On Strong Bounds for Trotter and Zeno Product Formulas with Bosonic Applications

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The Trotter product formula and the quantum Zeno effect are both indispensable tools for constructing time-evolutions using experimentally feasible building blocks. In this work, we discuss assumptions under which quantitative bounds can be proven in the strong operator topology on Banach spaces and provide natural bosonic examples. Specially, we assume the existence of a continuously embedded Banach space, which relatively bounds the involved generators and creates an invariant subspace of the limiting semigroup with a stable restriction. The slightly stronger assumption of admissible subspaces is well-recognized in the realm of hyperbolic evolution systems (time-dependent semigroups), to which the results are extended. By assuming access to a hierarchy of continuously embedded Banach spaces, Suzuki-higher-order bounds can be demonstrated. In bosonic applications, these embedded Banach spaces naturally arise through the number operator, leading to a diverse set of examples encompassing notable instances such as the Ornstein-Uhlenbeck semigroup and multi-photon driven dissipation used in bosonic error correction.

Table of Contents

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1 Introduction

The experimental motivation for operator product formulas, particularly the (symmetrized) Trotter product formula and the quantum Zeno effect, is the approximation of a time evolution by simpler building blocks. An instance from quantum dynamics is the approximation of a global Hamiltonian evolution by simpler locally defined Hamiltonians [42]. On an abstract structure-wise level, we establish convergence rates for differences like

$$\| \prod_{j=1}^{n} F_j(t,s)x - T(t,s)x\| \leq ?,$$

where $F_j(t,s)_{j=1}^n$ is a sequence of contractions, $T(t,s)$ a contractive strongly continuous (C_0-) evolution system (time-dependent C_0-semigroup), and x an element of the underlying Banach space. In the present paper, the building blocks $F_j(t,s)_{j=1}^n$ exhibit the structure given by the Trotter product formula or the quantum Zeno effect. For example, the former is given by $T(t,0) = e^{t(A+B)}$ and $F_j = e^{t\frac{A}{n}}e^{t\frac{B}{n}}$ for two generators A, B of contractive C_0-semigroups, for which Trotter proved that

$$\left\| \left(e^{t\frac{A}{n}}e^{t\frac{B}{n}} \right)^n x - e^{t(A+B)}x \right\| \to 0 \quad \text{for} \quad n \to \infty \tag{1}$$

if the closure of $A + B$ generates a contractive C_0-semigroup. The books [59, 45, Appx. D] provide a comprehensive overview of the extensive literature on the topic,. Here, only closely related results are covered.

The seminal work by Sophus Lie in 1875 proves Equation 1 as well as a symmetrized product formula (see Eq. 28) for real matrices (Lie's product formula). Following the symmetrization, which improves the convergence rate to n^{-2}, Suzuki investigated different families of product formulas achieving higher powers in the convergence rate [52, 54, 53, 23]. In the direction of the mentioned result by Trotter for C_0-semigroups, Kato proved a convergence result for non-negative self-adjoint operators on a separable Hilbert space with weak assumptions on the common domain as well as a generalization beyond semigroups [28, 30]. In the direction of Trotter-Suzuki product formulas, the work [2] proves explicit convergence rate for Hamiltonian evolutions on quantum lattices systems in the infinite volume limit. Recently, the quantitative result [8] addresses Trotter-Suzuki formulas in infinite closed quantum systems, leveraging knowledge of the eigenstates of the limiting dynamics. Combining the search for a good convergence rate and the generalization to C_0-semigroups, many interesting works have investigated the interplay of the domain of the generators with the convergence rate in the uniform operator topology [46, 40, 25, 55, 41, 60] and recently in the strong operator topology [5, 33]. Additionally, the extension to evolution systems (time-dependent semigroups) can, for example, be found in [58, 39, 26, 49].

The structural relation to the quantum Zeno effect becomes clear at the example of the projective Zeno effect in closed quantum systems introduced by Beskow and Nilsson. Here, $T(t,0) = e^{iPHP}P$ and $F_j = Pe^{t\frac{A}{n}}P$ for any Hermitian matrix H and projection P represent the result by Misra and Sudarshan, who named the effect after the greek philosopher Zeno of Elea,

$$\|(Pe^{\frac{it}{n}H})^n - e^{it\,PHP}P\| \to 0 \qquad \text{for} \qquad n \to \infty\,. \tag{2}$$

