Measurement events relative to temporal quantum reference frames

Ladina Hausmann[1], Alexander Schmidhuber[1], and Esteban Castro-Ruiz[1,2,3]

[1]Institute for Theoretical Physics, ETH Zurich, Switzerland
[2]Université Paris-Saclay, Inria, CNRS, LMF, 91190 Gif-sur-Yvette, France
[3]Institute for Quantum Optics and Quantum Information (IQOQI) Vienna, Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria

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The Page-Wootters formalism is a proposal for reconciling the background-dependent, quantum-mechanical notion of time with the background independence of general relativity. However, the physical meaning of this framework remains debated. In this work, we compare two consistent approaches to the Page-Wootters formalism to clarify the operational meaning of evolution and measurements with respect to a temporal quantum reference frame. The so-called "twirled observable" approach implements measurements as operators that are invariant with respect to the Hamiltonian constraint. The "purified measurement" approach instead models measurements dynamically by modifying the constraint itself. While both approaches agree in the limit of ideal clocks, a natural generalization of the purified measurement approach to the case of non-ideal, finite-resource clocks yields a radically different picture. We discuss the physical origin of this discrepancy and argue that these approaches describe operationally distinct situations. Moreover, we show that, for non-ideal clocks, the purified measurement approach yields a time non-local evolution equation, which can lead to non-unitary evolution. Moreover, it implies a fundamental limitation to the operational definition of the temporal order of events. Nevertheless, unitarity and definite temporal order can be restored if we assume that time is discrete.

1 Introduction

In standard quantum mechanics, systems evolve via the Schrödinger equation relative to an external background time. In general relativity, on the other hand, the metric determines the reading of rods and clocks, irrespective of any external background. The contrast between these two notions of time calls for a conceptual framework in which quantum systems evolve in the absence of an external time parameter.

The Page-Wootters formalism proposes a solution to this problem in non-relativistic quantum mechanics. Motivated by the Wheeler-De-Witt equation [1],

$$\hat{H}\,|\Psi\rangle = 0, \tag{1}$$

the Page-Wotters formalism [2, 3] poses that time evolution emerges from the correlations between the system of interest and a temporal reference frame – a clock. The total system, composed of the system of interest and the frame, satisfies the so-called Hamiltonian constraint, eq. (1). Because $|\Psi\rangle$ does not depend on any external background time, the approach based on eq. (1) is referred to as "timeless" quantum mechanics. For a review of the Page-Wootters formalism, see appendix A.

Although the Page-Wootters formalism reproduces time evolution in a timeless setting, the formalism has been criticized [4, 5]. In particular, Kuchař noted that the post-measurement states of the formalism do not satisfy eq. (1) and predict inconsistent probabilities for two-time measurements [4]. A more detailed review of Kuchař's criticisms is given in appendix B.

In the current literature, there are two different ways to resolve Kuchař's criticisms. The first route, which we call the twirled observable (TO) approach, has been taken by [6] (see also [7]). It is

based on time-translation invariant operators, which are inspired by Dirac quantization [8, 9] and time-reparametrization invariant observables in quantum gravity [10–16]. The second approach, which we call the purified measurement (PM) approach, was pursued by [17]. In this approach, the idea is to modify the constraint to describe the measurement dynamically (the same idea was also pursued in [18–20]). More specifically, the modified constraint couples an ancillary system to the clock and the system to be measured, and a measurement on the ancilla reveals the outcome of the measurement on the system.

Both approaches produce a theory that addresses Kuchař 's criticisms. This leads to the question: are these two proposals physically equivalent? In this work, we compare both approaches and shed light on the operational meaning of each.

Our analysis is motivated by recent literature on quantum reference frames transformations [21–28]. In these works, an important goal is to understand, intuitively speaking, how physics "looks like" as described from the perspective of a quantum reference frame. We explore this question in the case of temporal reference frames, and in particular, in the case where the clocks have finite resources, that is, when clocks are not ideal.

After presenting the clock models used in this work (section 2), we review the twirled observable and purified measurement approaches in section 3. We show that in physical scenarios where the clocks can be treated as ideal, both models give the same predictions, and are equivalent to ordinary quantum mechanics with an external background time.

Usually, clocks have a large amount of resources at their disposal, so the ideal clock approximation holds in most practical situations. However, there are fundamental limitations to the ideal clock approximation, most notably from the perspective of gravity. Specifically, ideal clocks require an infinite energy spread, creating a large back reaction on spacetime. This leads not only to the well-known, fundamental operational limitations on spacetime measurements at the Plank scale [29], but also to limitations stemming from gravitationally-induced entanglement between clocks [30]. Motivated by these considerations, we construct a natural generalization of measurements in the PM approach for non-ideal clocks in 4. We obtain a time-non-local equation for the evolution of the system with respect to the clock, as reported in [31]. We compare our model with its TO counterpart [6, 32], and show that the TO and PM approaches lead to different physical predictions.

We argue that this difference arises because each approach has a different notion of measurement event[1], and therefore describes a different experimental situation. Moreover, because ideal clocks are nothing but a limiting case of non-ideal clocks, we argue that the TO and PM approaches should be considered operationally different even in the ideal-clock case.

In the TO approach, measurements and state-update proceed basically as in ordinary quantum mechanics with external time evolution. Importantly, time evolution with respect to the non-ideal clock is unitary, and the temporal order between events can be defined sharply. The extension of the PM model to non-ideal clocks leads to a radically different picture. Here, the clock can no longer resolve single time instances, leading to important measurable consequences. In particular, the time evolution of the system "as seen" by the clock is no longer unitary, and the probability rule is modified. Moreover, we show that measurement events, understood as an interaction between the system of interest and an ancillary device, can occur in an indefinite temporal order, pointing to a fundamental limitation to the notion of the temporal order of events, with potential implications to the notion of indefinite causal structure [33–35].

Finally, in 5, we construct a discrete-time version of the PM model and show that unitarity is recovered for general measurement interactions. If unitary evolution is to be held at all costs, our analysis can be seen as an argument in favour of the idea that time is fundamentally discrete.

2 Ideal and non-ideal clocks

Before analysing the TO and PM approaches, we summarize the clock models we use in this paper. Our summary is based on Ref. [6].

[1]Throughout the paper, we use the term measurement event, or simply event, to refer to the notion of measuring a system at a certain time.

In this work, a clock is a quantum system C with Hilbert space $\mathcal{H}_C$[2] and a Hamiltonian

$$\hat{H}_C = \int_\sigma \mathrm{d}e\, e\, |e\rangle\langle e|_C\,, \tag{2}$$

where $\sigma = [-E, E]$ is the spectrum of the clock, for some energy E[3]. The measurement of time corresponds to a covariant positive operator-valued measure (POVM). In the case of time, a covariant POVM consists of a set of positive operators $\hat{\phi}_t$, for $t \in \mathbb{R}$, satisfying $e^{-it\hat{H}_C}\hat{\phi}_{t'}e^{it\hat{H}_C} = \hat{\phi}_{t+t'}$ and forming a resolution of the identity. In this work, we take $\hat{\phi}_t = |\phi_t\rangle\langle\phi_t|_C$, with

$$|\phi_t\rangle_C = \frac{1}{\sqrt{2E}}\int_{-E}^{E} e^{-iet}\,|e\rangle_C\,. \tag{3}$$

The states $|\phi_t\rangle_C$ transform as $e^{-it'\hat{H}_C}\,|\phi_t\rangle_C = |\phi_{t'+t}\rangle_C$, and resolve the identity as

$$N_C \int dt\, |\phi_t\rangle\langle\phi_t|_C = \mathbb{1}_C, \tag{4}$$

for a normalization constant $N_C = E/\pi$.

The overlap between two clock states at different times $\langle\phi_t|\phi_{t'}\rangle_C$ is

$$N_C\,\langle\phi_t|\phi_{t'}\rangle_C = \frac{\sin(E(t-t'))}{\pi(t-t')}. \tag{5}$$

It is easy to see from eq. (5) that the distinguishability of clock states for two different times, $|\phi_t\rangle_C$ and $|\phi_{t'}\rangle_C$, increases as the energy E increases. In this sense, energy is a resource for temporal quantum reference frames. A clock with finite E is then an instance of a quantum reference frame with finite resources [36]. In the limit $E \to \infty$, the spectrum of the clock becomes $\sigma_C = \mathbb{R}$. In this special case, we denote the clock states by $|t\rangle_C$. In contrast to the case of finite E, the states $|t\rangle_C$ can be perfectly distinguished, and we can write their inner product as

$$\langle t|t'\rangle_C = \delta(t-t'). \tag{6}$$

In this work, we refer to a clock defined by the states $|t\rangle$ as *perfect* or *ideal*. For ideal clocks, the Hamiltonian in eq. (2) has a Hermitian conjugate *time operator* $\hat{T}_C$, satisfying $[\hat{T}_C, \hat{H}_C] = i$, and $\hat{T}_C\,|t\rangle_C = t\,|t\rangle_C$. Naturally, ideal clocks do not exist. Whenever we say a clock is ideal, what we really mean is that we are considering an experiment in a regime where the approximation of eq. (6) is physically meaningful. On the other hand, we refer to the (more realistic) clocks corresponding to states $|\phi_t\rangle_C$ as *imperfect* or *non-ideal*.

3 Review of the twirled observable and purified measurement approaches

We briefly summarize the TO and PM approaches for the case of ideal clocks, as defined in section 2. We show that, in this idealized setting, both approaches lead to the same physical predictions.

3.1 Twirled observable approach

Let us start with the TO approach, introduced in [6]. For a more detailed account, see appendix C. The key point of this approach is to substitute the standard Schrödinger equation with a constraint

$$(\hat{H}_C + \hat{H}_S)\,|\Psi\rangle = 0. \tag{7}$$

[2]Throughout the paper we assume that the Hilbert space of the clock is given by $L^2[E, -E]$.

[3]In the main text, we use a symmetric spectrum as the overlap $\langle\phi_t|\phi_{t'}\rangle_C$ is the widely-known sinc function, thus making formulas where $\langle\phi_t|\phi_{t'}\rangle_C$ appears more intuitive. The derivations of the results of sections section 4 are valid for a general spectrum, except the explicit solution calculated in eq. (18), which can easily be generalized to general spectra. In section 5, we use the symmetric spectrum for constructing a clock with a bounded spectrum whose time states are orthogonal at the desired values.

The dynamics of the system S, with Hamiltonian $\hat{H}_S$, is described by its correlations with a clock C. The Hamiltonian of the clock, $\hat{H}_C$, is ideal in the sense of section 2.

The constraint recovers the Schrödinger equation by contracting eq. (7) with $\langle t|$. Solving the resulting equation for the initial condition $|\psi_0\rangle_S$ at $t = 0$, we find the solution to eq. (7):

$$|\Psi\rangle = \int_{\mathbb{R}} \mathrm{d}t \; |t\rangle_C \otimes e^{-it\hat{H}_S} \, |\psi_0\rangle_S \, . \tag{8}$$

Note that the evolution of S with respect to C is unitary. By this, we mean that, for any pair of times t_1 and t_2, the state of S at t_1 is transformed into the state of S at t_2, by a unitary operator. That is, $|\psi(t_2)\rangle_S = U(t_2, t_1) \, |\psi(t_1)\rangle_S$. Moreover, we require that the operators $U(t, t')$ satisfy the so-called group property: $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$. Measurable physical quantities correspond to operators that commute with the constraint $\hat{H}_S + \hat{H}_C$, as they should leave the state in the space of states respecting the constraint. For any operator $\hat{F}_S$ on S and any time τ we define the relational Dirac operator

$$\hat{F}_{CS}(\tau) = \int_{\mathbb{R}} dt \;\; e^{-it\hat{H}_C} \;\; |\tau\rangle\langle\tau|_C \;\; e^{it\hat{H}_C} \otimes e^{-it\hat{H}_S} \;\; \hat{F}_S \, e^{it\hat{H}_S} . \tag{9}$$

Note that $\hat{F}_{CS}(\tau)$ relies on the well-known $\mathcal{G}$-twirl procedure (see, e.g. [37]) for constructing invariant observables – hence the name "twirled observable (TO)" approach. By construction, $\hat{F}_{CS}(\tau)$ commutes with the constraint, so the state $\hat{F}_{CS}(\tau) \, |\Psi\rangle$ satisfies eq. (7) as well.

