uncoupling a bipartite state between Alice and Charlie containing the entire resource of quantum non-Markovianity of the original tripartite state is not possible using free operations, as it requires either $\mathbb{LOCC}_{\mathbf{B}\rightarrow\mathbf{C}}$ or $\mathbb{LOCC}_{\mathbf{B}\rightarrow\mathbf{A}}$ [85, 86], which are not free operations for us. We leave the topic of how to narrow the difference between the achievable and converse bounds for the interconversion between generic tripartite states an open question.

3.2 Quantum non-Markovianity and communication cost

For any tripartite state of the parties Alice, Bob, and Charlie with Bob as the condition, the maximally non-Markov state is any bipartite maximally entangled state shared between Alice and Charlie. For instance, ϕ^+_{AC} is a maximally entangled state of Schmidt rank d. We assume that Bob does not possess any state. One may create state ϕ^+_{AC} in the following steps. First, Alice and Bob share a maximally entangled state ϕ^+_{AB}. Note this is allowed as it does not increase the sQNM. In the second step, Bob transfers his share of state to Charlie via a perfect $(\log_2 d)$-qubit of quantum communication to Charlie. As a result, Alice and Charlie share a maximally entangled state. Note that performing quantum communication from Bob to Charlie is not free and requires a quantum communication cost. We refer to an operation involving a perfect quantum communication of $(\log_2 d)$-qubit from Bob to Charlie (or Alice) as a maximally resourceful operation that can potentially create a maximally resourceful state. We highlight here that while a maximally resourceful operation can create a maximally resourceful state by consuming a free state, the converse, i.e., implementing a maximally resourceful operation enabling perfect quantum communication from Bob to Charlie (or Alice) consuming a maximally resourceful state (ϕ^+_{AC}) and free states, is not possible. This is contrary to what we see in the resource theory of entanglement, at least for the bipartite cases. A maximally entangling operation can yield a maximally entangled state while applied to a non-entangled state. In reverse, using a maximally entangled state and local operation and classical communications (LOCC), a free operation in the resource theory of entanglement, one can implement arbitrary bipartite operations, including maximally entangling operations. Thus, not satisfying the inter-convertibility between maximally resourceful states and operations within the framework developed for sQNM may indicate two possibilities. First, the set of operations proposed in Lemma 3 as the free operations is not a complete set of operations that do not increase sQNM. Second, there is fundamental irreversibility in the inter-conversion between sQNM resources present in the states and operations. We conjecture the second possibility. With this, we now study the quantum communication cost required to implement an operation resulting in the desired state transformation.

Consider Alice, Bob, and Charlie want to prepare a quantum state ρ_{ABC} using free operations and quantum communication. Initially, Alice locally prepares the state $\rho_{AA'A''}$ in her lab, where $A' \simeq B$ and $A'' \simeq C$. She then distributes systems A' and A'' to Bob and Charlie, respectively. Note, the latter can be done in two steps: (i) Alice sends $A'A''$ to Bob via noiseless quantum channel, i.e., the identity channel, $\mathrm{id}_{A'A''\rightarrow BB'}$, where $A''{\simeq}B'{\simeq}C$; then, (ii) Bob sends B' to Charlie via $\mathrm{id}_{B'\rightarrow C}$.

Both steps exploit the protocol of quantum state redistribution. Step (i) can be implemented using a free operation and does not consume any resources. But, step (ii) cannot be executed using free operations, and there is a cost of quantum communication to realize it.

Lemma 4. *The minimum quantum communication cost Q_c to prepare an arbitrary quantum state ρ_{ABC} via quantum state redistribution protocol is*

$$Q_c \geq N_{\rm sq}(A;C|B)_\rho. \tag{33}$$

Here, Alice and Bob initially have access to systems A and BB', respectively with $B' \simeq C$. Bob uses a quantum communication cost of Q_c to transfer B' to Charlie C via the identity channel $\mathrm{id}_{B'\rightarrow C}$, $B' \simeq C$. This results in Alice, Bob, and Charlie having access to systems A, B, and C, respectively.

Proof. The proof relies on the quantum state redistribution protocol [37]. In particular, here, for an initial state $\rho_{ABB'}$, we quantify the cost of quantum communication for the distribution of $B' \simeq C$ from Bob to Charlie, i.e., $\mathrm{id}_{B'\to C}$, where A shared with Alice remains undisturbed. For that, we consider the purified state $|\psi\rangle_{ABCR}$ of ρ_{ABC}, where R represents a reference system required for the extension. The purification is not unique. There may be infinitely many such purifications interrelated through unitary rotations on the reference R. The state redistribution using this purified state must not alter the systems A and R.

In general, implementation of $\mathrm{id}_{B'\to C}$ for a state $\rho_{ABB'}$ requires a quantum communication cost of $Q_c = I(AR;C)_\psi/2$, where R is the purifying extension with the corresponding pure state $|\psi\rangle\langle\psi|$. As $B' \simeq C$, in the following entropic inequalities, we will use them interchangeably for the notational convenience.

$$\begin{aligned} Q_c = \frac{1}{2}I(AR;C)_\psi &\geq \frac{1}{2}I(A;C)_\rho \\ &\geq \frac{1}{2}\inf_{\rho_{ABCE}} I(A;C|BE)_\rho \\ &= N_{\mathrm{sq}}(A;C|B)_\rho, \end{aligned} \tag{34}$$

the first inequality is due to data-processing inequality, and the second inequality is due to the definition of sQNM. Note that the second inequality saturates when ρ_{ABC} is a pure state.

Above, we have considered quantum communication from Bob to Charlie. Can there be less quantum communication cost if we consider quantum channels from Alice to Charlie instead? We answer this negatively.

Consider for an initial state $\rho_{AA'B}$, with $A' \simeq C$, the state redistribution protocol to achieve $\mathrm{id}_{A'\to C}$ without loss of generality. This protocol requires pre-shared entanglement cost of $I(A;C)_\rho/2$ and quantum communication cost of $I(C;BR)_\rho/2$ where R is the purifying extension of ρ_{ABC} [37]. As preparing shared entanglement between Alice and Charlie is not a free operation unlike the case between Bob and Charlie, we have to invest $I(A;C)_\rho/2$ amount of quantum communication cost as well, i.e., the total quantum communication cost is $I(A;C)_\rho/2{+}I(C;BR)_\rho/2{=}S(C)_\rho$. In general, $S(C)_\rho{\geq}N_{\mathrm{sq}}(A;C|B)_\rho$ (Corollary 1), with the equality is achieved when ρ_{ABC} is a pure bipartite state in AC. In contrast, the minimum quantum communication cost for the previously discussed protocol implementing $\mathrm{id}_{B'\to C}$, for a state $\rho_{ABB'}$ when ρ_{ABC} is pure tripartite state, in which case $Q_c{=}I(A;C)_\rho/2{\leq}S(C)_\rho$. This means quantum communication from Bob to Charlie requires lesser cost. □

Above, we have considered preparing an arbitrary quantum state from scratch. Now, we delve into the transformation between quantum states and associated quantum communication costs.

Lemma 5. *Consider a state transformation, from ρ_{ABC} to $\sigma_{A'B'C'}$, with $N_{\mathrm{sq}}(A';C'|B')_\sigma{\geq}N_{\mathrm{sq}}(A;C|B)_\rho$ via reversible local operations and perfect quantum communication. Then, the quantum communication cost Q_c satisfies*

$$Q_c \geq N_{\mathrm{sq}}(A';C'|B')_\sigma - N_{\mathrm{sq}}(A;C|B)_\rho. \tag{35}$$

Proof. We first recall that the sQNM remains invariant under reversible local transformation. Let us start with an extension ρ_{ABCE} of ρ_{ABC} so that $2\ N_{\mathrm{sq}}(A;C|B)_\rho = I(A;C|BE)_\rho$. Let us say the desired transformation can be executed in the steps involving: (i) application of an isometry $V_B : B \to B'B_1B_2$ leading to $\rho_{ABCE} \to \rho_{AB'B_1B_2CE}$; (ii) perfect quantum communication of B_1 and B_2 from Bob to Alice and Charlie, via channel $\mathrm{id}_{B_1\to A_1}$ and $\mathrm{id}_{B_2\to C_1}$ respectively; and (iii) application of isometries $V_A : AA_1 \to A'$ and $V_C : CC_1 \to C'$ resulting in $\rho_{AA_1B'CC_1E} \to \sigma_{A'B'C'E}$. Following the chain rule of QCMI,

$$\begin{aligned} I(A;C|BE)_\rho &= I(A;C|B'B_1B_2E)_\rho \\ &= I(AA_1;CC_1|B'E)_\rho - 2Q_c, \end{aligned} \tag{36}$$

where $Q_c = (I(A_1;CC_1|B'E)_\rho + I(A;C_1|B'E)_\rho)/2$ is the quantum communication cost for the state redistribution $A|B'B_1B_2E|C \to AA_1|B'B_2E|C \to AA_1|B'E|CC_1$. The notation A|B|C signifies a partitioned arrangement where the first segment belongs to Alice's lab, the second to Bob, and the third to Charlie. Now with $I(AA_1;CC_1|B'E)_\rho = I(A';C'|B'E)_\sigma \geq 2\ N_{\mathrm{sq}}(A';C'|B')_\sigma$, we have $Q_c \geq N_{\mathrm{sq}}(A';C'|B')_\sigma - N_{\mathrm{sq}}(A;C|B)_\rho$. Note, for pure states ρ_{ABC} and $\sigma_{A'B'C'}$, the equality is attained. □

We now consider a transformation to more general cases. Consider an initial state ρ_{ABC} is transformed to σ_{ABC} by utilizing free operations and quantum communication from Bob to Alice and Charlie. Then, the cost of quantum communication is given by the following theorem.