Since then, the Zeno effect was generalized in several different directions (see [16, 47, 27] for an overview). Building upon the research contributions of [9, 38, 3], the quantum Zeno effect has been extended to encompass open and infinite-dimensional quantum systems, featuring general quantum operations but uniformly continuous time evolutions. Especially, we allowed for evolution systems in the work [38]. Going beyond uniformly continuous semigroups, which are always generated by an bounded operator, to C_0-semigroups, Exner and Ichinose have recently demonstrated the convergence of the projective Zeno effect in the strong operator topology for unbounded Hamiltonians under the weak assumption that $H^{1/2}P$ is densely defined and $(H^{1/2}P)^{\dagger}H^{1/2}P$ self-adjoint [14]. In open quantum systems, the recent work by Becker, Datta, and Salzmann [4] and our follow up [37] show that the optimal convergence rate can be achieved in the strong operator topology: Let $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ be the generator of a contractive C_0-semigroup on a Banach space $\mathcal{X}$ and M a contraction satisfying the uniform power convergence, i.e. there is a projection P such that $\|M^n - P\|_{\mathcal{X}\to\mathcal{X}} \leq \delta^n$. If $P\mathcal{L}P$ is a generator of a contractive C_0-semigroup and the asymptotic Zeno condition, i.e. there is a $b \geq 0$ such that $\|Pe^{t\mathcal{L}}(\mathbb{1}-P)\|_{\mathcal{X}\to\mathcal{X}}, \|(\mathbb{1}-P)e^{t\mathcal{L}}P\|_{\mathcal{X}\to\mathcal{X}} \leq tb$ for all $t \geq 0$ holds true, then there is an explicit constant $c(t,b) > 0$ independent of n so that for any $t \geq 0$ and all $x \in \mathcal{D}((\mathcal{L}P)^2)$

$$\left\|\left(Me^{\frac{t}{n}\mathcal{L}}\right)^n x - e^{tP\mathcal{L}P}Px\right\| \leq \frac{c(t,b)}{n}\left(\|x\| + \|\mathcal{L}Px\| + \left\|(\mathcal{L}P)^2 x\right\|\right) + \delta^n\|x\|\,.$$

Both works ([4, 37]) are proven in the setup of Banach spaces and can be applied to open or closed quantum systems choosing the underlying space to be an separable Hilbert space $\mathcal{H}$ or the space of trace-class operators $\mathcal{T}_1(\mathcal{H})$.
As already mentioned, the projective quantum Zeno effect frequently measures an evolving quantum state and thereby forces the evolution to remain within the projective space. This concept is, for example, used to construct the $Z(\theta)$-gate in bosonic error correction [34, 56, 22, 21]. Moreover, certain measurements can even decouple the system from its environment [15, 9] or protect the initial state from decoherence [11, 13].

In this work, we generalize the Trotter product formula and the quantum Zeno effect in the following directions. First, we prove quantitative bounds for both by assuming that there exists a stronger Banach space $\mathcal{Y}$, on which the reduction of the limiting C_0-semigroup is well-defined, stable, and relatively bounds all involved generators. This assumption is not only motivated by hyperbolic C_0-evolution systems (time-dependent C_0-semigroups) but also by so-called Sobolev-preserving semigroups [20], which is the main example of the present paper. Nevertheless, we also extend the product formulas to evolution systems and present a large class of bosonic semigroups, for which the results can be easily applied. Moreover, we manage to prove higher-order approximations for the Trotter product formula.

For that, the paper is divided into four parts: First, we briefly recap the necessary notation and statements in Section 2. Then, we give an overview over the main results and some examples in Section 3. Next, we discuss the results and proofs for Trotter-like product formulas in Section 4. Here, we begin with a bound for the Trotter product formula in the strong operator topology and extend this to C_0-evolution systems and Suzuki's higher-order approximations. We then apply the Trotter-like results to Sobolev-preserving semigroups. Finally, we generalize the quantum Zeno effect in Section 5, starting with a bound for the projective Zeno effect, followed by an extension to C_0-evolution systems and more general measurements. Once again, the results are applied to Sobolev-preserving semigroups, particularly, the $Z(\theta)$-gate in bosonic error correction.

2 Preliminaries

We start by revisiting certain tools from Banach space theory used in the present paper. More details can be found in [29, Ch. III, 10, Ch. IV, 48, Ch. 2-3, 24, Ch. 2-3]. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ denote Banach spaces and $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$, which defines a Banach space again if it is equipped with the operator norm:

$$\|A\|_{\mathcal{X}\to\mathcal{Y}} \coloneqq \sup_{\|x\|_{\mathcal{X}}=1} \|A(x)\|_{\mathcal{Y}} \qquad \text{for } A \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \,. \tag{3}$$