We can then define probabilities for consecutive measurements analogously to ordinary quantum mechanics. For example, let $\hat{\Pi}^k_S$ and $\hat{\Pi}^q_S$ be projectors corresponding to measurement outcomes k and q, respectively, of two observables $\hat{K}_S$ and $\hat{Q}_S$. Assume that we have obtained the outcomes k and q at times τ_1 and τ_2, respectively. Then, in this approach, the operators $|\tau_1\rangle\langle\tau_1|_C \otimes \hat{\Pi}^k_S$ and $|\tau_2\rangle\langle\tau_2|_S \otimes \hat{\Pi}^q_S$ cannot be directly applied to the state as they are not relational, and would lead to the problems Kuchař pointed out. Instead, we should use the corresponding relational Dirac operators from eq. (9), $\hat{\Pi}^k_{CS}(\tau_1)$ and $\hat{\Pi}^q_{CS}(\tau_2)$, labelled by times τ_1 and τ_2. The joint probability of measuring p followed by q, at times τ_1 and τ_2 respectively, is given by

$$P(k, q|\tau_1\tau_2) = \langle t = 0|_C \, \langle\psi_0|_S \, \hat{\Pi}^k_{CS}(\tau_1)\hat{\Pi}^q_{CS}(\tau_2)\hat{\Pi}^k_{CS}(\tau_1) \, |\Psi\rangle \, . \tag{10}$$

A straightforward calculation yields

$$P(k, q|\tau_1\tau_2) = \left\| \hat{\Pi}^q_S \exp(-i(\tau_2 - \tau_1)\hat{H}_S)\hat{\Pi}^k_S \exp(-i\tau_1\hat{H}_S) \, |\psi_0\rangle \right\|^2 . \tag{11}$$

Therefore, if $\tau_2 > \tau_1$, eq. (10) gives the standard quantum mechanical probabilities for measuring k at time τ_1 and then q at time τ_2.

3.2 Purified measurement approach

Instead of constructing covariant observables, the PM approach introduced in [17] describes measurements dynamically, at the level of the constraint. In this sense, we say that the measurement is "purified" – hence the name "purified measurement (PM)" approach. Specifically, the measurement is modelled by adding to the constraint an interaction term between the clock, the system and an ancilla:

$$\hat{H} \, |\Psi\rangle_{CSA} = \left(\hat{H}_C + \hat{H}_S + \int dt \, |t\rangle\langle t|_C \otimes \hat{K}_{SA}(t) \right) |\Psi\rangle_{CSA} = 0. \tag{12}$$

The ancillary system A[4] can be thought of as the pointer of a measurement device that records the measurement outcome. $\hat{H}_C$ is the Hamiltonian of a perfect clock.

A standard choice for the measurement interaction is $\hat{K}_{SA}(t) = \delta(t - \tau)\hat{H}_{SA}$, which models a measurement interaction happening sharply at time $t = \tau$. If we choose $\hat{H}_{SA}$ such that

[4]The Hilbert space of the system A is the span of all possible measurement outcomes $|r\rangle_A$ that are all mutually orthogonal.

$e^{-i\hat{H}_{SA}} |\psi(\tau)\rangle_S \otimes |r\rangle_A \mapsto \sum_a \hat{K}_S^a |\psi(\tau)\rangle_S \otimes |a\rangle_A$, then this interaction is a purification of the POVM $\{K_S^a\}_a$. We assume this interaction for the rest of this section. The states satisfying the constraint $\hat{H}|\Psi\rangle = 0$ and boundary conditions such that $\langle t|_C |\Psi\rangle_{CSA} = |\psi(t)\rangle_S \otimes |r\rangle_A$, for $t \leq \tau$, have the form

$$\begin{aligned} |\Psi\rangle_{CSA} &= \int_{-\infty}^{\tau} dt\, |t\rangle_C \otimes |\psi(t)\rangle_S \otimes |r\rangle_A \\ &\quad + \int_{\tau}^{\infty} dt\, |t\rangle_C \otimes \hat{U}_S(t-\tau) e^{-i\hat{H}_{SA}} |\psi(\tau)\rangle_S \otimes |r\rangle_A \\ &= \int_{-\infty}^{\tau} dt\, |t\rangle_C \otimes |\psi(t)\rangle_S \otimes |r\rangle_A \\ &\quad + \int_{\tau}^{\infty} dt\, |t\rangle_C \otimes \hat{U}_S(t-\tau) \sum_a \hat{K}_S^a |\psi(\tau)\rangle_S \otimes |a\rangle_A , \end{aligned} \tag{13}$$

which can be interpreted as follows. The system first evolves freely according to $\hat{H}_S$. The ancilla is assumed to start in a "ready" state $|r\rangle_A$ and evolve trivially until the measurement interaction, which occurs when the clock shows time τ, resulting in the application of the unitary $\exp(-i\hat{H}_{SA})$. Afterwards, the system evolves freely again. By adding more measurement interactions with additional memories to the Hamiltonian we can model multi-time measurements.

The probability that the measurement outcome a has been written into the memory of the measurement apparatus at time t is given by

$$P(a|\tau)_t = \frac{\langle\Psi| \, |t\rangle\langle t|_C \otimes \mathbb{1}_S \otimes |a\rangle\langle a|_A \, |\Psi\rangle}{\langle\Psi| \, |t\rangle\langle t|_C \otimes \mathbb{1}_S \otimes \mathbb{1}_A \, |\Psi\rangle}. \tag{14}$$

For ideal clocks, the denominator of eq. (14) is independent of t. As we will see in Section 4, this is no longer the case when the perfect-clock approximation is not valid. (For alternative probability rules applied to Wigner's Friend scenarios within the Page and Wootters formalism, see [38, 39].) The rule of eq. (14) can be naturally extended to multiple-time measurements by adding one ancillary system for each measurement [17]. If we construct a purification for the observables K_S and Q_S, set $t > \tau_2 > \tau_1$, it is easy to see that eq. (14) leads to eq. (11), showing that both frameworks give the same predictions for ideal clocks. A constructive proof of this result, showing how the two approaches are formally related, can be found in appendix D.

Note that the time t is not the time of the modelled measurement interaction, but the time at which the register is measured. This only needs to happen once, after all of the measurement interactions have occurred.

From the joint probability distribution for all outcomes, we can construct the conditional probabilities $P((a|\tau)|(b|\tau'))_t$, which is the probability of measuring a at time τ, given that we measured b at time τ'. If we omit the subscript t in $P((a|\tau)|(b|\tau'))_t$, then mean that we evaluate $P((a|\tau)|(b|\tau'))_t$ at a time $t \gg \tau, \tau'$. Both the TO and the PM approaches yield the same conditional probabilities in the case of perfect clocks. More precisely, consider two subsequent measurements on a system and let $P_I((b|\tau')|(a|\tau))$ denote the probability of observing the measurement result $|b\rangle$ at time τ' conditioned on measuring outcome $|a\rangle$ at time τ. Here, $I \in \{\text{PM} \, , \text{TO} \}$ is an index determining whether the probability was computed in the PM or TO approach. Then, as we show in appendix D, $P_{PM}((b|\tau')|(a|\tau)) = P_{TO}((b|\tau')|(a|\tau))$. However, as our analysis of the non-ideal case below will show, that the approaches are operationally different.

4 Non-ideal clocks

In this section, we study the case where the idealization of a perfect clock is no longer valid. How do we describe the dynamics of a system with respect to a finite-resource temporal reference frame? To answer this question, we focus on the clock model defined by eq. (2) in the case where the energy E is finite. We show that the answer depends on whether we use the TO or the PM approach, as in this finite case the two approaches give different physical predictions. The non-ideal TO approach was studied in [6]; the non-ideal PM approach is developed in section 4.2.

4.1 Twirled observable approach for non-ideal clocks

The central aspects of the TO approach do not depend on whether the clocks are ideal or not. In particular, the constraint is still given by eq. (7), but with the bounded-spectrum Hamiltonian $\hat{H}_C$ replacing the ideal one. Likewise, covariant extensions of operators are defined by eq. (9), and the probability rule remains the one of eq. (10) replacing the ideal clock with a non-ideal one. This replacement results in a restriction of the system's energy values to $\sigma(\hat{H}_S) \cap (-\sigma)$, where $\hat{H}_S$ is the Hamiltonian of the system appearing in the constraint. Consequently, also the evolution of the system with respect to the clock is generated by a Hamiltonian which has a spectrum given by $\sigma(\hat{H}_S) \cap (-\sigma)$. This restriction is derived in detail in Section IV A of [6]. Broadly, it is an example of the system being constrained to the physical system Hilbert space as a result of a non-ideal quantum reference frame in the perspective-neutral approach [32]. As in the ideal case, the evolution of the system with respect to the clock is unitary.

4.2 Purified measurement approach for non-ideal clocks

In contrast to the TO approach, the PM approach is fundamentally different when clocks are non-ideal. Even with a natural extension of eq. (12), as the one presented below, the evolution equation changes substantially, and as a result the evolution can become non-unitary. The probability rule remains formally the same as eq. (14) (changing $|t\rangle_C \to |\phi_t\rangle_C$), with the important difference that the denominator is now not always equal to 1.

4.2.1 The Hamiltonian constraint

A natural generalization of eq. (12), for the case of non-ideal clocks reads

$$\hat{H}\,|\Phi\rangle_{CSA} = \left(\hat{H}_C + \hat{H}_S + N_C \int dt\, |\phi_t\rangle\langle\phi_t|_C \otimes \hat{K}_{SA}(t)\right) |\Phi\rangle_{CSA} = 0, \tag{15}$$

where $N_C\,|\phi_t\rangle\langle\phi_t|_C$ is an element of the clock's covariant POVM described in section 2, which corresponds to the measurement of time. We can then interpret the last term of eq. (15) as a simplified description that an external observer would give of an internal observer looking at her clock and triggering a measurement interaction $\hat{K}_{SA}(t)$, which depends on the time t that she observes.

4.2.2 Evolution equation

As in the case of perfect clocks, we can also consider the state $|\Psi\rangle_{CSA}$ conditioned on the clock showing the time t, $|\psi(t)\rangle_{SA} = \langle\phi_t|_C\,|\Psi\rangle_{CSA}$. Similar to the ideal case, we show in appendix E that $|\Psi\rangle_{CSA} = N_C \int \mathrm{d}t\, |\phi_t\rangle_C \otimes |\psi(t)\rangle_{SA}$ fulfils the constraint of eq. (15) if and only if the conditioned state $|\psi(t)\rangle_{SA} = \langle\phi_t|_C\,|\Psi\rangle_{CSA}$ satisfies the equation

$$i\frac{d}{dt}\,|\psi(t)\rangle_{SA} = \hat{H}_S\,|\psi(t)\rangle_{SA} + N_C \int dt'\, \langle\phi_t|\phi_{t'}\rangle_C\, \hat{K}_{SA}(t')\,|\psi(t')\rangle_{SA}\,. \tag{16}$$

If the clock is non-ideal, then eq. (16) is non-local in time. By this, we mean that $\frac{d}{dt}\,|\psi(t)\rangle_{SA}$ does not depend only on the state of SA at time t, but also on the state evaluated at different times.[5]

We now turn to the solution of eq. (16) and show that, for non-ideal clocks, the TO and PM approaches are physically inequivalent. For simplicity, we consider the case where $H_S = 0$, as this case already captures the conceptual issues relevant to this work. For the general case, we can simply substitute $\hat{K}_{SA}(t)$ with $\hat{K}_{SA}(t) + \hat{H}_S$.

[5]Note that a time non-local *equation* does not necessarily mean a time non-local or a non-unitary *evolution*. Indeed, as shown in [31], in some cases a time non-local equation can be recast in the form of an ordinary Schrödinger equation, which is local in time and generates unitary time evolution in the sense of section 3, but features a different Hamiltonian $\hat{h}(t)$. However, to find the Hamiltonian $\hat{h}(t)$, in general, one needs to solve the time non-local equation first.