Lemma 6. *If a state ρ_{ABC} is transformed to a final state σ_{ABC}, with $N_{\mathrm{sq}}(A;C|B)_\sigma{\geq}N_{\mathrm{sq}}(A;C|B)_\rho$, by exploiting free operations and quantum communications from Bob to Alice and Charlie, then the necessary amount of quantum communication Q_c required for the transformation satisfies*

$$Q_c \geq N_{\mathrm{sq}}(A;C|B)_\sigma - N_{\mathrm{sq}}(A;C|B)_\rho. \tag{37}$$

Proof. Note we can restrict quantum communication from Bob to Charlie without loss of generality. Now, from the converse bound in Lemma 2, it is clear that when we want to convert $\rho^{\otimes n}{\to}\sigma^{\otimes n}$, the necessary condition is $N_{\mathrm{sq}}(A;C|B)_\rho{\geq}N_{\mathrm{sq}}(A;C|B)_\sigma$. The scenario where the condition is not met, i.e., $N_{\mathrm{sq}}(A;C|B)_\rho{<}N_{\mathrm{sq}}(A;C|B)_\sigma$, Bob needs to perform quantum communication $\mathbb{QC}_{\mathbf{B}\to\mathbf{C}}$ to increase the sQNM and make the transformation possible. The optimal one qubit quantum communication $\mathbb{QC}_{\mathbf{B}\to\mathbf{C}}$ increases the overall sQNM by at most one qubit. The protocol is similar to the examples in Section 3.2. Consider Alice and Bob initially share a Bell state, which is a free state. Now Bob transfers his share to Charlie via a one-qubit channel. This introduces a maximal resourceful state between Alice and Charlie and increases the overall sQNM by one qubit. Let us denote this operation by Φ_1, where the subscript denotes the number of quantum communications and

$$\Phi_1(\rho_{ABC}) = |\phi^+\rangle\langle\phi^+|_{A'C'} \otimes \rho_{ABC}, \tag{38}$$

where the state $|\phi^+\rangle\langle\phi^+|_{A'C'}$ is a Bell state shared between Alice and Charlie. Similarly, when Bob utilises k-qubit quantum channel to increase k-bit sQNM by successive application of Φ_1, we denote the overall operation as Φ_k where

$$\begin{aligned}\Phi_k(\rho_{ABC}) &= \underbrace{\Phi_1 \circ \Phi_1 \circ \cdots \circ \Phi_1}_{k \text{ times}}(\rho_{ABC}) \\ &= |\phi^+\rangle\langle\phi^+|^{\otimes k}_{A'C'} \otimes \rho_{ABC}.\end{aligned} \tag{39}$$

Clearly, we have

$$N_{\rm sq}(A;C|B)_{\Phi_k(\rho^{\otimes n})} = k + nN_{\rm sq}(A;C|B)_\rho. \tag{40}$$

Now consider Bob has access to nQ_c quantum channels to Charlie. Is there a better protocol to increase sQNM than Bob sending all nQ_c ebits simultaneously, i.e., applying Φ_{nQ_C}? To answer this, we consider a general operation Ω involving total nQ_C quantum communication where each operation Φ_1 is applied after a free operation:

$$\Omega = \Omega_{(nQ_C)} \circ \cdots \circ \Omega_{(2)} \circ \Omega_{(1)}, \tag{41}$$

where $\Omega_{(i)} = \Phi_1 \circ \Lambda^{(i)}_{\rm free}$ represents one step of quantum communication. As the operation Φ_1 is being applied on a Bell state independent of $\rho^{\otimes n}$, we can apply the monotonicity of sQNM:

$$N_{\rm sq}(A;C|B)_{\Phi_{nQ_c}(\rho^{\otimes n})} \geq N_{\rm sq}(A;C|B)_{\Omega(\rho^{\otimes n})}. \tag{42}$$

This suggests Φ_{nQ_c} is the operation causing the maximum increase in sQNM for a given number of quantum communications. Now invoking the converse bound in Lemma 2, the necessary condition for nQ_c quantum channels to allow the conversion from $\rho^{\otimes n}$ to $\sigma^{\otimes n}$ is

$$\begin{aligned}nQ_c + nN_{\rm sq}(A;C|B)_\rho &= N_{\rm sq}(A;C|B)_{\Phi_{nQ_c}(\rho^{\otimes n})} \\ &\geq nN_{\rm sq}(A;C|B)_\sigma.\end{aligned} \tag{43}$$

The first equality is due to Eq. (40). Rearranging the above equation and dividing by n, we have

$$Q_c \geq N_{\rm sq}(A;C|B)_\sigma - N_{\rm sq}(A;C|B)_\rho. \tag{44}$$

This completes the proof. □

So far, we have only provided a bound for the necessary quantum communication. In other words, satisfying the bound in Theorem 6 does not guarantee the relevant state transformation. Hence, it is interesting to have a bound for quantum communication, ensuring a successful state transformation. Below, we provide a similar bound motivated from our achievability protocol in Lemma 3.

Lemma 7. *If a state ρ_{ABC} is transformed to a final state σ_{ABC}, with $N_{\rm sq}(A;C|B)_\sigma \geq N_{\rm sq}(A;C|B)_\rho$, by exploiting free operations and quantum communications from Bob to Alice and Charlie, then it is sufficient to have a quantum communication rate of Q_c satisfying*

$$\begin{aligned}Q_c \geq& \min\{S(A)_\sigma, S(C)_\sigma\} \\ &- \max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho, 0\}.\end{aligned} \tag{45}$$

Proof. The relation $N_{\rm sq}(A;C|B)_\sigma \geq N_{\rm sq}(A;C|B)_\rho$, due to Corollary 1, implies $\min\{S(A)_\sigma, S(C)_\sigma\} \geq \max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho\}$. The theorem is motivated by the achievable bound and the corresponding protocol in Lemma 3. Consider the task is to convert $\rho^{\otimes n}$ to $\sigma^{\otimes n}$ for a large n, then firstly Alice and Charlie distill $\approx n \max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho\}$ Bell pairs from $\rho^{\otimes n}$. This distillation is only successful when the relevant coherent information is non-zero. In the scenario where $\max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho\} \leq 0$, the parties prepare the state $\sigma^{\otimes n}$ from scratch. Now, to prepare $\sigma^{\otimes n}$, Alice and Charlie need to establish $\approx n \min\{S(A)_\sigma, S(C)_\sigma\}$ Bell-pairs. In the scenario when $\min\{S(A)_\sigma, S(C)_\sigma\} \geq \max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho\}$, the desired state transformation is not possible with free operation as Alice and Charlie still need $n\Big(\min\{S(A)_\sigma, S(C)_\sigma\} - \max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho, 0\}\Big)$ amount of extra Bell pairs. However, the quantum communication from Bob to Charlie/ Alice can establish these extra Bell Pairs. Hence, whenever the rate of quantum communication satisfies

$$\begin{aligned}Q_C \geq& \min\{S(A)_\sigma, S(C)_\sigma\} \\ &- \max\{I(A\rangle C)_\rho, I(B\rangle C)_\rho, 0\},\end{aligned} \tag{46}$$

a successful conversion from $\rho^{\otimes n}$ to $\sigma^{\otimes n}$ is guaranteed. □

To summarise, we have provided rigorous characterizations of resourceful operations to perform transformation from a less resourceful state to a more resourceful state. Without the loss of generality, two particular resourceful operations are quantum communications from Bob to Charlie and quantum communications from Alice to Charlie.

Firstly, we have shown that to increase the same amount of sQNM, the former resourceful operation generally consumes fewer quantum channels than the latter. Next, motivated by the converse bound in Lemma 2, in Theorem 6, we have shown that the quantum communication cost must be greater than the extra resource possessed by the final state, given by the difference in the sQNM between the final and the initial state. Finally, invoking the achievability bound in Lemma 3, in Theorem 7, we have provided a protocol and a lower bound on quantum communication cost from Bob to Alice/ Charlie that guarantees a transformation of a less resourceful state into a more resourceful state.

4 Operational meaning of squashed QNM

Now we consider two important quantum information theoretic tasks, i.e., conditional quantum one-time pad [40], and quantum state deconstruction [38, 64], to provide some operational meanings to sQNM. The operational interpretations are in the same spirit as the operational interpretations for squashed entanglement provided in Refs. [87, 40, 64].

4.1 Conditional quantum one-time pad

We first consider the conditional one-time pad protocol introduced in [40]. In this protocol, Alice sends a private message to Charlie over an ideal quantum channel accessible to both Charlie and an eavesdropper, Bob.