We denote the identity map in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ by $\mathbb{1}$ and the space of bounded endomorphisms by $\mathcal{B}(\mathcal{X})$. More generally, an unbounded operator $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$ is a linear map defined on a domain $\mathcal{D}(A)$ and is called densely defined if its domain is a dense subspace of $\mathcal{X}$. The addition and concatenation of two unbounded operators $(A, \mathcal{D}(A))$ and $(B, \mathcal{D}(B))$ is defined on $\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(AB) = B^{-1}(\mathcal{D}(A))$. An operator $(A, \mathcal{D}(A))$ is closed iff its graph $\{(x, A(x)) \,|\, x \in \mathcal{D}(A)\}$ is a closed set in the product space $\mathcal{X} \times \mathcal{X}$. By convention, we extend all densely defined and bounded operators by the bounded linear extension theorem to bounded operators on $\mathcal{X}$ [31, Thm. 2.7-11]. Here, $\overline{A}$ is an extension of A if $\mathcal{D}(A) \subset \mathcal{D}(\overline{A})$ and $Ax = \overline{A}x$ for all $x \in \mathcal{D}(A)$. A densely defined operator is called closable if there is a closed extension — its smallest is called closure. Given operators[1] $(A, \mathcal{D}(A)), (B, \mathcal{D}(B))$ on $\mathcal{X}$, the operator $(B, \mathcal{D}(B))$ is relatively A-bounded if $\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and there are $a, b \geq 0$ for all $x \in \mathcal{D}(A)$ such that

$$\|B(x)\|_{\mathcal{X}} \leq a\|A(x)\|_{\mathcal{X}} + b\|x\|_{\mathcal{X}} \,. \tag{4}$$

Similarly, we call $(A, \mathcal{D}(A))$ relatively $\mathcal{Y}$-bounded w.r.t. a Banach space $\mathcal{Y} \subset \mathcal{D}(\mathcal{A})$ if $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is continuously embedded, i.e. there is a $b \geq 0$ such that $\|x\|_{\mathcal{X}} \leq b\|x\|_{\mathcal{Y}}$ for all $x \in \mathcal{Y}$, and

$$\|A\|_{\mathcal{Y}\to\mathcal{X}} < \infty \tag{5}$$

Besides the convergence w.r.t. the operator norm (i.e. uniform convergence), a sequence of operators $\{A_k\}_{k\in\mathbb{N}}$ defined on a common domain $\mathcal{D}(A)$ converges strongly to $(A, \mathcal{D}(A))$ if

$$\lim_{k\to\infty} \|A_k x - Ax\|_{\mathcal{X}} = 0$$

for all $x \in \mathcal{D}(A)$. Next, specific weighted Banach spaces are introduced, which are important for the examples of this work. Let $(\mathcal{W}, \mathcal{D}(\mathcal{W}))$ be an injective operator, then $\mathcal{D}(\mathcal{W})$ equipped with

$$\|X\|_{\mathcal{W}} \coloneqq \|\mathcal{W}(X)\|_{\mathcal{X}} \tag{6}$$

defines a normed space, which is additionally complete if $(\mathcal{W}, \mathcal{D}(\mathcal{W}))$ is surjective. This can be seen by the closed Graph theorem, which shows that $\mathcal{W}$ is a closed operator [20, Lem. 2.2].

In our examples, $\mathcal{W}$ plays the role of a reference operator which relatively bounds the considered generators and defines a stable or admissible subspace required for our theorems.

2.1 Strongly continuous semigroups

A quantum mechanical evolution is often described by a differential equation called the master equation, for which strongly continuous $(C_0$-)semigroups and (C_0)-evolution systems constitute an important toolbox for its solution theory. More details can for example be found in

[1] As a short reminder, in this work we use the convention that an operator is a linear map on a Banach space.

[12, Chap. II][29, Chap. 9] or [24, Chap. X-XIII]. A C_0-semigroup is defined by a family of operators $(T_t)_{t\geq 0} \subset \mathcal{B}(\mathcal{X})$, which satisfies

$$T_t T_s = T_{t+s}, \quad T_0 = \mathbb{1}, \quad \text{and} \quad \lim_{t\downarrow 0} T_t x = x \tag{7}$$

for all $t, s \geq 0$ and $x \in \mathcal{X}$. Equivalently, every semigroup is uniquely determined by its generator [12, Thm. II.1.4] — a possible unbounded, densely defined, and closed operator given by

$$\mathcal{L}(x) = \lim_{t\to 0^+} \frac{1}{t}(T_t(x) - x)\,, \tag{8}$$

for any

$$x \in \mathcal{D}(\mathcal{L}) = \{x \in \mathcal{X} : t \mapsto T_t(x) \text{ differentiable on } \mathbb{R}_{\geq 0}\}\,. \tag{9}$$