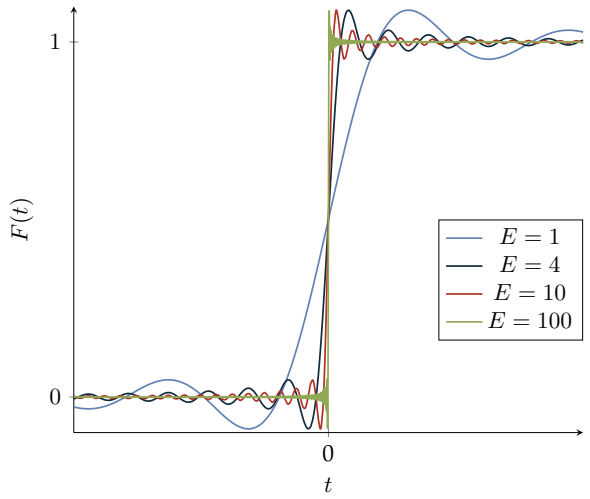


Figure 1: The function $F(t)$ for a clock with spectrum $\sigma_C = [-E, E]$, for different values of E. $F(t)$ parametrizes the solutions of eq. (16) in terms of the spectrum of the clock and characterizes the time-non-locality of the evolution equation relative to C.

We solve eq. (16) perturbatively in appendix E for the case of periodic and non-periodic clocks, using similar methods to those of [31]. We find

$$|\psi(t)\rangle = \sum_{N=0}^{\infty} (-i)^N \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dt_N \dots dt_1 F(t - t_N) \dots F(t_2 - t_1) \hat{K}_{SA}(t_N) \dots \hat{K}_{SA}(t_1) |\psi_0\rangle \,, \tag{17}$$

where $F(t) = N_C \int_{-\infty}^{t} dt' \, \langle \phi_{t'} | \phi_0 \rangle$. In general, the evolution corresponding to eq. (17) is non-unitary. Consider the case $\hat{K}_{SA}(t) = \hat{K}_{SA} \delta(t - \tau)$. Then, we have

$$|\psi(t)\rangle = |\psi_0\rangle + 2F(t - \tau) \sum_{N=1}^{\infty} (-i)^N \frac{1}{2^N} \hat{K}_{SA}^N |\psi_0\rangle \tag{18}$$

$$= \left(1 - F(t - \tau) \frac{i\hat{K}_{SA}}{1 + \frac{i}{2}\hat{K}_{SA}} \right) |\psi_0\rangle \,. \tag{19}$$

The evolution $1 - F(t-\tau)\frac{i\hat{K}_{SA}}{1+\frac{i}{2}\hat{K}_{SA}}$ is only unitary for $F(t-\tau) = 0, 1$, which is the case for $t \gg \tau$ or $t \ll \tau$. In other words, the evolution is unitary far away from the measurement. This already shows that the TO and PM approaches are physically inequivalent, as evolution in the TO approach is always unitary.

Moreover, $F(t)$ can be interpreted as a measure of the time-non-locality of the evolution equation (see eq. (16)) and its solution, eq. (17). Taking the $E \longrightarrow \infty$ in eq. (17) enforces $F(t) = \Theta(t)$, where $\Theta(t)$ denotes the Heaviside theta function, and yields unitary evolution:

$$|\psi(t)\rangle = \overleftarrow{T} e^{-i \int_0^t \mathrm{d}s K_{SA}(s)} |\psi_0\rangle \,, \tag{20}$$

where $\overleftarrow{T}$ denotes the time-ordering operator. After taking the limit $E \longrightarrow \infty$, it is valid to take the limit of $\hat{K}_{SA}(t)$ being an instantaneous interaction, i.e. $\hat{K}_{SA}(t) = \delta(t - \tau)\hat{H}_{SA}$.[6] In this limit, we obtain the solution of section 3.2, which gives the same predictions of the TO approach. However, away from this limit, $F(t)$ induces measurement events that are "spread out" in time. This is another aspect in which the PM approach departs from the TO approach in the non-ideal case, as measurement events are always "sharp" in the TO approach.

[6] Importantly, the limit $E \longrightarrow \infty$ and $\hat{K}_{SA}(t) \to \delta(t - \tau)\hat{H}_{SA}$ should be taken precisely in that order, as a clock with finite resources cannot control an operation at a sharp time τ.

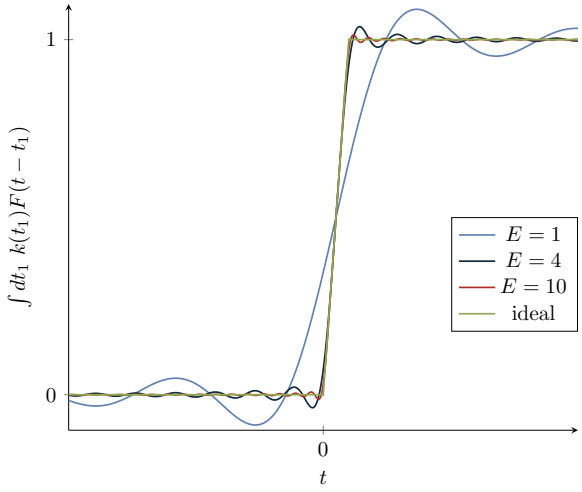


Figure 2: Magnitude of the order term of the time evolution $\int dt_1 \hat{K}_{SA}(t_1)F(t-t_1)$ for $\hat{K}_{SA}(t) = k(t)\hat{K}$, for clocks with spectrum $\sigma_C = [-E, E]$, with $k(t) = \chi_{[0,1]}(t)$, where χ is the characteristic function. As expected, we see that for a broader spectrum of the clock, the non-ideal solutions approximate the ideal case better.

In fig. 1, we plot $F(t)$ for different values of E and illustrate how it approaches Heaviside's function. As E increases, the spectrum broadens and the clock becomes more resourceful. As a consequence, time-non-local effects in the evolution equation are suppressed. Note, however, that E cannot fundamentally increase indefinitely, due to the back-action of the clock on spacetime. While the problem is not apparent in this work, where gravity is not modelled, it is known that gravitational effects will eventually become relevant [29, 30]. This is one of our motivations for considering non-ideal clocks. Therefore, the limit $E \longrightarrow \infty$ should be interpreted only as an approximation. Roughly, this approximation is suitable when the minimum time δt such that $\langle\phi_0|\phi_{\delta t}\rangle \approx 0$ is much smaller that the minimum time Δt such that $\langle\psi(t)|\psi(t+\Delta t)\rangle_{SA} \approx 0$, if we assume that $|\psi(t)\rangle_{SA}$ evolves with the time-dependent Hamiltonian $\hat{K}_{AS}(t) + H_S$.

4.2.3 Indefinite temporal order of events

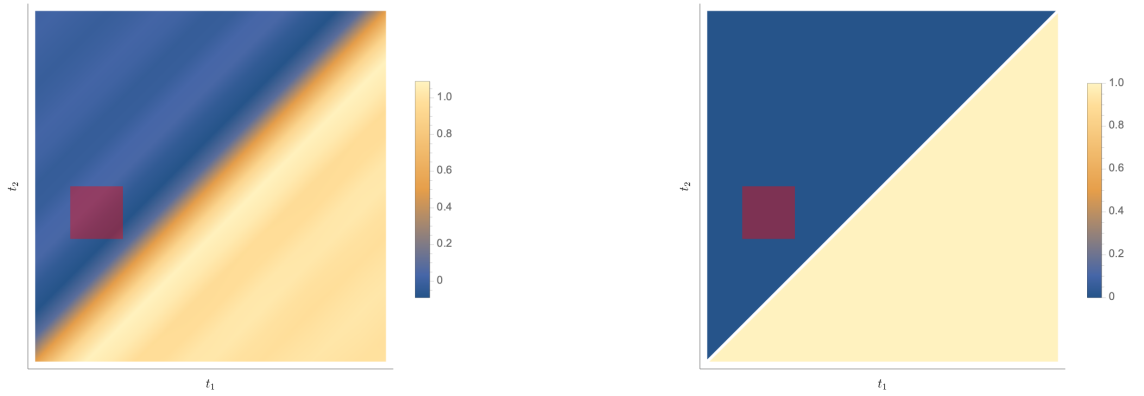
The bounds on the energy of our clock have important consequences on the time ordering of events, understood as the measurement interactions triggered by the clock. To see this, consider two measurement interactions, modelled by $\hat{K}_{SA}(t) = k(t-\tau_1)\hat{K}_1 + k(t-\tau_2)\hat{K}_2$ in eq. (15). For the following discussion, we assume that $\tau_1 < \tau_2$. We now look at the state of the system "as seen" by the clock in this scenario. To second order in the expansion of eq. (17), we have

$$\begin{aligned}|\psi(t)\rangle = |\psi_0\rangle + \Big[&\int dt_1 F(t-t_1)\Big(k(t_1-\tau_1)\hat{K}_1 + k(t_1-\tau_2)\hat{K}_2\Big) \\ &+ \int dt_1 dt_2 F(t-t_1)F(t_1-t_2)\big(k(t_1-\tau_1)\hat{K}_1 \\ &+ k(t_1-\tau_2)\hat{K}_2\big)\Big(k(t_2-\tau_1)\hat{K}_1 + k(t_2-\tau_2)\hat{K}_2\Big)\Big]\,|\psi_0\rangle\end{aligned} \tag{21}$$

In fig. 2 we plot the magnitude of the first order term for one of the measurements in the case where $k(t) = \chi_{[0,1]}(t)$. To second order, this expansion contains terms where $\hat{K}_1$ is applied before $\hat{K}_2$, that is, terms of the form

$$\int dt_1 dt_2 F(t-t_1)F(t_1-t_2)k(t_1-\tau_1)k(t_2-\tau_2)\hat{K}_1\hat{K}_2. \tag{22}$$

This on its own is not yet a feature stemming only from the imperfection of the clocks, as even for perfect clocks these terms appear if the support of $k(t-\tau_1)$ and $k(t-\tau_2)$ overlap. However, if we choose $k(t-\tau_1)$ and $k(t-\tau_2)$ such that their supports do not overlap, then in the ideal case this term should not contribute. An illustration of this case is shown in fig. 3 for perfect and imperfect



(a) Non-ideal clocks

(b) Ideal clocks

Figure 3: The heat map shows the expression $F(t-t_1)F(t_1-t_2)$ occurring in eq. (21) in the case of non-ideal (left) and ideal (right) clocks. The red square illustrates a region where the contribution of the term $k(\tau_1-t_1)\hat{K}_1 k(\tau_2-t_2)\hat{K}_2$ of eq. (21), with $k(t)=\chi_{[0,1]}(t)$, is not zero. Here, the two times τ_1 and τ_2 are far apart enough to ensure that the support of $k(t-\tau_1)$ is disjoint from the support of $k(t-\tau_2)$. As these supports do not overlap in time, a non-zero contribution of $k(\tau_1-t_1)\hat{K}_1 k(\tau_2-t_2)\hat{K}_2$ means an indefinite temporal order of events. The integration over t_1, t_2 of $F(t-t_1)F(t_1-t_2)k(\tau_1-t_1)k(\tau_2-t_2)\hat{K}_1\hat{K}_2$ gives the contribution of this term. In the ideal clock case, we can see that, because there is no overlap in time of the two measurements, this term does not contribute. In the non-ideal case, the situation is different. Because the clock states are not orthogonal, even if there is no overlap of the two measurements in time, the contribution $k(\tau_1-t_1)\hat{K}_1 k(\tau_2-t_2)\hat{K}_2$ to eq. (21) is not zero.

clocks. As $F(t_1-t_2)$ approximates but is not equal, to Heaviside's theta function $\Theta(t_1-t_2)$, the term in eq. (22) is not zero even if $t_1-t_2<0$.

We conclude that, if the clocks are no longer perfect, and events are defined according to the PM approach, then there is a limitation to the notion of time ordering of events, stemming from the time-non-locality of the evolution equation. This effect is another example of the inequivalence between the TO and PM approaches.

As discussed above, the PM definition of event corresponds to the description that an external observer would give of an internal experimenter looking at a clock and, conditioned on measuring a certain time, performing a measurement on a quantum system. From this perspective, these considerations might be relevant to the study of indefinite causal structure in the sense of process matrices and related operational frameworks [33–35]. The causal inequalities introduced in the field of indefinite causal orders would provide an interesting test for how strong the limitations on time-ordering of events are that stem from a non-ideal clock. We currently do not know if this evolution equation would lead to a violation of such inequalities.