Suppose the parties share many copies of the state ρ_{ABC}, with the subsystems A, B, and C possessed by Alice, Bob, and Charlie, respectively. In that case, the optimal rate at which Alice can send the message to Charlie keeping it hidden from Bob is given by the conditional quantum mutual information $I(A;C|B)_\rho$ [40]. The protocol fails when the state ρ_{ABC} is a Markov state corresponding to the Markov chain $(A - B - C)$. This is because, for a Markov state, the quantum conditional mutual information vanishes, $I(A;C|B)_\rho{=}0$. Operationally, such a Markov state allows Bob to access Charlie's subsystem by applying a recovery map $\mathcal{R}_{B\to BC}$ on her share to fully reconstruct the state ρ_{ABC}:

$$\mathrm{id}_A{\otimes}\mathcal{R}_{B\to BC}(\rho_{AB}) = \rho_{ABC}. \tag{47}$$

Here id is the identity channel and $\rho_{AB} = \mathrm{Tr}_B\, \rho_{ABC}$ is the state composed of Alice's system (A), which is accessible to Bob via the ideal quantum channel, and her system (B). In contrast, Bob cannot construct a perfect recovery map for a non-Markov state to fully reconstruct the state, resulting in a non-zero private communication rate from Alice to Charlie.

Let us now introduce a variant of the conditional quantum one-time pad protocol where we have allowed the eavesdropper (Bob) to take the help of a powerful eavesdropper, Eve, who could provide Bob with the system that extends ρ_{ABC}. In this scenario, Alice has a bigger challenge: concealing her message from Bob and Eve while successfully transferring it to Charlie.

We now give the following operational interpretation of sQNM.

Theorem 4. *Alice wants to share a private message with Charlie in the presence of eavesdropper Bob who may access a powerful eavesdropper, Eve. Alice is to communicate with Charlie over a noiseless quantum channel that is also accessible to Bob. If Alice (A), Bob (B), and Charlie (C) share state ρ_{ABC} and Eve (E) has access to all the remaining systems, then the (asymptotic) optimal rate $\widetilde{R}$ at which Alice can communicate with Charlie is*

$$\widetilde{R} = 2N_{\mathrm{sq}}(A;C|B)_\rho. \tag{48}$$

Proof. Consider we have many copies of the state ρ_{ABCE} with Eve in possession of the extension E of the state $\rho_{ABC}{=}\mathrm{Tr}_E[\rho_{ABCE}]$. In this scenario, using Ref. [40], the private communication rate from Alice to Charlie is $I(A;C|BE)_\rho$. Considering Eve to be a smart adversary who chooses the optimal extension to reduce the private communication rate, we conclude that the optimum asymptotic rate of private communication for this variant of the conditional one-time pad is

$$\widetilde{R} = \inf_{\rho_{ABCE}} I(A;C|BE)_\rho = 2N_{\mathrm{sq}}(A;C|B)_\rho. \tag{49}$$

This proves the theorem. □

To further explain the scenario, we consider the state ρ_{ABC} to be a probabilistic mixture of different Markov states, resulting in a classical non-Markov state. In such a scenario, Bob alone cannot access Alice's secret message, as $I(A;C|B)_\rho{\neq}0$. However, Bob can always collaborate with another malicious party, Eve, who has the optimum extended subsystem E such that $2N_{\mathrm{sq}}(A;C|B)_\rho = \inf_{\rho_{ABCE}} I(A;C|BE)_\rho{=}0$. In other words, Bob and Eve can apply the global recovery map $\mathcal{R}_{BE\to BCE}$ to access Charlie's subsystem. In contrast, if the state ρ_{ABC} is a quantum non-Markov state ($N_{\mathrm{sq}}{\neq}0$), Eve cannot jeopardize the protocol even in the presence of another adversary. This gives an operational interpretation of our measure of the squashed quantum non-Markovianity of a state.

4.2 State deconstruction

Let us now consider state deconstruction protocol as introduced in [38, 64]. In a deconstruction protocol, the goal is to decouple subsystem A in a density matrix ρ_{ABC} from the BC system. The decoupling is done over asymptotically large copies of the quantum state i.e., $\rho^{\otimes n}_{A^nB^nC^n}$ with a large n, via the action of a quantum channel Λ acting on the quantum systems A^nB^n and an auxiliary state $\alpha_{A'}$ initially uncorrelated with the rest of the systems. The channel Λ represents uniform randomisation over unitary operations $\{U_i\}$ with $i\in\{1,2,\cdots,M_n\}$, i.e., $\Lambda(\cdot)=1/M_n\sum_{i=1}^{M_n}U_i(\cdot)U_i^\dagger$,

$$\sigma_{A^nB^nC^nA'}=(\mathrm{id}_{C^n}\otimes\Lambda_{A^nB^nA'})\left[\rho_{A^nB^nC^n}^{\otimes n}\otimes\alpha_{A'}\right]. \tag{50}$$

In the process of deconstruction of system A, the marginal state BC should remain asymptotically intact, a condition known as *negligible disturbance*:

$$F\left(\mathrm{Tr}_{A^n}(\rho_{ABC}^{\otimes n}),\mathrm{Tr}_{A^nA'}(\sigma_{A^nB^nC^nA'})\right)\geq 1-\varepsilon. \tag{51}$$

The deconstruction protocol also asymptotically obeys the *local recoverability* condition with respect to system B, i.e. there is a recovery map $\mathcal{R}_{B^n\to A^nB^nA'}$ such that

$$F\left(\sigma_{A^nB^nC^nA'},\mathcal{R}_{B^n\to A^nB^nA'}\left(\mathrm{Tr}_{A^nA'}(\sigma_{A^nB^nC^nA'})\right)\right)\geq 1-\varepsilon. \tag{52}$$

What is the minimum number of unitaries required to achieve a successful state deconstruction? To answer this, the *deconstruction cost* $D(A;C|B)_\rho=\inf_\Lambda\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{\log_2 M_n}{n}$ have been introduced [38, 64].It has been shown the deconstruction cost is given by QCMI of the state ρ conditioned over B: $D(A;C|B)_\rho=I(A;C|B)_\rho$. Clearly when ρ is a Markov state, $D(A;C|B)_\rho=I(A;C|B)_\rho=0$—only one unitary is enough.

We now introduce a variant of the state deconstruction protocol where Bob is allowed to have access to an arbitrary extension ρ_{ABCE} of ρ_{ABC}. In this case, the random unitary channel for the deconstruction protocol is acting on the systems ABE in the asymptotic limit and A' of an auxiliary state $\alpha_{A'}$. This allows us to give the following operational interpretation of state deconstruction protocol.

Theorem 5. *When Bob is allowed to have access to system E from any possible state extension ρ_{ABCE} of the state ρ_{ABC}, then the minimum deconstruction cost, $\widetilde{D}(A;C|B)_\rho$, is given by twice the sQNM of the state:*

$$\widetilde{D}(A;C|B)_\rho=\inf_{\rho_{ABCE}}D(A;C|BE)_\rho=2N_{\mathrm{sq}}(A;C|B)_\rho. \tag{53}$$

Proof. The theorem is the simple consequence of the fact that $D(A;C|BE)_\rho=I(A;C|BE)_\rho$. Hence,

$$\begin{aligned}\widetilde{D}(A;C|B)_\rho&=\inf_{\rho_{ABCE}}D(A;C|BE)_\rho\\&=\inf_{\rho_{ABCE}}I(A;C|BE)_\rho=2N_{\mathrm{sq}}(A;C|B)_\rho.\end{aligned}$$

□

As a consequence of the above theorem, when ρ is a classically non-Markov state, we can still achieve zero deconstruction cost as there exists an extension ρ_{ABCE} such that $I(A;C|BE)_\rho=0$.

5 Discussion and Conclusion

Quantum non-Markovianity, seen in tripartite quantum systems, is a widely discussed quantum correlation. It finds many applications in quantum information processing. It is thus pertinent to study the quantum origin of non-Markovianity in quantum non-Markovian states, i.e., states with strictly positive quantum conditional mutual information (QCMI). In this regard, we have introduced a resource-theoretic framework to characterize genuine quantum non-Markovianity as a resource. We have shown that quantum non-Markovianity in states can have both classical and quantum contributions, and the leftover non-Markovianity can quantify the genuine quantum non-Markovianity after squashing out all non-quantum contributions. We denote this as the squashed quantum non-Markovianity (sQNM), and it satisfies several desirable information-theoretic properties. With this in hand, we have characterized the free states, i.e., states with no resources or vanishing sQNM. The set of free states forms a convex set, and hence, our resource theory of genuine quantum non-Markovianity is a convex quantum resource theory [88, 89]. Thus, by probabilistic mixing of the free states, one cannot create a resource state, i.e., a state with genuine quantum non-Markovianity. We have also characterized the free quantum operations, i.e., physical transformations that cannot increase or create sQNM in states. Equipped with free states and free operations, we have studied the conditions for state transformations under free operations and derived various bounds on the rate of transformations. We explored the interrelation between quantum communication and sQNM in state transformations and have demonstrated that the quantum communication cost of creating an arbitrary quantum state is lower bounded by the sQNM of the state. Further, the quantum communication cost for any arbitrary state transformation is lower bounded by the change in sQNM. We have discussed a generalization of sQNM, expressed in terms of generalized divergence and fidelity of recovery, and possible implications of our results in quantum information theoretic tasks, namely conditional quantum one-time pad and quantum state deconstruction.

Our results, particularly the resource-theoretic framework, are expected to deepen the understanding of quantum non-Markovianity and find wide-range implications in quantum information theory. For instance, the genuine quantum non-Markovianity quantified by sQNM represents a correlation that may be understood as conditional quantum entanglement, i.e., representing entanglement between two parties while conditioning on a third one. It unravels a different kind of 'bipartite' entanglement in a tripartite system. Thus, studying the genuine quantum non-Markovianity, which lies between bipartite and tripartite entanglement, may help understand qualitative differences between bipartite and multipartite entanglements. Our results find that quantum communication, a non-free (or resourceful) operation, can create or increase genuine quantum non-Markovianity in states. This substantiates our general intuition that quantum non-Markovianity and quantum communication are interconnected. However, we also note that non-Markovianity as a resource present in a state and the same in an operation are not inter-convertible. For instance, one cannot simulate quantum communication by utilizing a quantum non-Markovian state and free operations, in general. This raises the question of whether there is a fundamental irreversibility in the resource transformations. The answer to this requires further studies, which we leave to future works.

Beyond states, non-Markovianity is also widely explored in the context of operations and dynamics, represented by stochastic processes involving sequences of events. Markovianity, in general, signifies the memoryless property of such stochastic processes where the present event of a system fully determines the evolution to the next event. Non-Markovianity, on the other hand, signifies that the dynamics of a system cannot completely be characterized by ignoring the event(s) it underwent in the past. In the last decades, the non-Markovian properties have been studied extensively in quantum dynamical processes [90, 91, 58, 51, 92, 93, 52, 94, 95, 96, 97, 98, 99]. We anticipate that, as the dynamical properties are often characterized in terms of states, the resource theoretical insights gained in studying non-Markovianity in states will find important implications to acquire improved understanding of non-Markovianity in quantum processes. Furthermore, Markov states, i.e., the states represented by short Markov chains, often appear in the causal context of quantum processes [100, 101, 102, 103, 104, 105, 106]. In particular, it is known that the common-cause quantum processes are characterized by Markov states [59]. With the notion of genuine quantum non-Markovianity in states, our results are expected to introduce a new approach to these studies, particularly exploring the genuine quantum aspects in the causal structures and their possible technological applications.

An executive summary of the main results is the following.

- Squashed quantum non-Markovianity (sQNM) is introduced to measure genuine quantum non-Markovian correlations in tripartite quantum states. It is based on quantum conditional mutual information. For finite-dimensional systems, the sQNM is convex, monogamous, continuous, faithful, and super-additive. For infinite-dimensional systems, measures of genuine quantum non-Markovianity based on generalized divergence and recoverability of states via universal recovery maps are formulated.

- The relation of genuine QNM with entanglement and unextendibility, two of the important correlations in quantum information theory, is explored. In particular, it is shown that the genuine QNM is monogamous. The sQNM of an arbitrary tripartite state is always lower bounded by the squashed entanglement between non-conditioning systems. Further, it is limited by the extendibility of any of the non-conditioning systems.

- A framework for a convex resource theory is introduced to study genuine quantum non-Markovianity as a resource after characterizing (resource-)free states and operations. The resourceful states have non-zero sQNM and are genuine quantum non-Markovian states. State transformations, in relation to sQNM and quantum communication, are studied within this framework.

- Two quantum information-theoretic tasks, quantum one-time pad and state deconstruction, are re-investigated to provide an operational meaning to sQNM.

Note.— In a recent work [60], the squashed quantum non-Markovianity in states has found application in characterizing the open quantum system dynamics in terms of its information revival being either due to a genuine information backflow from the environment or due to a non-causal revival.

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A Preliminaries

In this section, we briefly describe the standard notations and definitions of some basic concepts used in this paper to discuss and arrive at the main results.

A quantum system A is associated with a separable Hilbert space $\mathcal{H}_A$ and $|A| := \dim(\mathcal{H}_A)$. The Hilbert space of a composite system AB is given by $\mathcal{H}_A \otimes \mathcal{H}_B$. Wherever there is no confusion, we label the Hilbert space of a system with the same label of the system. A density operator ρ_A represents a state of a quantum system A. The density operator ρ_A is positive semi-definite defined on $\mathcal{H}_A$ with a unit trace, i.e., $\mathrm{Tr}[\rho_A] = 1$. If ρ_{AB} is a joint state of a composite system AB, then its reduced state on system B is $\rho_B := \mathrm{Tr}_B[\rho_{AB}]$. An arbitrary pure state ψ_A is a density operator of rank-one, where if $\psi_A = |\psi\rangle\langle\psi|_A$ then $|\psi\rangle_A \in \mathcal{H}_A$. For finite-dimensional system A or B, a maximally entangled state ϕ^+_{AB} between A and B is of Schmidt-rank $d = \min\{|A|, |B|\}$ and denoted as $\phi^+_{AB} := |\phi^+\rangle\langle\phi^+|_{AB}$, where

$$|\phi^+\rangle_{AB} := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle_{AB}, \tag{54}$$

and $\{|i\rangle\}_i$ is an orthonormal basis in the d-dimensional Hilbert space.

A quantum channel $\mathcal{N}_{A\to B} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is a completely positive, trace-preserving (CPTP) map that takes input in the system A and outputs in the system B. If a quantum channel $\mathcal{N}_{A\to A} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$, then we simply denote it as $\mathcal{N}_A$. We denote the identity operator on $\mathcal{H}_A$ as $\mathbb{1}_A$. Let $\mathrm{id}_{A\to B}$ denote the identity quantum channel with input in system A and output in system B.

A quantum instrument is relevant in the context of a general evolution which takes an input quantum state ρ_A and gives rise to both quantum and classical outputs, and is defined by a set of complete positive (CP), trace non-increasing maps $\{p_x\Lambda^x_{A\to B}\}$, such that $\mathrm{Tr}(p_x\Lambda^x_{A\to B}(\rho_A)) \leq \mathrm{Tr}(\rho_A)$, and the sum map $\sum_x p_x\Lambda_x$ is a quantum channel. Each CP map, $p_X\Lambda^x$, is associated with a classical outcome $|x\rangle\langle x|_X$ resulting in an overall quantum channel $\mathcal{E}_{A\to XB}$ transforming the quantum state ρ_A to

$$\mathcal{E}_{A\to XB}(\rho_A) = \sum_x p_x|x\rangle\langle x|_X \otimes \Lambda^x{}_{A\to B}(\rho_A). \tag{55}$$

The fidelity $F(\rho, \sigma)$ between two quantum states ρ and σ is $F(\rho, \sigma) := \left\| \sqrt{\rho}\sqrt{\sigma} \right\|_1^2 = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]\right)^2$, where $\|M\|_1 := \mathrm{Tr}[|M|] = \mathrm{Tr}\left[\sqrt{M^\dagger M}\right]$ is the trace-norm of an operator M. The fidelity between the two states is always between 0 and 1. $F(\rho, \sigma) = 0$ if and only if ρ and σ are orthogonal and $F(\rho, \sigma) = 0$ if and only if $\rho = \sigma$. The trace-distance between two states ρ and σ is given by $\|\rho - \sigma\|_1/2$. The trace distance between any two states is always between 0 and 1. $\|\rho - \sigma\|_1/2 = 1$ if and only if ρ and σ are orthogonal and $\|\rho - \sigma\|_1/2 = 0$ if and only if $\rho = \sigma$. The following relation between the trace distance and fidelity between arbitrary states ρ and σ holds [107]:

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}. \tag{56}$$

The von Neumann entropy $S(A)_\rho$ for a quantum state ρ_A is defined as

$$S(A)_\rho := S(\rho_A) = -\mathrm{Tr}(\rho_A \log_2(\rho_A)). \tag{57}$$

For any pure pure state ψ_A, $S(A)_\psi = 0$. $0 \leq S(A)_\rho \leq \log_2 |A|$ always holds for arbitrary system A. When $|A| < \infty$, the maximum is attained for a maximally mixed state, i.e., $S(A)_\rho = \log_2 |A|$ if and only if $\rho_A = \mathbb{1}_A/|A|$. If a bipartite state ψ_{AB} is pure then $S(A)_\psi = S(B)_\psi$.

Quantum conditional entropy $S(A|B)_\rho$ of an arbitrary bipartite state ρ_{AB} when conditioned on B is given as

$$S(A|B)_\rho := S(AB)_\rho - S(B)_\rho. \tag{58}$$

For a bipartite state ρ_{AB}, it quantifies the average uncertainty of quantum state ρ_A when the state ρ_B is known. The negative quantum conditional entropy $S(AB)_\rho$, i.e., $S(AB)_\rho < 0$, implies that the state ρ_{AB} is entangled, but the converse need not be true.