4.2.4 Comparison with the non-ideal twirled observable approach

It is clear from the analysis above that, away from the limit of ideal clocks, the TO and PM approaches are physically different. We now propose an operational interpretation of each, accounting for this discrepancy.

In the TO approach, we interpret events in the following way: there is an external experimenter who can perform measurements relative to the clock on the system. The joint state of the clock and the system is "frozen". Moreover, the order of the measurements of the experimenter is not bound by the clock, that is, they can apply the measurements in any order they like. This is not only a feature of TO but also in ordinary quantum mechanics where it is possible to measure an operator $\hat{A}_S(t)$ at any time t' by evolving the corresponding operator to compensate for the time difference. The state of the system is given by eq. (8) and the operations are given by eq. (9) (suitably modifying each of them to the case of non-ideal clocks). Interestingly, for an imperfect yet sufficiently "good" clock (see [6] for the exact conditions), this description is mathematically equivalent, by means of conditioning on the clock to be on a state $|\phi_t\rangle_C$, to the standard quantum

mechanical description of a system evolving in an external background time. Specifically,

$$\langle\phi_t|_C \hat{F}_{CS}(\tau_2)\hat{G}_{CS}(\tau_1)\,|\Psi\rangle_{CS} = e^{-i\hat{H}_S(t-\tau_2)}\hat{F}_S e^{-i\hat{H}_S(\tau_2-\tau_1)}\hat{G}_S e^{-i\hat{H}_S\tau_1}\,|\psi_0\rangle_S\,. \tag{23}$$

However, the probe used to perform such measurements and the order of events in this approach are given *externally* (for instance using an external clock) and are, operationally, unrelated to the time τ in eq. (49) (or τ_1 and τ_2 in eq. (23)). Therefore, the internal spacetime interpretation of "events at a time τ" in the TO approach is a *reconstruction* based on observations "from the outside".

Importantly, the resources of the clock do not play any role in deciding the order in which the operations $\hat{F}_{CS}(\tau_2)$ and $\hat{G}_{CS}(\tau_1)$ (or equivalently $\hat{F}_S$ and $\hat{G}_S$) are applied. It is as if the external experimenter had access to a continuous set of operators labelled by a parameter τ, and she had the power to choose unambiguously specific operators from the set and apply them in any order she wants. In this sense, she has infinite resources, even if these are not provided by a temporal reference frame.

On the other hand, the PM approach can be interpreted as the external description of an internal observer who looks at a clock and triggers a measurement interaction on the system when they see the clock showing a certain time. This action constitutes the event. By contracting the solution to eq. (15) with $\langle\phi_t|$, we obtain eq. (16), which we can interpret as the description of events "as seen" from the perspective of the quantum clock C (although in a "purified" sense). In this case, the time order between events is determined *dynamically*, by the interaction between the clock and the systems. This is clearly operationally different from the notion of event in the TO approach. Therefore, the inequivalence between the two frameworks is due to a different implicit notion of what an "event at time t" means operationally.

In particular, the resources of the clock become all important in the PM approach. For clocks with finite resources, the measurement interaction induces time-non-local effects, which imply that the evolution of the system "as seen" by the clock is in general not unitary and that there is a fundamental limitation to the operational time order of events, as seen in subsection 4.2.3. (Non-unitarity can also arise in the $\sigma = \mathbb{R}$ case, see [31, 40–42].) Moreover, the denominator of the probability rule, eq. (14) becomes non-trivial, which is a departure from the Born rule of ordinary quantum mechanics.

To translate between one formalism and the other, in appendix F we show how to compute PM probabilities in terms of TO conditional probabilities, where evolution is unitary. Not surprisingly, the resulting invariant observables in the TO approach have time-non-local support. Thus, the PM approach can be interpreted in terms of states which are solutions to the constraint given by the TO approach, but with a restriction on the available class of operators. This restriction, which also implies a form of time-non-locality, is a consequence of the limited resources of the clock. Finally, to illustrate the difference between the two approaches in an extreme case, imagine that the clock has no dynamics, that is, set $\hat{H}_C = 0$. In the TO approach, it is still meaningful to talk about measurements. Even if the state of the system does not evolve relative to the clock, the operator $\hat{F}_{CS}(\tau)$ will modify the state in a non-trivial way. In contrast, setting $\hat{H}_C = 0$ in the PM approach makes measurements meaningless, as a working clock is necessary to trigger the interaction by which the ancilla A gathers information about the measurement outcome.

5 Non-unitarity and discreteness of time

In the above discussion, we tacitly assumed that time, and consequently time evolution, is continuous. This, and the assumption that the clock has a bounded spectrum, led us to conclude that, when measurements are modelled dynamically, as in the PM approach, the evolution equation governing the dynamics is time-non-local, in general leading to a non-unitary evolution.

In this Section, we drop the assumption that time evolution is continuous while keeping the resources of the clock bounded. We construct a model for the PM approach to measurements and show that it is consistent both with unitary evolution and bounded resources. A similar model was previously constructed in [19].

If we consider only a discrete set of times, we are only interested in the system being invariant under discrete time evolution, rather than an infinitesimal time evolution generated by a Hamil-

tonian. We define the discrete time evolution operator by

$$\hat{U} = \sum_k |t_{k+1}\rangle\langle t_k|_C \otimes \hat{U}_{SA}^{(k)}, \tag{24}$$

with $\{|t_k\rangle \, | t_k = \frac{k\pi}{E},\ k \in \mathbb{Z}\}$ being an orthonormal basis spanning the Hilbert space of a clock with spectrum $[-E, E]$, such that $|t_{k+1}\rangle = e^{-i\hat{H}_C \frac{\pi}{E}} |t_k\rangle$. Note that making time discrete is crucial here, as then such an orthonormal basis exits for clocks with bounded energy spectrum. This is not possible if time is continuous, i.e., k can take any value in $\mathbb{R}$. The unitary operators $\hat{U}_{SA}^{(k)}$ act on the system S and an ancilla A. They model a general measurement interaction in which the ancilla records the result of a measurement performed on the system at a specified time. In this way, the measurement process forms part of the dynamics, making this model a discrete version of the PM approach. Note, however, that this model is not obtained by directly discretising the continuous-time model of section section 4.2.

As noted above, the analogue of the Hamiltonian constraint in the discrete case is invariance under finite time evolution. Therefore, we demand that the global state $|\Psi\rangle_{CSA}$ satisfies

$$\hat{U} \, |\Psi\rangle_{CSA} = |\Psi\rangle_{CSA} \, . \tag{25}$$

Note that this constraint is weaker than the constraint of eq. (15), as we only require that the system is invariant under the discrete evolution.

As we show in appendix G, the solution to eq. (25) reads

$$\sum_k |t_k\rangle_C \otimes |\psi(t_k)\rangle_{SA} \, , \tag{26}$$

with

$$|\psi(t_{k+1})\rangle_{SA} = \prod_{j=0}^{k} \hat{U}_{SA}^{(j)} \, |\psi(t_0)\rangle_{SA} \, . \tag{27}$$

Note that, in eq. (26), the evolution of the system with respect to the clock is completely unitary.

Based on this observation, we arrive at the following statement: if, (*i*) every physical clock has a bounded energy spectrum, (*ii*) measurements of a system at a certain time couple the clock to the system, and (*iii*) we demand that the system evolves locally and unitarily with respect to the clock, then our analysis provides evidence that the underlying time, understood operationally as what a clock can show, has to be discrete. Note, however, that this model does not exclude the possibility that discrete time evolution could be non-unitary.

6 Discussion

We have analysed two recent approaches to measurements in the Page-Wootters formalism – the TO approach and the PM approach. Each approach solves Kuchař's criticisms differently: the TO approach defines measurements in terms of time-translation invariant operators, and the PM approach includes explicitly a dynamical description of the measurement process. We have shown that, whereas the two approaches produce the same predictions in the limit of ideal clocks, their predictions differ in the non-ideal case. We have compared the two solutions to reach a better operational understanding of measurement events in the "timeless" approach to quantum mechanics.

The TO approach, closely related to Dirac quantization, is based on time translation invariant operators applied to a static state. The operators are applied in an order that is chosen externally, using resources other than the clock's to distinguish between them. The resulting picture can be "internalized" and reinterpreted as ordinary quantum mechanics with external background time. In this reconstruction of events and time evolution, we have infinite resources to distinguish between different operations and determine their order, even if the clock is not ideal.

The PM approach describes measurements dynamically, and the resources of the clock are important to determine the order in which measurement interactions happen. It can be interpreted as the external description of an internal observer who measures a system when she "sees" her clock

reading a certain time. Note that both the internal and external descriptions are relative to the same clock. However, while the measurement event consists of a detector "click" in the internal description, it takes the form of a correlation between the system, the clock and an ancilla in the external description. When the clocks are far from the ideal case, time evolution looks very different to the unitary evolution of ordinary quantum mechanics. Specifically, it leads to a time non-local evolution equation, which generically implies a non-unitary evolution. In this approach, the fact that the clock has finite resources becomes very important. Most strikingly, the lack of resources to distinguish perfectly between different clock states leads to fundamental limitations to the notion of time ordering.

Therefore, the reconstruction of measurement events in the TO approach coincides with the operational notion of "looking" at a clock and measuring the system only in the limit where clocks can be treated as ideal. Intriguingly, even if both approaches are operationally different, they predict the same results in this limit. A possible explanation for this is that a clock with infinite resources in the PM approach is not affected by its coupling to the system and the ancilla, becoming effectively the external resource deciding the measurements in the TO approach. However, a full explanation of this fact lies beyond the scope of this work.

Our analysis opens the question of how to properly describe the purification of measurements in timeless approaches to quantum theory. Measurement interactions carry energy, and this has non-negligible consequences at the level of the constraint. The question of purification is important given the role of purification as a foundational principle for quantum mechanics [43]. Furthermore, it would be interesting to see how the analysis presented here extends to the case where we have multiple clocks describing a single system, as done in recent works on quantum reference frames, e.g. [21, 24].

In the non-ideal case, what are the implications of the PM model for spacetime? If time is defined operationally, in terms of what a clock measures, then a fundamental limit on our clock's resources would also imply a fundamental limit on our notion of time as we understand it classically. Considering the gravitational back reaction of a quantum clock on spacetime strengthens the operational point of view. In such scenarios, there are physical situations that fundamentally prohibit the measurement of time with more than a certain accuracy, limiting the resources of our clocks.

This operational stance resonates with the analysis of Peres in [44], who noted that:

> It thus seems that the Schrödinger wave function $\psi(t)$, with its continuous time evolution given by $i\hbar\dot{\psi} = H\psi$, is an idealization rooted in classical theory. It is operationally ill-defined (except in the limiting case of stationary states) and should probably give way to a more complicated dynamical formalism, perhaps one non-local in time. Thus, in retrospect, the Hamiltonian approach to quantum physics carries the seeds of its own demise.

The time-non-local evolution and the indefinite time order of measurement events explored here are instances of such a description.

Time non-locality and indefinite temporal order are, however, not the only possibilities. Returning to Peres' remarks, if continuous time evolution is operationally ill-defined, perhaps a discrete treatment of time would be the right thing to do. This resonates with our findings: in a timeless setting where the Hamiltonian evolution is generated by the relative degrees of freedom between a clock and the system, unitary evolution, definite time order and a clock with a bounded spectrum can be recovered if the emergent time model is discrete.

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A Review of the Page-Wootters Formalism

Here we give a pedagogical introduction to the Page-Wootters formalism [2, 3]. The relational nature and reparametrization-invariance of general relativity suggest that the physical states $|\Psi\rangle$ describing a theory of quantum gravity should satisfy a Wheeler-DeWitt equation

$$\hat{H}\,|\Psi\rangle = 0, \tag{28}$$

where $\hat{H}$ is the total *Hamiltonian constraint*. An immediate consequence is that states evolving according to the Schrödinger equation

$$i\frac{\partial}{\partial t}\,|\Psi\rangle = \hat{H}\,|\Psi\rangle = 0, \tag{29}$$

appear frozen in time. For this reason, theories satisfying eq. (28) are often called *timeless*. Don Page and William Wootters showed that one can nevertheless recover the correct dynamics using a mechanism that is now known as Page-Wootters formalism. In the following, we give a concise summary of this approach.