Quantum mutual information $I(A; B)_\rho$ quantifies the total correlations between A, B when they are in the joint state ρ_{AB},

$$\begin{aligned} &I(A; B)_\rho \\ &\quad := S(A)_\rho + S(B)_\rho - S(AB)_\rho = I(B; A)_\rho. \end{aligned} \tag{59}$$

Quantum conditional mutual information $I(A; C|B)_\rho$ for a tripartite state ρ_{ABC} quantifies the total correlation between A, C when conditioned on B,

$$\begin{aligned} I(A; C|B)_\rho &:= S(AB)_\rho + S(BC)_\rho - S(ABC)_\rho - S(B)_\rho \\ &= I(AB; C)_\rho - I(C; B)_\rho \\ &= I(C; A|B)_\rho. \end{aligned} \tag{60}$$

The coherent information $I(A\rangle B)_\rho$ of a bipartite state ρ_{AB} between subsystems A and B is

$$I(A\rangle B)_\rho := S(B)_\rho - S(AB)_\rho = -S(A|B)_\rho. \tag{61}$$

B Proof of Lemma 1

Here, we prove the Lemma 1 elaborating on the inter-relation between squashed quantum non-Markovianity (N_{sq}) and squashed entanglement

($E_{\rm sq}$). The squashed entanglement for a state ρ_{AC} is defined as,

$$E_{\rm sq}(A;C)_\rho = \inf_{\rho_{ABF}} \frac{1}{2} I(A;C|F)_\rho, \tag{62}$$

where the infimum is carried out over all extensions of $\rho_{AB} = \mathrm{Tr}_F[\rho_{ABF}]$. For any state ρ_{ABC} with $\rho_{AC} = \mathrm{Tr}_B[\rho_{ABC}]$, we can easily see that

$$\inf_{\rho_{ABCE}} \frac{1}{2} I(A;C|BE)_\rho = N_{\rm sq}(A;C|B)_\rho \geq E_{\rm sq}(A;C)_\rho.$$

Because the infimum taken to find out $N_{\rm sq}$ utilizes a more restricted set of extended states than that for $E_{\rm sq}$. From this, it is clear that $E_{\rm sq}(A;C)_\rho > 0$ implies $N_{\rm sq}(A;C|B)_\rho > 0$, but the reverse statement is not necessarily true.

For any state ρ_{ABC}, we can easily see that

$$E_{\rm sq}(AB;C)_\rho \geq N_{\rm sq}(A;C|B)_\rho, \tag{63}$$

by using the definitions of squashed entanglement and sQNM, and $I(AB;C)_\rho \geq I(A;C|B)$, a consequence of the chain rule of QCMI. Similarly, we have $E_{\rm sq}(A;BC)_\rho \geq N_{\rm sq}(A;C|B)_\rho$.

Consider an isometry $V : B \to B_L \otimes B_R$, leading to a transformation $\rho_{ABC} \to \sigma_{AB_LB_RC}$. Note this operation does not alter $E_{\rm sq}(AB;C)_\rho$, $E_{\rm sq}(A;BC)_\rho$, and $N_{\rm sq}(A;C|B)_\rho$ of the initial state ρ_{ABC}. Nevertheless, it can be easily checked that

$$E_{\rm sq}(AB_L;B_RC)_\sigma \geq N_{\rm sq}(A;C|B_LB_R)_\sigma, \tag{64}$$

following the definitions of squashed entanglement, sQNM, and applying the chain rule of QCMI. Again, $N_{\rm sq}(A;C|B_LB_R)_\sigma = N_{\rm sq}(A;C|B)_\rho$. Now, combining the equations, we have

$$\min \{E_{\rm sq}(AB_L;B_RC)_\sigma,\ E_{\rm sq}(AB;C)_\rho,\ E_{\rm sq}(A;BC)_\rho\} \geq N_{\rm sq}(A;C|B)_\rho \geq E_{\rm sq}(A;C)_\rho. \tag{65}$$

C Proof of Lemma 2

(P1) *Convexity*: For any state $\rho_{ABC} = \sum_m p_m\, \rho^m_{ABC}$,

$$N_{\rm sq}(A;C|B)_\rho \leq \sum_m p_m\, N_{\rm sq}(A;C|B)_{\rho^m}. \tag{66}$$

To prove the convexity, consider an extension $\rho_{ABCE} = \sum_m p_m\, \rho^m_{ABCE}$ for which $N_{\rm sq}(A;C|B)_{\rho^m} = I(A;C|BE)_{\rho^m}$ for all m. By adding an ancilla F, we have a state

$$\rho_{ABCEF} = \sum_m p_m\, |m\rangle\langle m|_F \otimes \rho^m_{ABCE}, \tag{67}$$

where $\{|m\rangle_F\}$ are the orthonormal bases of F. The QCMI of the state ρ_{ABCEF} satisfies

$$\begin{aligned} I(A;C|BEF)_\rho &= \sum_m p_m\, I(A;C|BE)_{\rho^m}, \\ &= \sum_m p_m\, 2N_{\rm sq}(A;C|B)_{\rho^m}. \end{aligned} \tag{68}$$

Now with $I(A;C|BEF)_\rho \geq 2N_{\rm sq}(A;C|B)_\rho$, we recover the convexity relation (66). As a corollary, the states with $N_{\rm sq} = 0$, i.e., Markov states and states with CNM, form a convex set.

(P2) *Super-additive:* For every state $\rho_{AA'BB'CC'}$, the sQNM is super-additive

$$\begin{aligned} &N_{\rm sq}(AA';CC'|BB')_\rho \\ &\quad \geq N_{\rm sq}(A;C|B)_\rho + N_{\rm sq}(A';C'|B')_\rho, \end{aligned} \tag{69}$$

and additive on the tensor product, i.e.,

$$\begin{aligned} &N_{\rm sq}(AA';CC'|BB')_\rho \\ &\quad = N_{\rm sq}(A;C|B)_\rho + N_{\rm sq}(A';C'|B')_\rho, \end{aligned} \tag{70}$$

for $\rho_{AA'BB'CC'} = \rho_{ABC} \otimes \rho_{A'B'C'}$.

We prove the super-additivity property of sQNM, from which the additivity is easily followed. Consider a state $\rho_{AA'BB'CC'}$ and its no-signaling extension $\rho_{AA'BB'CC'E}$ such that $2N_{\rm sq}(AA';CC'|BB')_\rho = I(AA';CC'|BB'E)_\rho$. Now, using the chain rule, monotonicity of QCMI under partial tracing, and definition of $N_{\rm sq}$, we have

$$\begin{aligned} 2N_{\rm sq}(AA';CC'|BB')_\rho &= I(AA';CC'|BB'E)_\rho \\ &= I(A;CC'|BB'EA')_\rho + I(A';CC'|BB'E)_\rho \\ &\geq I(A;C|BB'EA')_\rho + I(A';C'|BB'E)_\rho \\ &\geq 2N_{\rm sq}(A;C|B)_\rho + 2N_{\rm sq}(A';C'|B')_\rho. \end{aligned} \tag{71}$$

(P3) *Monogamy:* For every state $\rho_{AA'BC}$, the measure of quantum non-Markovianity, the quantum non-Markovianity satisfies a monogamy relation

$$\begin{aligned} &N_{\rm sq}(AA';C|B)_\rho \\ &\quad \geq N_{\rm sq}(A;C|B)_\rho + N_{\rm sq}(A';C|B)_\rho. \end{aligned} \tag{72}$$

To prove the relation, we consider an extension $\rho_{AA'BEC}$ of the state for which $N_{\rm sq}(AA';C|B)_\rho = I(AA';C|BE)_\rho$. Now, using the chain rule of QCMI

$$\begin{aligned} N_{\rm sq}(AA';C|B)_\rho &= I(AA';C|BE)_\rho, \\ &= I(A;C|BEA')_\rho + I(A';C|BE)_\rho, \\ &\geq N_{\rm sq}(A;C|B)_\rho + N_{\rm sq}(A';C|B)_\rho. \end{aligned} \tag{73}$$

This monogamy relation can be generalized to n-party systems possessed by Alice.

(P4) *Continuity*: For any two states ρ_{ABC} and σ_{ABC} with $\frac{1}{2}\|\rho_{ABC} - \sigma_{ABC}\|_1 \leq \varepsilon$,

$$|\ N_{\rm sq}(A;C|B)_\rho - N_{\rm sq}(A;C|B)_\sigma\ | \leq \frac{f(\varepsilon)}{2}, \tag{74}$$

where $f(\varepsilon) \to 0$, for $\varepsilon \to 0$.

Our proof is similar to the continuity bound of squashed entanglement in Ref. [63]. We first use the continuity bound for state purification. For two states, ρ_{ABC} and σ_{ABC}, with a trace distance

$\|\rho-\sigma\|_1/2 \leq \varepsilon$, we can find respective purifications, $|\psi_{ABCR}\rangle$ and $|\phi_{ABCR}\rangle$, for which the trace distance has the upper bound

$$\frac{\| \, |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \, \|_1}{2} \leq 2\sqrt{\varepsilon}. \qquad (75)$$

We obtain the above inequality utilizing the Uhlman theorem [108, 109] and Fuchs-Van de Graaf inequality [110]. Alternatively, one can directly use the continuity of Stinespring dilations [111, 112]. Note, without loss of generality, we have considered identical purifying systems (R) for both ρ and σ as we can always consider the smaller purifying system a subspace of the larger one.