In the Page-Wootters formalism, the evolution of a system S is described by considering states $|\Psi\rangle$ not on the Hilbert space $\mathcal{H}_S$ of the system, but on an extended Hilbert space $|\Psi\rangle \in \mathcal{H}_C \otimes \mathcal{H}_S$ [7]. Here, C is an ancilla system referred to as the clock. The clock is equipped with a canonical pair of time and energy coordinates $\hat{T}_C, \hat{H}_C$, satisfying

$$[\hat{T}_C, \hat{H}_C] = i. \tag{30}$$

The Hamiltonian constraint eq. (28) translates to

$$\hat{H}\,|\Psi\rangle = 0, \quad \hat{H} = \hat{H}_C + \hat{H}_S \equiv \hat{H}_C \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_S. \tag{31}$$

Page and Wootters consider the case of an *ideal clock* C, meaning $\hat{T}_C$ is a self-adjoint operator with generalized orthogonal eigenvectors $|t\rangle_C$, such that $\hat{T}_C\,|t\rangle_C = t\,|t\rangle_C$ and $\langle t'|t\rangle_C = \delta(t-t')$. In that case, we can interpret $|\psi(t)\rangle_S = \langle t|_C\,|\Psi\rangle_{CS}$ as the state of the system S when clock C shows time t. Indeed, the Hamiltonian constraint in eq. (28) recovers the correct dynamics for $|\psi(t)\rangle_S$, that is the Schrödinger equation,

$$\langle t|_C\,\hat{H}\,|\Psi\rangle_{CS} = 0 \quad \leftrightarrow \quad i\frac{\partial}{\partial t}\,|\psi(t)\rangle_S = \hat{H}_S\,|\psi(t)\rangle_S\,. \tag{32}$$

This follows from the commutation relation given in eq. (30) and the Stone-von Neumann theorem, which together imply that $\hat{H}_C$ is represented as a differential operator $\hat{H}_C\,|\psi(t)\rangle_S = -i\partial_t\,|\psi(t)\rangle_S$. In the $|t\rangle_C$ basis, the state $|\Psi\rangle$ then takes the form

$$|\Psi\rangle = \int dt\,|t\rangle_C \otimes |\psi(t)\rangle_S\,, \tag{33}$$

which is why it is often referred to as the history state.

One-time measurements of an observable $\hat{O}_S = \sum_a O_a\,|a\rangle\langle a|$ at time τ can be defined in the Page-Wootters mechanism by first conditioning on finding the clock in state $|\tau\rangle_C$: The probability to find the system S in state $|a\rangle_S$ at time τ is thus given by

$$P(a|\tau) = \frac{\langle\Psi|\ |\tau\rangle\langle\tau|_C \otimes |a\rangle\langle a|_S\ |\Psi\rangle_{CS}}{\langle\Psi|\ (|\tau\rangle\langle\tau|_C)\ |\Psi\rangle_{CS}} = \frac{\left|\langle\psi(\tau)|a\rangle_S\right|^2}{\left\|\,|\psi(\tau)\rangle\,\right\|^2}. \tag{34}$$

This probability is in agreement with Born's rule and thus recovers the correct dynamics of the system S for one-time measurements; however, for consecutive measurements, one should be more careful, as we will see below. The issues with consecutive measurements give rise to the different approaches to measurements in the Page-Wootters mechanism discussed in this paper.

[7] Alternatively, one can introduce the Page-Wootters mechanism by considering a global Hilbert space $\mathcal{H}$, choosing one of the subsystems as a clock, and looking at the evolution of the system relative to this choice of clock.

B Review of Kuchař's criticisms

While eq. (34) is in agreement with Born's rule for one-point measurements, it immediately leads to issues when multiple subsequent measurements are performed. These issues were first raised by Kuchař [4] as follows:
1. Violation of constraints: In the Page-Wootters mechanism, the measurement of an observable $\hat{O}_S$ at time τ is described by a projective operator

$$\hat{\Pi}_{CS}^{a,\tau} = |\tau\rangle\langle\tau|_C \otimes |a\rangle\langle a|_S \tag{35}$$

acting on $|\Psi\rangle_{CS}$, where $|a\rangle_S$ is an eigenstate of $\hat{O}_S$. However, this operator, in general, does not commute with the constraint $\hat{H}$. Indeed, as the constraint is the total Hamiltonian, any operator commuting with it must be stationary, while there is obvious experimental evidence that the above projector is time-dependent. In that case, applying the measurement operator throws $|\Psi\rangle$ out of the constrained Hilbert space.
2. Wrong propagators: As a consequence, the Page-Wootters mechanism does not give the correct propagators for multiple subsequent measurements. For example, the dynamical question "If the system is in state $|a\rangle_S$ at time τ, what is the probability to find it in state $|b\rangle_S$ at time $\tau' > \tau$?" cannot be answered by applying the conditional probability in equation (34) twice. This would yield

$$P((b|\tau')|(a|\tau)) = \frac{\langle\Psi|\,\hat{\Pi}_{CS}^{a,\tau}\hat{\Pi}_{CS}^{b,\tau'}\hat{\Pi}_{CS}^{a,\tau}\,|\Psi\rangle}{\langle\Psi|\,\hat{\Pi}_{CS}^{a,\tau}\,|\Psi\rangle} = |\delta(\tau'-\tau)|^2|\langle b|a\rangle|, \tag{36}$$

which is non-zero only for $\tau' = \tau$ and not in accordance with Born's rule.

Kuchař's first criticism concerns the fact that projective operators and the constraint H do not commute in general. To resolve this issue, it seems natural to either modify the constraint (and therefore the history state $|\Psi\rangle$) or change the way operators are represented. These two approaches exactly correspond to the formalisms studied in the main body of this paper, the purified measurement [17] approach and twirled observable [6] approach respectively.

C Further details of the purified measurement and twirled observable approaches

C.1 Review of ideal purified measurement approach

Giovannetti *et al.* [17] proposed an extension to the Page-Wootters mechanism which overcomes the deficiencies raised by Kuchař . They do so by associating a Hamiltonian to each measurement, and including it in the constraint as follows: Let us consider an additional purifying ancilla A, which allows the description of a projective measurement at time τ through its Kraus representation as a unitary mapping

$$\hat{V} : |\psi(\tau)\rangle_S \otimes |r\rangle_A \mapsto \sum_a \hat{K}_S^a\,|\psi(\tau)\rangle_S \otimes |a\rangle_A \tag{37}$$

acting on $\mathcal{H}_S \otimes \mathcal{H}_A$, where the ancilla A is initialized in a ready state $|r\rangle_A$ and stores the measurement outcome. Here, $\hat{K}_S^a$ are Kraus operators satisfying $\sum_a (\hat{K}_S^a)^\dagger \hat{K}_S^a = \mathbb{1}_S$. Let $\hat{H}_{SA}$ be the Hamiltonian responsible for the measurement unitary $\hat{V}$, i.e. $\hat{V} = e^{-i\hat{H}_{SA}}$. Giovannetti *et al.* [17] consider the case where $\hat{H}_A$ contributes to the total Hamiltonian constraint in the form

$$\hat{H} = \hat{H}_C + \hat{H}_S + \delta(\hat{T}_C - \tau) \otimes \hat{H}_{SA}, \tag{38}$$

where $\delta(\hat{T}_C - \tau) \otimes \hat{H}_{SA}$ acts on $\mathcal{H}_C \otimes \mathcal{H}_S \otimes \mathcal{H}_A$. The solution to the constraint $\hat{H}\,|\Psi\rangle = 0$, where $\hat{H}$ is given by eq. (38), can be found by coherent group averaging [45]. This yields the history state

$$|\Psi\rangle = \int_{-\infty}^{\tau} dt\,|t\rangle_C \otimes |\psi(t)\rangle_S \otimes |r\rangle_A \tag{39}$$

$$+ \int_{\tau}^{\infty} dt\,|t\rangle_C \otimes \sum_a \hat{U}_S(t-\tau)\hat{K}_S^a\,|\psi(\tau)\rangle_S \otimes |a\rangle_A\,. \tag{40}$$

The probability that outcome a will be registered by the memory A at a given time t is given by projecting onto the measurement ancilla

$$P(a|t) = \frac{\langle\Psi| \, |a\rangle\langle a|_A \otimes |t\rangle\langle t|_C \, |\Psi\rangle}{\langle\Psi| \, |t\rangle\langle t|_C \, |\Psi\rangle}. \tag{41}$$

By construction, Kuchař's first criticism is resolved because the measurement projector now acts on the measurement ancilla and not on the system S. In the special case of projective, non-degenerate von Neumann measurements, the Kraus operators are $\hat{K}^a_S = |a\rangle\langle a|_S$ and the probability reduces to

$$P(a|\tau)_t = \left\| \langle a|_S \, |\psi(t)\rangle_S \right\|^2, \tag{42}$$

in accordance with the Born rule if $t \geq \tau$.

In contrast to the original Page-Wootters proposal, the PM approach allows the discussion of multiple subsequent measurements. Indeed, eq. (38) can be readily extended by a second measurement ancilla A'

$$\hat{H} = \hat{H}_C + \hat{H}_S + \delta(\hat{T}_C - \tau) \otimes \hat{H}_{SA} + \delta(\hat{T}_C - \tau') \otimes \hat{H}_{SA'}. \tag{43}$$

Group averaging results in the history state

$$|\Psi\rangle = \int_{-\infty}^{\tau} dt \, |t\rangle_C \otimes |\psi(t)\rangle_S \otimes |r\rangle_A \otimes |r\rangle_{A'} \tag{44}$$

$$+ \int_{\tau}^{\tau'} dt \, |t\rangle_C \otimes \sum_a \hat{U}_S(t-\tau)\hat{K}^a_S \, |\psi(\tau)\rangle_S \otimes |a\rangle_A \otimes |r\rangle_{A'} \tag{45}$$

$$+ \int_{\tau'}^{\infty} dt \, |t\rangle_C \otimes \sum_{a,b} \hat{U}_S(t-\tau')\hat{K}^b_S\hat{U}_S(\tau'-\tau)\hat{K}^a_S \, |\psi(\tau)\rangle_S \otimes |a\rangle_A \otimes |b\rangle_{A'} \, . \tag{46}$$

It is important to note that the probability $P(a|t)$ in eq. (41) is not affected by the presence of the second ancilla. However, the probability of observing the measurement result $|b\rangle_{A'}$ at time τ' conditioned on outcome $|a\rangle_A$ at time τ can now be expressed as

$$P\left((b|\tau')|(a|\tau)\right)_t = \frac{\left\| \langle t|_C \otimes \langle b|_{A'} \otimes \langle a|_A \, |\Psi\rangle \right\|^2}{\left\| (\langle t|_C \otimes \langle a|_A \, |\Psi\rangle \right\|^2}, \tag{47}$$

which results in

$$P\left((b|\tau')|(a|\tau)\right)_t = \frac{\left\| \hat{\Pi}^b_S \hat{U}(\tau'-\tau)\hat{\Pi}^a_S \, |\psi(\tau)\rangle \right\|^2}{\left\| \hat{\Pi}^a_S \, |\psi(\tau)\rangle \right\|^2}. \tag{48}$$

if $t \geq \tau' > \tau$. This is the correct expression for two-point measurements, in agreement with ordinary quantum mechanics. Kuchař's second criticism is thus resolved as well.

In eq. (40) and (46), the measurement outcome is entirely encoded in the record kept by the measurement ancillas. From a philosophical viewpoint, Giovannetti *et al.* [17] therefore adopt Wheeler's operationalist stance [1] that "the past has no existence except as it is recorded in the present".

C.2 Review of the twirled observable approach

Höhn *et al.* [6] propose a different extension of the Page-Wootters mechanism, which we call the twirled observable approach or TO approach. As in the Page-Wootters formalism evolution of S is described by a time-independent state $|\Psi\rangle$ solving the constraint

$$\hat{H} \, |\Psi\rangle = 0, \quad \hat{H} = \hat{H}_C + \hat{H}_S. \tag{49}$$

The constraint once again recovers the Schrödinger equation for $|\psi(t)\rangle_S = \langle t|_C \, |\Psi\rangle_S$. However, the requirement eq. (49) results in a different representation of operators acting on $|\Psi\rangle$: An operator f_S acting on $|\psi(\tau)\rangle_S$ is now represented by the incoherent group twirl

$$\hat{F}_{SC}(\tau) = \int_{\mathbb{R}} dt \;\; e^{-it\hat{H}_C} \;\; |\tau\rangle\langle\tau|_C \;\; e^{it\hat{H}_C} \otimes e^{-it\hat{H}_S} \, \hat{F}_S \, e^{it\hat{H}_S} \tag{50}$$

acting on $|\Psi\rangle$, where again $\hat{U}(t) = e^{-i\hat{H}t}$ is the free time evolution. By construction, $\hat{F}_{SC}(\tau)$ is invariant under $\hat{U}(t)$ and as a result, $[\hat{F}_{SC}(\tau), \hat{H}] = 0$, resolving Kuchař's first criticism.

The states $|\Psi\rangle$, solutions to the constraint eq. (49) are normalized with respect to the so-called *physical* scalar product $\langle\cdot|\cdot\rangle_{\text{phys}}$ [6, 31], which is defined as

$$\langle\Psi|\Phi\rangle_{\text{phys}} = \langle\psi(t)|_S \otimes \langle t|_C \left(\int ds\ \hat{U}(s)\right) |t\rangle_C \otimes |\phi(t)\rangle_S\,, \tag{51}$$

for any time t. It is straightforward to verify that this scalar product is well-defined and in particular independent of the choice of t.

The probability to find the system in state $|a\rangle_S$ at time τ is defined by

$$P(a|\tau) = \langle\Psi|\,\hat{\Pi}^a_{CS}(\tau)\,|\Psi\rangle_{\text{phys}}\,, \tag{52}$$

where $\hat{\Pi}^a_{CS}(\tau) = \int_{\mathbb{R}} dt\ \ e^{-it\hat{H}_C}\ |\tau\rangle\langle\tau|_C\ e^{it\hat{H}_C} \otimes e^{-it\hat{H}_S}\,|a\rangle\langle a|_S\,e^{it\hat{H}_S}$. This coincides with the TO probability introduced in section 3.1. This indeed gives the correct measurement statistics, as readily verified (setting $t = 0$):

$$P(a|\tau) = \langle\Psi|\left[\int dt\ e^{-it(\hat{H}_C+\hat{H}_S)}(|\tau\rangle\langle\tau|_C \otimes \hat{\Pi}^a_S)e^{it(\hat{H}_C+\hat{H}_S)}\right]|\Psi\rangle_{\text{phys}} \tag{53}$$

$$= \langle\psi(0)|_S \otimes \langle 0|_C \left[\int ds\ e^{-is(\hat{H}_C+\hat{H}_S)} \int dt\ e^{-it(\hat{H}_C+\hat{H}_S)}(|\tau\rangle\,\langle\tau|_C \otimes \hat{\Pi}^a_S)\hat{e}^{it(\hat{H}_C+\hat{H}_S)}\right]|0\rangle_C \otimes |\psi(0)\rangle_S \tag{54}$$

$$= \langle\psi(0)|_S \otimes \langle 0|_C \left[\int ds\ e^{-i(s-\tau)(\hat{H}_C+\hat{H}_S)} \otimes \Pi^a_S\right]|\tau\rangle_C \otimes |\psi(\tau)\rangle_S \tag{55}$$

$$= \langle\psi(\tau)|\,\hat{\Pi}^a_S\,|\psi(\tau)\rangle_S\,, \tag{56}$$

in agreement with Born's rule.

Because measurements in the TO picture commute with $\hat{H}$, they do not throw $|\Psi\rangle$ out of the constrained Hilbert space. Consequently, eq. (52) can be readily extended to more than one measurement. Furthermore, one can define conditional probabilities of the form

$$P((b|\tau')|(a|\tau)) = \frac{\langle\Psi|\,\hat{\Pi}^a_{SC}(\tau)\hat{\Pi}^b_{SC}(\tau')\hat{\Pi}^a_{SC}(\tau)\,|\Psi\rangle_{\text{phys}}}{\langle\Psi|\,\hat{\Pi}^a_{SC}(\tau)\,|\Psi\rangle_{\text{phys}}}. \tag{57}$$

We will show in the next section that this expression indeed recovers the correct measurement statistics for two-point measurements by relating it to the PM approach. Hence the TO approach also resolves Kuchař's second criticism.

D Equivalence of predictions for perfect PM vs TO

The PM and TO approach presented above are seemingly different from both a mathematical and philosophical perspective. Giovanetti *et al.* [17] overcome Kuchař's criticisms by considering ancillas storing the measurement outcomes and by modifying the constraint operator $\hat{H}$. Their procedure corresponds to an operationalist viewpoint. Höhn *et al.* [6] overcome them by representing observables in a way that is invariant under the flow generated by the constraint, without modifying the constraint itself. We now show that these two *a priori* different approaches are equivalent for multiple-time measurements in the case of perfect clocks, meaning they result in the same measurement outcome probabilities. Note that both approaches agree for single-time measurements already by construction.

To prove this, we want to show that the conditional probabilities eq. (47) and (57) of measuring System S in state $|b\rangle_S$ at time τ', given that the system is measured to be in state $|a\rangle_S$ at time $\tau < \tau'$, are equal for both the PM and TO approach. To simplify the calculation, let us define the reduction map $R(\tau)$, which projects $|\Psi\rangle$ onto $|\psi(\tau)\rangle$, and its inverse on the constrained Hilbert space:

$$\hat{R}(\tau) = \langle\tau|_C\,, \quad \hat{R}^{-1}(\tau) = \int dt\,|t\rangle_C\,\hat{U}_S(t-\tau) \tag{58}$$

Indeed, a change of variable yields

$$\hat{R}^{-1}(\tau)\hat{R}(\tau)\left|\Psi\right\rangle = \int dt\ \ \left|t\right\rangle\!\left\langle t\right|_C \left|\Psi\right\rangle = \left|\Psi\right\rangle. \tag{59}$$

With this notation at hand, and using $\hat{U}^\dagger(t)\left|\Psi\right\rangle = \left|\Psi\right\rangle$ the operator acting on the constrained Hilbert space can be written as

$$\begin{aligned}\hat{F}_{CS}(\tau)\left|\Psi\right\rangle &= \int dt\ \hat{U}(t)\ \left(\left|\tau\right\rangle\left\langle\tau\right| \otimes \hat{f}_S\right)\left|\Psi\right\rangle &(60)\\ &= \int dt\ \ \left|t\right\rangle\!\left\langle\tau\right| \otimes \hat{U}_S(t-\tau)\hat{f}_S\left|\Psi\right\rangle &(61)\\ &= (\hat{R}^{-1}(\tau)\hat{f}_S\hat{R}(\tau))\left|\Psi\right\rangle. &(62)\end{aligned}$$

This allows us to expand the conditional probability eq. (57) in terms of ordinary projection operators Π^a_S, Π^b_S :

$$\begin{aligned}P((b|\tau')|(a|\tau)) &= \frac{\left\langle\Psi\right|\hat{\Pi}^a_{SC}(\tau)\hat{\Pi}^b_{SC}(\tau')\hat{\Pi}^a_{SC}(\tau)\left|\Psi\right\rangle_{\text{phys}}}{\left\langle\Psi\right|\hat{\Pi}^b_{SC}(\tau)\left|\Psi\right\rangle_{\text{phys}}}. &(63)\\ &= \frac{\left\langle\Psi\right|\hat{R}^{-1}(\tau)\hat{\Pi}^a_S R(\tau)\hat{\Pi}^b_{SC}(\tau')\hat{R}^{-1}(\tau)\hat{\Pi}^a_S R(\tau)\left|\Psi\right\rangle_{\text{phys}}}{\left\langle\Psi\right|\hat{R}^{-1}(\tau)\hat{\Pi}^b_S\hat{R}(\tau)\left|\Psi\right\rangle_{\text{phys}}} &(64)\\ &= \frac{\left\langle\psi(\tau)\right|\hat{\Pi}^a_S R(\tau)\hat{\Pi}^b_{SC}(\tau')\hat{R}^{-1}(\tau)\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle}{\left\langle\psi(\tau)\right|\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle} &(65)\\ &= \frac{\left\langle\psi(\tau)\right|\hat{\Pi}^a_S R(\tau)\hat{R}^{-1}(\tau')\hat{\Pi}^b_S R(\tau')\hat{R}^{-1}(\tau)\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle}{\left\langle\psi(\tau)\right|\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle} &(66)\\ &= \frac{\left\langle\psi(\tau)\right|\hat{\Pi}^a_S\hat{U}(t-\tau')\hat{\Pi}^b_S U(\tau'-\tau)\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle}{\left\langle\psi(\tau)\right|\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle} &(67)\end{aligned}$$

where we have used equation eq. (62) and $\hat{R}(\tau')\hat{R}^{-1}(\tau) = \hat{U}_S(\tau'-\tau)$. We have defined the "physical inner product" as

$$\left\langle\Phi|\Psi\right\rangle_{phys} = \left\langle Phi\right|\left|s\right\rangle\!\left\langle s\right|_C \otimes \mathbb{1}_S\left|\Psi\right\rangle. \tag{68}$$

This inner product was introduced in [6, 31] and shown to be independent of the time s. Eq. (67) is indeed the correct two time conditional measurement probability as postulated by Born's rule. We now show that the PM approach yields the same result: substituting the state in eq. (44) (with $\hat{K}_a = \hat{\Pi}^a_S$) into equation eq. (47) yields

$$\begin{aligned}P((b|\tau')|(a|\tau)) &= \frac{\left|\left\langle\tau'\right|_C \otimes \left\langle b\right|_{M'} \otimes \left\langle a\right|_M)\left|\Psi\right\rangle\right|^2}{\left|(\left\langle\tau\right|_C \otimes \left\langle a\right|_M)\left|\Psi\right\rangle\right|^2} &(69)\\ &= \frac{\left\|\hat{\Pi}^b_S\hat{U}(\tau'-\tau)\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle\right\|^2}{\left\|\hat{\Pi}^a_S\left|\psi(\tau)\right\rangle\right\|^2}, &(70)\end{aligned}$$

identical to eq. (67). It is straightforward to generalize this result to more than two subsequent measurements. The PM and TO approach hence result in the same physical predictions for perfect clocks, despite constructing a different physical picture!

E Calculations for the purified measurement approach with non-ideal clocks

E.1 The evolution equation for non-ideal clocks

In a similar manner to how the Schrödinger equation was derived in the ideal clocks case, we can also derive the Schrödinger equation in the non-ideal case. Let us remember that the Hamiltonian

we consider in the non-ideal case is given by

$$\hat{H} = \hat{H}_C + \hat{H}_S + N_C \int dt \, |\phi_t\rangle\langle\phi_t|_C \otimes \hat{K}_{SA}(t) \tag{71}$$

where N_C is a normalization constant such that $\{N_C \, |\phi_t\rangle\langle\phi_t|_C\}_{t\in\mathbb{R}}$ forms a POVM.