Next, we introduce the continuity bound of arbitrary extensions of ρ and σ. We obtain an arbitrary extension of a state by applying a CPTP map $\Lambda_{R\to E}$ acting on the purifying extension R of the state, i.e.,

$$\rho_{ABCE} = \left[\mathrm{id}_{ABC} \otimes \Lambda_{R\to E}\right] (|\psi\rangle\langle\psi|_{ABCR}), \qquad (76)$$

$$\sigma_{ABCE} = \left[\mathrm{id}_{ABC} \otimes \Lambda_{R\to E}\right] (|\phi\rangle\langle\phi|_{ABCR}). \qquad (77)$$

Next, we use the contractivity of trace distance [113] to obtain

$$\frac{\|\rho_{ABCE} - \sigma_{ABCE}\|_1}{2} \leq 2\sqrt{\varepsilon}. \qquad (78)$$

Having established Eq. (78), we can now use the tight uniform continuity bound for quantum conditional mutual information (See Corollary 1 of [83], also [114]):

$$\begin{aligned} &|I(A;C|BE)_\rho - I(A;C|BE)_\sigma| \\ &\leq 4\sqrt{\varepsilon}\log_2 \min\{|A|,|C|\} + 2(1+2\sqrt{\varepsilon})\; h_2\left(\frac{2\sqrt{\varepsilon}}{1+2\sqrt{\varepsilon}}\right) \\ &= f(\varepsilon). \end{aligned} \qquad (79)$$

Here, $h_2(.)$ is the binary Shannon entropy. Note this is a tighter bound compared to the one introduced in [63] by combining the continuity of von Neumann entropy [115, 116] and continuity of conditional von Neumann entropy.

Now, we are ready to prove the continuity bound of the measure of quantum non-Markovianity N_{sq}. Without loss of generality, we consider $N_{\mathrm{sq}}(A;C|B)_\rho \geq N_{\mathrm{sq}}(A;C|B)_\sigma$. Let us assume the CPTP map $\widetilde{\Lambda}$ achieves the infimum value of $I(A;C|BE)$ for σ, i.e., $\inf_{\widetilde{\Lambda}} I(A;C|BE)_\sigma = I(A;C|BE)_{\widetilde{\Lambda}(|\phi\rangle\langle\phi|)}$. Then, one can write,

$$\begin{aligned} &2\;|N_{\mathrm{sq}}(A;C|B)_\rho - N_{\mathrm{sq}}(A;C|B)_\sigma| \\ &= \inf_{\widetilde{\Lambda}} I(A;C|BE)_\rho - I(A;C|BE)_{\widetilde{\Lambda}(|\phi\rangle\langle\phi|)} \\ &\leq I(A;C|BE)_{\widetilde{\Lambda}(|\psi\rangle\langle\psi|)} - I(A;C|BE)_{\widetilde{\Lambda}(|\phi\rangle\langle\phi|)} \\ &\leq f(\varepsilon). \end{aligned} \qquad (80)$$

Here, Eq. (80) indicates the state $\widetilde{\Lambda}(|\psi\rangle\langle\psi|)$ may not achieve the infimum value of QCMI, $I(A;C|BE)_\rho$. The final inequality is due to Eq. (79). This completes the proof.

(P5) *Faithfulness*: It follows from combining Eqs. (2) and (4) (see also Remark 1) as the fidelity of recovery is vanishing if and only if the state is perfectly recoverable via a universal recovery map.

D Proof of Lemma 3

D.1 Monotonicity under local quantum channel on Bob ($\mathbb{LO}_{\mathbf{B}}$)

Here, we aim to demonstrate the monotonicity under local quantum channel local trace-preserving and completely positive operations (i.e., CPTP maps), also known as the local quantum channel applied to Bob's system. In previous works [62, 61], the authors established that only reversible operations on Bob's system are allowable operations that do not increase non-Markovianity. However, their measure of non-Markovianity includes both classical and quantum aspects. In contrast, using the squashed quantum non-Markovianity measure N_{sq}, we establish that any CPTP map will not increase quantum non-Markovianity.

First, let us establish that the application of the partial tracing-out operation on the conditioned system possessed by Bob does not lead to an increase in quantum non-Markovianity. We consider two states, ρ_{ABC} and $\rho_{ABB'C}$, where $\rho_{ABC} = \mathrm{Tr}_{B'}[\rho_{ABB'C}]$. Now, the quantum non-Markovianity is expressed as follows:

$$2N_{\mathrm{sq}}(A;C|B)_\rho = \inf_{\rho_{ABCE}} I(A;C|BE)_\rho, \qquad (81)$$

$$2N_{\mathrm{sq}}(A;C|BB')_\rho = \inf_{\rho_{ABB'CF}} I(A;C|BB'F)_\rho, \qquad (82)$$

the infimums are carried out on the set of all state extensions ρ_{ABCE} and $\rho_{ABB'CF}$, respectively. A noticeable distinction between Eq. (81) and Eq. (82) is that in Eq. (82), due to the presence of system B', the conditional mutual information is subjected to a more constrained infimum over F than the one over E. This leads us to conclude that

$$N_{\mathrm{sq}}(A;C|BB')_\rho \geq N_{\mathrm{sq}}(A;C|B)_\rho. \qquad (83)$$

Thus, a partial tracing-out operation on the conditioning system does not increase quantum non-Markovianity.

Next, we prove monotonicity under the local CPTP operation applied to Bob's system B. Let us consider a state ρ_{ABC} and its extension ρ_{ABCE} for which $2N_{\mathrm{sq}}(A;C|B)_\rho = I(A;C|BE)_\rho$. Now we introduce an ancillary system B' in the state $|0\rangle\langle 0|$, and composite state becomes $\rho_{ABCEB'} = \rho_{ABCE} \otimes |0\rangle\langle 0|_{B'}$. Then any CPTP operation Λ on Bob, leading to $\sigma_{A\widetilde{B}CE} =$

$\Lambda_{B\to\widetilde{B}}\otimes \mathrm{id}_{ACE}(\rho_{ABC})$, can be simulated by applying a unitary operation on $BB'(U_{BB'\to\widetilde{B}\widetilde{B'}})$ and tracing out $\widetilde{B'}$ following Stinespring dilation, as

$$\begin{aligned}
&\sigma_{A\widetilde{B}CE} \\
&= \mathrm{Tr}_{\widetilde{B'}}\left[U_{BB'\to\widetilde{B}\widetilde{B'}}\left(\rho_{ABCE}\otimes|0\rangle\langle 0|_{B'}\right)U^\dagger_{BB'\to\widetilde{B}\widetilde{B'}}\right] \\
&= \mathrm{Tr}_{\widetilde{B'}}\left[\sigma_{A\widetilde{B}\widetilde{B'}CE}\right]. \qquad (84)
\end{aligned}$$

Noting that QCMI remains unchanged under unitary operations on the conditioning system, we have

$$\begin{aligned}
2N_{\mathrm{sq}}(A;C|B)_\rho &= I(A;C|BE)_\rho \\
&= I(A;C|\widetilde{B}\widetilde{B'}E)_\sigma \\
&\geq 2N_{\mathrm{sq}}(A;C|\widetilde{B}\widetilde{B'})_\sigma \\
&\geq 2N_{\mathrm{sq}}(A;C|\widetilde{B})_\sigma. \qquad (85)
\end{aligned}$$

We use the relation (83) in the last inequality.

D.2 Monotonicity under $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$

Here, we prove the monotonicity under $\mathbb{LOCC}_{\mathbf{A}\leftrightarrow\mathbf{C}}$. According to [73, 63], we need to show that N_{sq} is a strong monotone under local operations on Alice and Charlie, and it is convex. The latter we have demonstrated earlier.

Now, we show strong monotonicity under a local operation by Alice. Let us consider a state ρ_{ABC} and any unilocal quantum instruments $\{\Lambda_A^m\}$ applied on Alice's system A, where each Λ_A^m is a completely positive map and their sum is trace-preserving. Then, the strong monotonicity implies

$$N_{\mathrm{sq}}(A;C|B)_\rho \geq \sum_m p_m N_{\mathrm{sq}}(A;C|B)_{\sigma^m}, \qquad (86)$$

where $p_m = \mathrm{Tr}[\Lambda_A^m(\rho_{ABC})]$ and $\sigma^m_{ABC} = \Lambda_A^m(\rho_{ABC})/p_m$.

To prove the above, (i) we consider the state ρ_{ABCE}, serving as a non-signaling extension of the initial state ρ_{ABC}, for which $2N_{\mathrm{sq}}(A;C|B)_\rho = I(A;C|BE)_\rho$. (ii) Now we introduce two ancillary systems R and R' and both are initialized in the $|0\rangle_{R/R'}$ state. Thus, the composite becomes $\rho_{ARR'BCE} = \rho_{ABCE}\otimes|0\rangle\langle 0|_R\otimes|0\rangle\langle 0|_{R'}$. (iii) Next, we apply a global unitary operation $U_{ARR'}$ on ARR'. The rest of the systems will go through an identity channel. As a result, we have $\sigma_{ARR'BCE} = U_{ARR'}(\rho_{ARR'BCE})U^\dagger_{ARR'}$. (iv) Following this, we trace out the ancillary system R'. It can be written as,

$$\begin{aligned}
\sigma_{ARBCE} &= \sum_m |m\rangle\langle m|_R\otimes\Lambda_A^m(\rho_{ABCE}) \\
&= \sum_m p_m\ |m\rangle\langle m|_R\otimes\sigma^m_{ABCE}, \qquad (87)
\end{aligned}$$

where $p_m = \mathrm{Tr}[\Lambda_A^m(\rho_{ABCE})]$ is the probability, $\sigma^m_{ABCE} = \Lambda_A^m(\rho_{ABCE})/p_m$ is the normalized state after the instrument Λ_A^m applied by Alice, and $\{|m\rangle_R\}$ are the orthonormal bases on R. Further, we ensure that $\mathrm{Tr}_E[\sigma^m_{ABCE}] = \sigma^m_{ABC}$ as considered in Eq. (86). Our arguments proving strong monotonicity are based on QCMI, as in the following:

$$\begin{aligned}
2\ N_{\mathrm{sq}}(A;C|B)_\rho &\overset{(i)}{=} I(A;C|BE)_\rho \\
&\overset{(ii)}{=} I(ARR';C|BE)_\rho \\
&\overset{(iii)}{=} I(ARR';C|BE)_\sigma \\
&\overset{(iv)}{\geq} I(AR;C|BE)_\sigma \\
&\overset{(v)}{=} I(R;C|BE)_\sigma + I(A;C|BER)_\sigma \\
&\overset{(vi)}{\geq} I(A;C|BER)_\sigma \\
&\overset{(vii)}{=} \sum_m p_m\ I(A;C|BE)_{\sigma^m} \\
&\overset{(viii)}{\geq} 2\sum_m p_m\ N_{\mathrm{sq}}(A;C|B)_{\sigma^m}. \qquad (88)
\end{aligned}$$

The equality (i) is due to the choice of the extension ρ_{ABCE} of the initial state ρ_{ABC} as mentioned earlier. Equalities (ii) and (iii) are because the QCMI does not alter upon attaching (uncorrelated) ancilla RR' with Alice and a unitary $U_{ARR'}$ respectively. The inequality (iv) follows from the data-processing inequality of QCMI under partial tracing of R'. The equality (v) and the inequality are due to the chain rule and inequality (vi) due to the non-negativity of QCMI ($I(R;C|BE)_\sigma \geq 0$). For the state (87), the QCMI satisfies

$$I(A;C|BER)_\sigma = \sum_m p_m\ I(A;C|BE)_{\sigma^m},$$

as represented by the equality (vii). The inequality (viii) is the result of the fact that $I(A;C|BE)_{\sigma^m} \geq 2N_{\mathrm{sq}}(A;C|B)_{\sigma^m}$ for all m. Thus, we prove strong monotonicity of N_{sq} under local operation by Alice. Similar steps can show the strong monotonicity of local operations on Charlie.

As a corollary, we also prove the monotonicity of any trace-preserving local operation on Alice (also on Charlie)

$$\begin{aligned}
&N_{\mathrm{sq}}(A;C|B)_\rho \\
&\quad\geq \sum_m p_m N_{\mathrm{sq}}(A;C|B)_{\sigma^m} \overset{(ix)}{\geq} N_{\mathrm{sq}}(A;C|B)_\sigma. \qquad (89)
\end{aligned}$$

The inequality (ix) results from the convexity of the measure N_{sq}.

D.3 Monotonicity under $\mathbb{LOQC}_{\mathbf{A}\to\mathbf{B}}$ and $\mathbb{LOQC}_{\mathbf{C}\to\mathbf{B}}$

We first demonstrate the monotonicity of one-way local operation and quantum communication from

Alice to Bob ($\mathbb{LOQC}_{\mathbf{A}\to\mathbf{B}}$). Consider (i) a state ρ_{ABC} and its extension ρ_{ABCE} satisfying $2N_{\rm sq}(A;C|B)_\rho = I(A;C|BE)_\rho$. (ii) Now a local operation $\Lambda_A : A \to \widetilde{A}A'$ performed by Alice, leading to $\sigma_{\widetilde{A}A'BCE} = \Lambda_A \otimes \mathrm{id}_{BCE}(\rho_{ABCE})$. After this, (iii) the quantum information A', initially possessed by Alice, is transferred to Bob via the ideal quantum channel, $\mathrm{id}_{A'\to B'}$, i.e., $\widetilde{A}A'|BE|C \to \widetilde{A}|BB'E|C$. Finally, (iv) a local operation $\mathcal{E}_{BB'} : BB' \to \widetilde{B}$ is applied on Bob's systems, as a result we have $\tau_{\widetilde{A}\widetilde{B}CE} = \mathcal{E}_{BB'} \otimes \mathrm{id}_{\widetilde{A}CE}(\sigma_{\widetilde{A}BB'CE})$. Note, these are the generic steps to execute an arbitrary $\mathbb{LOQC}_{\mathbf{A}\to\mathbf{B}}$. To prove the monotonicity, we employ the following arguments based on the properties of QCMI:

$$\begin{aligned}
2\ N_{\rm sq}(A;C|B)_\rho &\overset{(i)}{=} I(A;C|BE)_\rho \\
&\overset{(ii)}{\geq} I(\widetilde{A}A';C|BE)_\sigma \\
&\overset{(iii)}{\geq} I(\widetilde{A};C|BB'E)_\sigma \\
&\overset{(iv)}{\geq} 2\ N_{\rm sq}(\widetilde{A};C|BB')_\sigma \\
&\overset{(v)}{\geq} 2\ N_{\rm sq}(\widetilde{A};C|\widetilde{B})_\tau .
\end{aligned} \tag{90}$$

The equality (i) is due to the choice of the extension ρ_{ABCE} of the state ρ_{ABC}, as mentioned above. The inequality (ii) is due to the data-processing inequality of QCMI. The inequality (iii) can be seen from the chain rule $I(\widetilde{A}A';C|BE)_\sigma = I(A';C|BE)_\sigma + I(\widetilde{A};C|BB'E)_\sigma$, where $B' \simeq A'$ and $I(A';C|BE)_\sigma \geq 0$. The inequalities (iv) and (v) are due to the definition of $N_{\rm sq}$ and monotonicity of $N_{\rm sq}$ under local trace-preserving operation (CPTP map) by Bob (see Section D.1). The monotonicity under $\mathbb{LOQC}_{\mathbf{C}\to\mathbf{B}}$ can be shown similarly.

E State transformations

E.1 Proof of the converse bound in Theorem 2.

Here, we present the maximum possible rate R_C at which one quantum state $(\rho_1)^{\otimes n}_{ABC}$ can be converted to the other $(\rho_2)^{\otimes m_n}_{ABC}$ using the free operations defined by our resource theory. The proof is similar to the converse of the entanglement manipulation theorem shown in Ref. [114]. Let us consider there exists a free operation $\Lambda_{\rm free}$ such that

$$\frac{1}{2}\left\|\Lambda_{\rm free}(\rho_1^{\otimes n}) - \rho_2^{\otimes m_n}\right\|_1 \leq \varepsilon. \tag{91}$$

Using our continuity relation (74), we have

$$\begin{aligned}
N_{\rm sq}&(A;C|B)_{\Lambda_{\rm free}(\rho_1^{\otimes n})} \geq N_{\rm sq}(A;C|B)_{\rho_2^{\otimes m_n}} - \frac{f(\varepsilon)}{2} \\
&= m_n N_{\rm sq}(A;C|B)_{\rho_2} - 2m_n\sqrt{\varepsilon}\log_2\min\{|A|,|C|\} \\
&\quad +(1{+}2\sqrt{\varepsilon})\ h_2\left(\frac{2\sqrt{\varepsilon}}{1{+}2\sqrt{\varepsilon}}\right).
\end{aligned} \tag{92}$$

The second equality is due to the additivity property of the measure $N_{\rm sq}$, and we expand $f(\varepsilon)$ in the continuity relation (74). Now, we can write down the following sets of equations:

$$\begin{aligned}
&nN_{\rm sq}(A;C|B)_{\rho_1} = N_{\rm sq}(A;C|B)_{\rho_1^{\otimes n}} \\
&\geq N_{\rm sq}(A;C|B)_{\Lambda_{\rm free}(\rho_1^{\otimes n})} \\
&\geq m_n N_{\rm sq}(A;C|B)_{\rho_2} - 2m_n\sqrt{\varepsilon}\log_2\min\{|A|,|C|\} \\
&\quad +(1{+}2\sqrt{\varepsilon})\ h_2\left(\frac{2\sqrt{\varepsilon}}{1{+}2\sqrt{\varepsilon}}\right).
\end{aligned} \tag{93}$$

The first equality is due to the additivity of the quantum non-Markovianity measure. The first inequality is due to the monotonicity of the quantum non-Markovianity measure under free operation. The second inequality is due to (92). Now dividing both sides by $nN_{\rm sq}(A;C|B)_{\rho_2}$ and rearranging the equations, we have

$$\begin{aligned}
&\frac{m_n}{n}\left(1 - \frac{2\sqrt{\varepsilon}\log_2\min\{|A|,|C|\}}{N_{\rm sq}(A;C|B)_{\rho_2}}\right) \\
&\leq \frac{N_{\rm sq}(A;C|B)_{\rho_1}}{N_{\rm sq}(A;C|B)_{\rho_2}} - \frac{(1{+}2\sqrt{\varepsilon})}{nN_{\rm sq}(A;C|B)_{\rho_2}}\ h_2\left(\frac{2\sqrt{\varepsilon}}{1{+}2\sqrt{\varepsilon}}\right).
\end{aligned} \tag{94}$$

Taking limit at $n \to \infty$ and $\varepsilon \to 0$, we have

$$R = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{m_n}{n} \leq \frac{N_{\rm sq}(A;C|B)_{\rho_1}}{N_{\rm sq}(A;C|B)_{\rho_2}} = R_C. \tag{95}$$

E.2 Proof of the achievability bound in Theorem 3

In this section, we establish a sufficient condition to convert a state $(\rho_1)_{ABC}$ to $(\rho_2)_{ABC}$ using free operations, i.e., local trace-preserving operations by all parties, $\mathbb{LOCC}_{\mathbf{A}\to\mathbf{C}}$, $\mathbb{LOCC}_{\mathbf{C}\to\mathbf{A}}$, and $\mathbb{QC}_{\mathbf{A}\to\mathbf{B}}$ or $\mathbb{QC}_{\mathbf{C}\to\mathbf{B}}$. Note that this protocol fails when $\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}<0$. The overall strategy follows three steps.