As $\{|\phi_t\rangle_C\}_{t\in\mathbb{R}}$ is a over-complete basis, if

$$\langle\phi_t|_C \, \hat{H} \, |\Psi\rangle_{CSA} = 0 \quad \forall t, \tag{72}$$

then this is equivalent to

$$\hat{H} \, |\Psi\rangle_{CSA} = 0. \tag{73}$$

Therefore, we consider the following problem, where for the sake of simplicity we absorbed the Hamiltonian of the system into the interaction term

$$\langle\phi_t|_C \left(\hat{H}_C + N_C \int d\tau \, |\phi_\tau\rangle\langle\phi_\tau|_C \otimes \hat{K}_{SA}(\tau) \right) |\Psi\rangle_{CSA} = 0. \tag{74}$$

As $|\phi_t\rangle_C = e^{-i\hat{H}_C t} \, |\phi_0\rangle_C$ is a coherent system of states it holds that $-i\frac{d}{dt} \langle\phi_t|_C = \langle\phi_t|_C \, \hat{H}_C$. Therefore, we find

$$-i\frac{d}{dt} \langle\phi_t|_C \, |\Psi\rangle_{CSA} + N_C \int d\tau \, \langle\phi_t|\phi_\tau\rangle_C \, \hat{K}_{SA}(\tau) \, \langle\phi_\tau|_C \, |\Psi\rangle_{CSA} = 0. \tag{75}$$

Defining $\langle\phi_t|_C \, |\Psi\rangle_{CSA} = |\psi(t)\rangle_{SA}$ we find the following evolution equation:

$$-i\frac{d}{dt} |\psi(t)\rangle_{SA} + N_C \int d\tau \, \langle\phi_t|\phi_\tau\rangle_C \, \hat{K}_{SA}(\tau) \, |\psi(\tau)\rangle_{SA} = 0. \tag{76}$$

When considering solutions to this equation we have to be careful, as contrary to the ideal case, it is not clear that one can directly construct a solution to the constraint from them because in the case of non-ideal clocks the basis $\{|\phi_t\rangle_C\}_t$ is over-complete. The natural way to extend a solution $|\psi(\tau)\rangle_{SA}$ to the evolution equation eq. (76) on SA to a solution for the constraint, would be to consider the state defined by

$$|\Psi\rangle_{CSA} = N_C \int d\tau \;\; |\phi_\tau\rangle_C \otimes |\psi(\tau)\rangle_{SA} \, . \tag{77}$$

It is now not clear that the state $|\Psi\rangle_{CSA}$ satisfies the constraint, even though $|\psi(\tau)\rangle_{SA}$ satisfies the evolution equation eq. (76) as it is not clear that

$$\left|\tilde{\psi}(t)\right\rangle_{SA} = N_C \, \langle\phi_t|_C \int d\tau \;\; |\phi_\tau\rangle_C \otimes |\psi(\tau)\rangle_{SA} \, . \tag{78}$$

is again a solution to eq. (76), even if $|\psi(t)\rangle_{SA}$ is.

In the case where $|\psi(t)\rangle_{SA} = \langle\phi_t|_C \, |\Psi\rangle_{CSA}$, with $|\Psi\rangle_{CSA}$ defined as in eq. (77). We find

$$\begin{aligned} \left|\tilde{\psi}(t)\right\rangle_{SA} &= N_C \, \langle\phi_t|_C \int d\tau \;\; |\phi_\tau\rangle_C \otimes |\psi(\tau)\rangle_{SA} && (79) \\ &= N_C \, \langle\phi_t|_C \int d\tau \;\; |\phi_\tau\rangle_C \otimes \langle\phi_\tau|_C \, |\Psi\rangle_{CSA} && (80) \\ &= \langle\phi_t|_C \, |\Psi\rangle_{CSA} = |\psi(t)\rangle_{SA} \, . && (81) \end{aligned}$$

For such a $|\psi(t)\rangle_{SA}$ we have that $|\Psi\rangle_{CSA}$, as defined in eq. (77), fulfills the constraint eq. (71). As we shall see in the next section, the solution $|\psi(t)\rangle_{SA}$ does satisfy $|\psi(t)\rangle_{SA} = \langle\phi_t|_C \, |\Psi\rangle_{CSA}$, with $|\Psi\rangle_{CSA}$ defined according to eq. (77), thereby being a solution to eq. (71).

E.2 Solving the evolution equation

To solve eq. (76) we use a perturbative procedure. We then show that in the case where the approximation converges the solution, $|\psi(t)\rangle_{SA}$ fulfils the evolution equation and also that $\langle\phi_t|_C \int d\tau\, |\phi_\tau\rangle_C |\psi(\tau)\rangle_{SA} = |\psi(t)\rangle_{SA}$.

Integrating the evolution equation leads us to the following equation

$$|\psi(t)\rangle_{SA} = -iN_C \int_{-\infty}^{t} d\sigma \int d\tau\, \langle\phi_\sigma|\phi_\tau\rangle_C \otimes \hat{K}_{SA}(\tau)\, |\psi(\tau)\rangle_{SA} + |\psi_0\rangle_{SA}\,. \tag{82}$$

To solve this equation, we make the approximation that the system does not change over time

$$|\psi(t)\rangle^0_{SA} := |\psi_0\rangle_{SA} \tag{83}$$

and then use the RHS of eq. (82) to obtain a (hopefully) better approximation to the solution

$$|\psi(t)\rangle^1_{SA} = |\psi_0\rangle_{SA} - iN_C \int_{-\infty}^{t} d\sigma \int d\tau\, \langle\phi_\sigma|\phi_\tau\rangle_C \otimes \hat{K}_{SA}(\tau)\, |\psi_0\rangle_{SA}\,. \tag{84}$$

To simplify this expression, we define

$$F(t) = N_C \int_{-\infty}^{t} d\sigma\, \langle\phi_\sigma|\phi_0\rangle_C\,. \tag{85}$$

Using the fact that

$$F(t-\tau) = N_C \int_{-\infty}^{t-\tau} d\sigma\, \langle\phi_\sigma|\phi_0\rangle_C = N_C \int_{-\infty}^{t} d\sigma\, \langle\phi_{\sigma-\tau}|\phi_0\rangle_C = N_C \int_{-\infty}^{t} d\sigma\, \langle\phi_\sigma|\phi_\tau\rangle_C \tag{86}$$

we find that

$$|\psi(t)\rangle^1_{SA} = |\psi_0\rangle_{SA} - i \int d\tau F(t-\tau) \hat{K}_{SA}(\tau)\, |\psi_0\rangle_{SA}\,. \tag{87}$$

We then use this new approximation as an Ansatz and repeat the procedure, to obtain approximations in higher orders of $\hat{K}_{SA}(t)$

$$\begin{aligned}
|\psi(t)\rangle^2_{SA} &= |\psi(t)\rangle^1_{SA} + (-i)^2 \int d\tau_1 \int d\tau_n F(t-\tau_1) F(\tau_1-\tau_2) \hat{K}_{SA}(\tau_1) \hat{K}_{SA}(\tau_2)\, |\psi_0\rangle_{SA} \\
&\dots \\
|\psi(t)\rangle^n_{SA} &= |\psi(t)\rangle^{n-1}_{SA} \\
&\quad + (-i)^n \int d\tau_1 \cdots \int d\tau_n F(t-\tau_1) \dots F(\tau_{n-1}-\tau_n) \hat{K}_{SA}(\tau_1) \dots \hat{K}_{SA}(\tau_n)\, |\psi_0\rangle_{SA}
\end{aligned} \tag{88}$$

If this recursive procedure converges we obtain the following expression

$$|\psi(t)\rangle_{SA} = \sum_{n=0}^{\infty} (-i)^n \int d\tau_1 \cdots \int d\tau_n F(t-\tau_1) \dots F(\tau_{n-1}-\tau_n) \hat{K}_{SA}(\tau_1) \dots \hat{K}_{SA}(\tau_n)\, |\psi_0\rangle_{SA}\,. \tag{89}$$

In the case when the clock is ideal, i.e. $\langle\phi_t|\phi_{t'}\rangle_C = \delta(t-t')$, this reduces to the solution to the usual time-dependent Schrödinger equation, as $F(t) = N_C \int_{-\infty}^{t} d\sigma\, \langle\phi_\sigma|\phi_0\rangle_C = \Theta(t)$, the Heaviside theta function.

Let us show that $|\psi(t)\rangle_{SA}$ is actually a solution to the evolution equation

$$\begin{aligned}
i\frac{d}{dt} |\psi(t)\rangle_{SA} &= \sum_{n=1}^{\infty} (-i)^{(n-1)} \int d\tau_1 \dots d\tau_n N_C\, \langle\phi_t|\phi_{\tau_1}\rangle_C\, F(\tau_1-\tau_2) \dots F(\tau_{n-1}-\tau_n) \\
&\qquad \hat{K}_{SA}(\tau_1) \dots \hat{K}_{SA}(\tau_n)\, |\psi_0\rangle_{SA} \\
&= N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C\, \hat{K}_{SA}(\tau) \sum_{n=1}^{\infty} (-i)^{(n-1)} \\
&\qquad \times \int d\tau_1 \dots d\tau_{n-1} F(\tau-\tau_1) \dots F(\tau_{n-2}-\tau_{n-1}) \hat{K}_{SA}(\tau_1) \dots \hat{K}_{SA}(\tau_{n-1})\, |\psi_0\rangle_{SA} \\
&= N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C\, \hat{K}_{SA}(\tau)\, |\psi(\tau)\rangle_{SA}\,.
\end{aligned} \tag{90}$$

We have now shown that the function $|\psi(t)\rangle_{SA}$ solves the evolution equation. However, we have argued above that this does not yet imply that $\int dt\, |\phi_t\rangle_C |\psi(t)\rangle_{SA}$ solves the constraint, as $|\psi(t)\rangle_{SA}$ does not necessarily have the form $\langle\phi_t|_C |\Psi\rangle_{CSA}$. For the type of Hamiltonian we consider here, we can show that this is indeed the case. Let us define $|\Psi\rangle_{CSA} = N_C \int dt\, |\phi_t\rangle_C |\psi(t)\rangle_{CSA}$ and let us calculate $\langle\phi_t|_C |\Psi\rangle_{CSA}$:

$$\begin{aligned}\langle\phi_t|_C |\Psi\rangle_{CSA} &= N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C |\psi(\tau)\rangle_{SA} \\ &= N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C |\psi_0\rangle_{SA} \\ &\quad + \sum_{n=1}^{\infty} (-i)^n N_C \int d\tau \int d\tau_1 \cdots \int d\tau_n\, \langle\phi_t|\phi_\tau\rangle_C F(\tau-\tau_1)\ldots F(\tau_{n-1}-\tau_n) \\ &\quad \times \hat{K}_{SA}(\tau_1)\ldots\hat{K}_{SA}(\tau_n)|\psi_0\rangle_{SA}\,.\end{aligned} \tag{91}$$

For the $n=0$ term, using the expression of $|\phi_\tau\rangle$ in the Fourier basis, we find

$$N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C |\psi_0\rangle_{SA} = N_C \int d\tau\, \langle\phi_0|\phi_\tau\rangle_C |\psi_0\rangle_{SA} = \frac{1}{2\pi}\int d\tau \int_\sigma d\epsilon\; e^{-i\tau\epsilon}|\psi_0\rangle_{SA} = |\psi_0\rangle_{SA} \tag{92}$$

For terms with $n > 0$, we consider the following expression, defining $f(t) = N_C \langle\phi_t|\phi_0\rangle_C = \frac{1}{2\pi}\int_\sigma de\; e^{iet}$:

$$\begin{aligned}N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C F(\tau-\tau_1) &= \int d\tau f(t-\tau)F(\tau-\tau_1) \\ &= \int d\tau f(t-\tau_1-\tau)F(\tau) = (f*F)(t-\tau_1).\end{aligned} \tag{93}$$

Using the relationship of convolution with integration, we find

$$(f*F)(t-\tau_1) = \int_{-\infty}^{t-\tau_1} (f*f)(\tau)d\tau \tag{94}$$

By definition of the function f, its Fourier transform is $\mathcal{F}(f) = \chi_\sigma$, which implies that $\mathcal{F}(f*f) = \mathcal{F}(f)^2 = \mathcal{F}(f)$ and thus $(f*f)(\tau) = f(\tau)$. Together this implies that

$$N_C \int d\tau\, \langle\phi_t|\phi_\tau\rangle_C F(\tau-\tau_1) = F(t-\tau_1). \tag{95}$$

Plugging this back into our expression for $\langle\phi_t|_C |\Psi\rangle_{CSA}$ we find

$$\langle\phi_t|_C |\Psi\rangle_{CSA} = |\psi(t)\rangle_{SA}\,, \tag{96}$$

which proves that the solution we found satisfies the constraint.

E.3 The evolution equation and its solution for periodic clocks

If we want to consider a finite-dimensional clock we need to consider a periodic time, where for the clock there exists a time T such that $|\phi_T\rangle_C = |\phi_0\rangle_C$ [6]. From integrating evolution equation

$$|\psi(t)\rangle_{SA} = (-i)N_C \int_0^t d\sigma \int_0^T d\tau\, \langle\phi_\sigma|\phi_\tau\rangle_C \hat{K}_{SA}(\tau)|\psi(\tau)\rangle_{SA} + |\psi_0\rangle_{SA}\,. \tag{97}$$

Applying the same technique as for the non-periodic case we find the following evolution if the recursive procedure converges

$$\begin{aligned}|\psi(t)\rangle_{SA} = \sum_{n=0}^{\infty}(-i)^n \int_0^t d\sigma_1 \int_0^T d\tau_1 \cdots \int_0^{\tau_{n-1}} d\sigma_n \int_0^T d\tau_n\, \langle\phi_{\sigma_1}|\phi_{\tau_1}\rangle_C \ldots \langle\phi_{\sigma_n}|\phi_{\tau_n}\rangle_C \\ \times \hat{K}_{SA}(\tau)_1 \ldots K\tau_n |\psi_0\rangle_{SA}\end{aligned} \tag{98}$$

For simplicity of notation, we define

$$F_\tau(t) := N_C \int_0^t d\sigma \, \langle \phi_\sigma | \phi_\tau \rangle_C \, . \tag{99}$$

With this definition, we can write the solution in a more compact form

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} (-i)^n \int_0^T d\tau_1 \cdots \int_0^T d\tau_n F_{\tau_1}(t) \dots F_{\tau_n}(\tau_{n-1}) \hat{K}_{SA}(\tau_1) \dots \hat{K}_{SA}(\tau_n) \, |\psi_0\rangle_{SA} \, . \tag{100}$$

As in the non-periodic case, we find a candidate for the solution to eq. (97) by appending the clock and integrating over time

$$|\Psi\rangle_{CSA} = N_C \int d\tau \, |\phi_\tau\rangle_C \otimes |\psi(\tau)\rangle_{SA} \, . \tag{101}$$

Note that $|\Psi\rangle_{CSA}$ does not necessarily need to fulfil the constraint. In this case, we will see that $\langle \phi_t |_C \, |\Psi\rangle_{CSA}$ is a solution to the evolution equation and, thus, $|\Psi\rangle_{CSA}$ satisfies the constraint if the solution $|\psi(t)\rangle_{SA}$ is periodic and $\langle \phi_0 |_C \, |\Psi\rangle_{CSA} = |\psi(0)\rangle_{SA}$. Let us calculate $\langle \phi_t |_C \, |\Psi\rangle_{CSA}$:

$$\langle \phi_t |_C \, |\Psi\rangle_{CSA} = N_C \int_0^T d\tau \, \langle \phi_t | \phi_\tau \rangle_C \, |\psi(\tau)\rangle_{SA} \, . \tag{102}$$

We derive this expression by t and find

$$\frac{d}{dt} \langle \phi_t |_C \, |\Psi\rangle_{CSA} = N_C \int_0^T d\tau \, \langle \phi_t | \phi_\tau \rangle_C \, \frac{d}{d\tau} |\psi(\tau)\rangle_{SA} - N_C \, \langle \phi_t | \phi_0 \rangle_C \, (|\psi(T)\rangle_{SA} - |\psi(0)\rangle_{SA}) \, . \tag{103}$$

Using that $|\psi(t)\rangle_{SA}$ is a solution to the evolution equation we find

$$\frac{d}{dt} \langle \phi_t |_C \, |\Psi\rangle_{CSA} = \frac{d}{dt} |\psi(\tau)\rangle_{SA} - N_C \, \langle \phi_t | \phi_0 \rangle_C \, (|\psi(T)\rangle_{SA} - |\psi(0)\rangle_{SA}) \, . \tag{104}$$

If we assume that the solution $|\psi(t)\rangle_{SA}$ is periodic, this implies

$$\frac{d}{dt} \langle \phi_t |_C \, |\Psi\rangle_{CSA} = \frac{d}{dt} |\psi(\tau)\rangle_{SA} \, . \tag{105}$$

Integrating both sides of the equation we find

$$\langle \phi_t |_C \, |\Psi\rangle_{CSA} = |\psi(t)\rangle_{SA} + |\psi(0)\rangle_{SA} - N_C \int_0^T d\tau \, \langle \phi_0 | \phi_\tau \rangle_C \, |\psi(\tau)\rangle_{SA} \, . \tag{106}$$

Therefore $\langle \phi_t |_C \, |\Psi\rangle_{CSA}$ satisfies the constraint if

$$|\psi(0)\rangle_{SA} - N_C \int_0^T d\tau \, \langle \phi_0 | \phi_\tau \rangle_C \, |\psi(\tau)\rangle_{SA} = 0. \tag{107}$$

F From purified measurements to twirled observables

The probability of obtaining measurement outcome a at time t for a measurement event at time τ in the PM approach can be written as

$$P(a|t,\tau) = \frac{\mathrm{Tr}(|\phi_t\rangle\langle\phi_t|_C \otimes \mathbb{1}_S \otimes |a\rangle\langle a|_A \, |\Psi\rangle\langle\Psi|_{CSA})}{\mathrm{Tr}(|\phi_t\rangle\langle\phi_t|_C \otimes \mathbb{1}_{SA} \, |\Psi\rangle\langle\Psi|_{CSA})}, \tag{108}$$

where $|\Psi\rangle$ is a solution to the constraint

$$\hat{H}_{CSA} \, |\Phi\rangle_{CSA} = \left(\hat{H}_C + \hat{H}_S + N_C \int dt \, |\phi_t\rangle\langle\phi_t|_C \otimes \hat{K}_{SA}(\,)SAt \right) |\Phi\rangle_{CSA} = 0, \tag{109}$$

with a measurement interaction at time τ.

We can express the solution as $|\Psi\rangle_{CSA} = \hat{\Pi}_{\text{PM}} \ |\psi_0\rangle_{CSA}$, where $\hat{\Pi}_{\text{PM}}$ is the "projector" onto the space of solutions to the constraint in eq. (109)

$$\hat{\Pi}_{PM} = \int_{\mathbb{R}} dt e^{-it\hat{H}_{CSA}}, \tag{110}$$

where $\hat{H}$ is given by eq. (109). The state $|\psi_0\rangle_{CSA}$ can be interpreted as an initial condition and is an arbitrary element of the kinematical Hilbert space. We extend the TO approach to obtain the ancillary system A, and assume that A contributes trivially to the TO constraint of eq. (7). Then, a solution $|\eta\rangle_{CSA}$ to the TO constraint now reads

$$(\hat{H}_C + \hat{H}_S)\,|\eta\rangle_{CSA} = 0, \tag{111}$$

with identities on A left implicit. We choose $|\eta\rangle_{CSA}$ such that $|\psi_0\rangle_{CSA} = |\phi_0\rangle\langle\phi_0|_C\,|\eta\rangle_{CSA}$. Then, using the cyclicity of the trace and the fact that $|\eta\rangle\langle\eta|_{CSA}$ is left unchanged under incoherent G-twirling $\mathcal{T}$ (with respect to the TO time translation), we can rewrite eq. (108) as

$$P(a|t,\tau) = \frac{\text{Tr}\left(\mathcal{T}[|\phi_0\rangle\langle\phi_0|_C\,\hat{\Pi}^\dagger_{\text{PM}}\ |\phi_t\rangle\langle\phi_t|_C \otimes |a\rangle\langle a|_A\,\hat{\Pi}_{\text{PM}}\ |\phi_0\rangle\langle\phi_0|_C]\,|\eta\rangle\langle\eta|_{CSA}\right)}{\text{Tr}\left(\mathcal{T}[|\phi_0\rangle\langle\phi_0|_C\,\hat{\Pi}^\dagger_{\text{PM}}\ |\phi_t\rangle\langle\phi_t|_C\,\hat{\Pi}_{\text{PM}}\ |\phi_0\rangle\langle\phi_0|_C]\,|\eta\rangle\langle\eta|_{CSA}\right)}, \tag{112}$$

where we used that $\mathcal{T}$ is self-adjoint and have left identity operators implicit for simplicity.

The operators in the numerator and denominator of eq. (112) are explicitly invariant under the action generated by the TO constraint. Moreover, the state $|\eta\rangle$ satisfies the TO constraint. Therefore, eq. (112) is a conditional probability in the spirit of the TO approach. It remains to be seen, however, whether this probability can be written exactly in terms of the "physical inner product" (see appendix D) of the TO approach.

In general, the operators in eq. (112) are not of the form

$$\hat{F}_{SC}(\tau) = \int_{\mathbb{R}} dt\ \ e^{-it\hat{H}_C}\ |\tau\rangle\langle\tau|_C\ e^{it\hat{H}_C} \otimes e^{-it\hat{H}_S}\,\hat{F}_S\,e^{it\hat{H}_S}, \tag{113}$$

which represents time-local measurements. Unless we are in the limit where clocks can be approximated as perfect, these operators represent measurements that are delocalised in time in the TO approach.

G Discrete time model

In the case the spectrum of the clock is the interval $[-E, E]$, even though, in general, clock states are not orthogonal, there still exists times $\{t_k\}_k$, where the clock states are mutually orthogonal $\langle t_k|t_{k'}\rangle = \delta_{k,k'}$.[8] Using the explicit form of the spectrum, we find

$$\langle t_k|t_{k'}\rangle = \frac{1}{2\pi}\int_{-E}^{E} de\ \mathrm{e}^{i(t_k - t_{k'})e} = \frac{E}{\pi}\,\text{sinc}(E(t_k - t_{k'})) \tag{114}$$

with $\text{sinc}(x) := \frac{\sin(x)}{x}$. So the clock states are orthogonal $\langle t_k|t_{k'}\rangle = 0$, if $t_k - t_{k'} = \frac{N\pi}{E}$ with $N \in \mathbb{Z}$. Therefore, we can choose the following orthonormal clock basis

$$\left\{ |t_k\rangle \,\middle|\, t_k = \frac{k\pi}{E}\ \ k \in \mathbb{Z} \right\}. \tag{115}$$

The unitary

$$\hat{U} = \sum_k |t_{k+1}\rangle\langle t_k| \otimes \hat{U}^{(k)}_{SA} \tag{116}$$

[8] This approach generalizes to other clock Hamiltonians that have times at which the clock states are mutually orthogonal.

can be interpreted as the unitary evolving the system and the clock for one time step, such that the system undergoes the evolution U_k when the clock shows time t_k. By choosing the unitary $U^{(k^*)}$ for some integer k^* to be an interaction for a von Neumann unitary, then we can interpret this as the system being measured at time t_{k^*}.

In this discrete model, we no longer have a Hamiltonian, but the evolution of the system for a single time step. The constraint we considered in the continuous case, is equivalent to the total system not evolving in time:

$$\mathrm{e}^{i\hat{H}_{CSA}t}\left|\Psi\right\rangle_{CSA} = \left|\Psi\right\rangle_{CSA} \quad \forall\, t. \tag{117}$$

So for the discrete case, we impose the following constraint:

$$\hat{U}^n\left|\Psi\right\rangle_{CSA} = \left|\Psi\right\rangle_{CSA} \quad \forall n \in \mathbb{Z}. \tag{118}$$

This is equivalent to

$$\hat{U}\left|\Psi\right\rangle_{CSA} = \left|\Psi\right\rangle_{CSA}. \tag{119}$$

Expanding the state $\left|\Psi\right\rangle_{CSA}$ in the discrete clock basis we find the condition

$$\sum_k \left|t_{k+1}\right\rangle_C \otimes \hat{U}_{SA}^{(k)}\left|\psi(t_k)\right\rangle_{SA} = \sum_k \left|t_k\right\rangle_C \otimes \left|\psi(t_k)\right\rangle_{SA}, \tag{120}$$

from which we find that the states $\left|\psi(t_k)\right\rangle_{SA}$ need to fulfil the relation

$$\left|\psi(t_{k+1})\right\rangle_{SA} = \hat{U}_{SA}^{(k)}\left|\psi(t_k)\right\rangle_{SA}. \tag{121}$$

Applying this relation recursively we find

$$\left|\psi(t_{k+1})\right\rangle_{SA} = \prod_{j=0}^{k} \hat{U}_{SA}^{(j)}\left|\psi(t_0\right\rangle_{SA}. \tag{122}$$

We then see that, in this model, the evolution of the system relative to the clock is unitary as long as we consider time steps of size $\frac{\pi}{E}$.

Moreover, for discrete times, it follows immediately that the PM approach constructed above and the discrete version of the TO approach, defined in the obvious way, produce equivalent predictions.