Firstly, Alice and Charlie want to distill as many shared ebits as possible via a state-merging protocol. The state distillation is achieved via a state-merging protocol, i.e., Alice wants to merge her system into Charlie's lab via $\mathbb{LOCC}_{\mathbf{A}\to\mathbf{C}}$. For a large n, this state-merging protocol on $(\rho_1)^{\otimes n}_{ABC}$ generates $\approx n.I(A\rangle C)_{\rho_1}$ shared ebits between Alice and Charlie (when $I(A\rangle C)_{\rho_1}\geq 0$). Considering the purification of ρ_1 to be $|\psi_1\rangle$ with the environment system R, the corresponding resource-inequality for state-merging follows (see Eq. 58 in [117])

$$\begin{aligned}
&(\psi_1)_{ABCR} + I(A:BR)_{\psi_1}[c\to c] \\
&\succeq (\psi_1)_{BC'CR} + I(A\rangle C)_{\rho_1}[qq].
\end{aligned} \tag{96}$$

Here, $A \simeq C'$, the above resource inequality indicates Alice's system A is now with Charlie which we denote it by C'. Note we can write $I(A\rangle C)_{\rho_1}$ and $I(A\rangle C)_{\psi_1}$ interchangeably. Alternatively, it is also possible for Charlie to merge his system into Alice's lab via $\mathbb{LOCC}_{\mathbf{C}\to\mathbf{A}}$ and establish $\approx n.I(C\rangle A)_{\rho_1}$ shared ebits between them (when $I(C\rangle A)_{\rho_1}\geq 0$).

$$\begin{aligned}&(\psi_1)_{ABCR}+I(C:BR)_{\psi_1}[c\to c]\\&\succeq(\psi_1)_{AA'BR}+I(C\rangle A)_{\rho_1}[qq].\end{aligned}\tag{97}$$

Here $A'\simeq C$. Depending on the amount of shared ebits, Alice and Charlie perform the state-merging in Eq. (96) or Eq. (97) to establish $\approx n.\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}$ shared ebits between them.

After the entanglement distillation, the state $\rho_1^{\otimes n}$ is discarded. Charlie now prepares the state $(\rho_2)^{\otimes m_n}_{C'C''C}$ with $C'\simeq A$ and $C''\simeq B$. Charlie transfers the system $(C'')^{m_n}$ to Bob via quantum communication $\mathbb{QC}_{\mathbf{C}\to\mathbf{B}}$, i.e. m_n copies of the map $\mathrm{id}_{C''\to B}$ is implemented. Now, Charlie aims to use the previously established shared ebits to perform a state-splitting protocol to transfer system C' to Alice's lab, with the map $\mathrm{id}_{C'\to A}$. Considering the purification of ρ_2 to be ψ_2 with purifying system R', and writing C' and A interchangeably in the entropic quantities, let us recall the relevant resource inequality for such state splitting [37]:

$$\begin{aligned}&(\psi_2)_{BC'CR'}+\frac{1}{2}I(A{:}BR')_{\psi_2}[q\to q]\\&\quad+\frac{1}{2}I(A;C)_{\psi_2}[qq]\succeq(\psi_2)_{ABCR'}.\end{aligned}\tag{98}$$

Although quantum communication from Charlie to Alice is not a free operation, Charlie can simulate some quantum channels with part of the shared ebits and classical communication via a teleportation protocol, i.e., he can perform the protocol by only using classical communication and consuming additional shared ebits. Replacing $[q\to q]$ with $[qq]+2[c\to c]$ (teleportation) in Eq: (98), we have

$$\begin{aligned}&(\psi_2)_{BC'CR'}+\frac{1}{2}\Big(I(A{:}BR)_{\psi_2}+I(A;C)_{\psi_2}\Big)[qq]\\&+I(A{:}BR)_{\psi_2}[c\to c]\succeq(\psi_2)_{ABCR}.\end{aligned}\tag{99}$$

Note $\frac{1}{2}\Big(I(A{:}BR)_{\psi_2}+I(A;C)_{\psi_2}\Big)=S(A)_{\psi_2}$, We can write $S(A)_{\psi_2}$ and $S(A)_{\rho_2}$ interchangeably. So to transfer m_n copies of the system A of ρ_2, Alice and Charlie need $\approx m_nS(A)_{\rho_2}$ ebits. The protocol is successful whenever previously distilled shared entanglement, $n.\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}$ is more than $m_nS(A)_{\rho_2}$, i.e.

$$n\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}\geq m_nS(A)_{\rho_2}\tag{100}$$

$$\Longrightarrow\frac{m_n}{n}\leq\frac{\max\Big(I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\Big)}{S(A)_{\rho_2}}.\tag{101}$$

Alternatively, instead of Charlie, Alice could produce the state $(\rho_2)^{\otimes m_n}_{AA''A'}$ with $A''\simeq B$ and $A'\simeq C$ (all the subsystems are with Alice), send $(A'')^{m_n}$ to Bob via $\mathbb{QC}_{\mathbf{A}\to\mathbf{B}}$ (implementing m_n copies of the map $\mathrm{id}_{A''\to B}$). She can then perform the state-splitting with the corresponding resource inequality:

$$\begin{aligned}&(\psi_2)_{AA'BR'}+\frac{1}{2}I(C{:}BR)_{\psi_2}[q\to q]\\&\quad+\frac{1}{2}I(A;C)_{\psi_2}[qq]\succeq(\psi_2)_{ABCR'}.\end{aligned}\tag{102}$$

Simulating the $[q\to q]$ channel via teleportation, we have the state-splitting protocol:

$$\begin{aligned}&(\psi_2)_{AA'BR'}+\frac{1}{2}\Big(I(C{:}BR)_{\psi_2}+I(A;C)_{\psi_2}\Big)[qq]\\&\quad+I(C{:}BR)_{\psi_2}[c\to c]\succeq(\psi_2)_{ABCR'}.\end{aligned}\tag{103}$$

Considering

$$\frac{1}{2}\Big(I(C{:}BR)_{\psi_2}+I(A;C)_{\psi_2}\Big)=S(C)_{\psi_2}=S(C)_{\rho_2},$$

In this case, the available shared ebits are $n.\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}$ need to exceed $m_n.S(C)_{\rho_2}$ to make the state-splitting in Eq. (103) successful:

$$n\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}\geq m_nH(C)_{\rho_2}\tag{104}$$

$$\Longrightarrow\frac{m_n}{n}\leq\frac{\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}}{S(C)_{\rho_2}}.\tag{105}$$

Combining Eq. (101) or Eq. (105), the maximum possible rate of inter-conversion is upper bounded by

$$\begin{aligned}R&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{m_n}{n}\\&\leq\frac{\max\left\{I(A\rangle C)_{\rho_1}, I(C\rangle A)_{\rho_1}\right\}}{\min\left\{S(A)_{\rho_2}, S(C)_{\rho_2}\right\}}=R_A.\end{aligned}\tag{106}$$

E.3 Proof of Corollary 1

Let us assume for ρ_{ABC}, E is the extension that gives the quantum non-Markovianity measure, $N_{\mathrm{sq}}(A;C|B)_\rho=I(A;C|BE)_\rho/2$. Also, we assume R is an extension such that ρ_{ABCER} is a pure state. Now we have

$$\begin{aligned}N_{\mathrm{sq}}(A;C|B)_\rho&=\frac{1}{2}I(A;C|BE)_\rho\\&=\frac{1}{2}(S(A|BE)_\rho-S(A|BCE)_\rho)\\&=\frac{1}{2}(S(A|BE)_\rho+S(A|R)_\rho)\leq S(A)_\rho.\end{aligned}\tag{107}$$

By symmetry, we also have $N_{\mathrm{sq}}(A;C|B)_\rho\leq S(C)_\rho$, hence

$$N_{\mathrm{sq}}(A;C|B)_\rho\leq\min\left\{S(A)_\rho, S(C)_\rho\right\}.\tag{108}$$

Now, we prove the lower bound. Consider $I(A\rangle C)_\rho = -S(A|C)_\rho = S(A|BER)_\rho \le S(A|BE)_\rho$, also we have $I(A\rangle C)_\rho = S(A|BER)_\rho \le S(A|R)_\rho$. The last inequality is because conditioning reduces entropy. Adding the inequalities and dividing by 2, we have

$$\begin{aligned} N_{\mathrm{sq}}(A;C|B)_\rho &= \frac{1}{2}(S(A|BE)_\rho + S(A|R)_\rho) \\ &\ge S(A|BER)_\rho = I(A\rangle C)_\rho. \end{aligned} \quad (109)$$

Similarly, we can show $N_{\mathrm{sq}}(A;C|B)_\rho \ge I(C\rangle A)_\rho$, hence we have

$$N_{\mathrm{sq}}(A;C|B)_\rho \ge \max\left\{I(A\rangle C)_\rho, I(C\rangle A)_\rho\right\}. \quad (110)$$

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