This justifies the notation $T_t = e^{t\mathcal{L}}$. Interestingly, one can show that every C_0-semigroup satisfies

$$\|T_t\|_{\mathcal{X}\to\mathcal{X}} \leq ce^{\omega t} \tag{10}$$

for a $c \geq 0$ and $\omega \geq 0$ [12, Prop. I.5.5], which is referred to as stability of the semigroup. In the special case $c \leq 1$ and $\omega = 0$, the semigroup is called contractive. Since the semigroup leaves the domain $\mathcal{D}(\mathcal{L})$ invariant and commutes with its generator, the following master equation admits the unique solution $T_t(x(0)) = x(t)$ [12, Prop. II.6.2]

$$\frac{d}{dt}x(t) = \mathcal{L}(x(t)) \quad x(0) \in \mathcal{D}(\mathcal{L}) \quad \text{and} \quad t \geq 0\,. \tag{11}$$

The above equation can be integrated in the following sense. For $[a, b] \subset [0, T]$, we denote the Bochner integral (see [24, Sec. 3.5-8]) of a vector-valued function $[0, T] \ni t \mapsto x(t) \in \mathcal{X}$ by

$$\int_a^b x(t)dt\,.$$

If the integral is well-defined, the usual properties like linearity, triangle inequality, additivity of disjoint sets, dominated convergence, as well as invariance under closed operators A if $[a, b] \ni t \mapsto Ax(t) \in \mathcal{X}$ is integrable, i.e.

$$A\int_a^b x(t)d\mu = \int_a^b Ax(t)dt\,, \tag{12}$$

can be shown [24, Sec. 3.7]. Moreover, the Fundamental theorem of calculus can be proven:

$$x(b) - x(a) = \int_a^b \frac{d}{dt}x(t)dt\,. \tag{13}$$

One way to verify that a vector-valued function on a compact interval is integrable is to prove continuity of the function [24, Thm. 3.7.4]. In the present work (see for instance Thm. 4.2, 5.3), the vector-valued map is usually given by a semigroup, an evolution system (see next paragraph), or a combination of both $t \mapsto G(t)x$ for $x \in \mathcal{D}(G(t))$. If the G is not just a semigroup which is by definition strongly continuous (see [12, Lem. 2.1.3]), we present the following slight extension of [12, Lem. B.16] as an auxiliary tool for the later proofs:

Lemma 2.1 *Let $\mathcal{Y} \subset \mathcal{X}$ be a continuously embedded subspace of $\mathcal{X}$, $[0, 1] \ni t \mapsto G_1(t) \in \mathcal{B}(\mathcal{X})$ and $[0, 1] \ni t \mapsto (G_2(t), \mathcal{Y})$ be two strongly continuous vector-valued maps. Then*

$$[0, 1] \ni t \mapsto (G_1(t)G_2(t), \mathcal{Y})$$

is again a strongly continuous vector-valued map on $\mathcal{Y}$.

Proof. The proof follows the proof of [12, Lem. B.15]. Let $x \in \mathcal{Y}$, $t \in [0,1]$, and $(t_n)_{n\in\mathbb{N}}$ a sequence converging to t. Then, there is an uniform bound on $\|G_1(t_n)\|_{\mathcal{X}\to\mathcal{X}}$ by the uniform boundedness theorem. Therefore,

$$\begin{aligned}\|G_1(t_n)G_2(t_n)x - G_1(t)G_2(t)x\|_{\mathcal{X}} = \|G_1(t_n)\|_{\mathcal{X}\to\mathcal{X}}\, &\|G_2(t_n)x - G_2(t)x\|_{\mathcal{X}} \\ &+ \|(G_1(t_n) - G_1(t))G_2(t)x\|_{\mathcal{X}}\end{aligned}$$

finishes the proof. □

As mentioned, we consider so-called evolution systems. These extend the master equation in Equation 11 by assuming that the generator is time-dependent. In this case, a two-parameter family of bounded operators $(T_{t,s})_{(t,s)\in I_T}$ with $I_T \coloneqq \{(t,s)\,|\,T \geq t \geq s \geq 0\}$ is called a C_0-evolution system (or time-dependent C_0-semigroup) if

$$T_{t,t} = \mathbb{1}\,, \quad T_{t,r}T_{r,s} = T_{t,s}\,, \quad \text{and} \quad (t,s) \mapsto T_{t,s}x \text{ is strongly continuous} \tag{14}$$

for all $0 \leq s \leq r \leq t \leq T$. Clearly, every C_0-semigroup T_t can be identified by an evolution system via $T_{t,s} = T_{t-s}$. However, a crucial difference is that evolution systems are not necessarily differentiable for any $x \neq 0$ [12, p. 478]. However, in the present paper all considered evolution systems are defined to be differentiable in the sense that there is a family of generators $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))_{0\leq s\leq T}$ such that for all $x \in \mathcal{D}(\mathcal{L}_s)$ and all $(t,s) \in I_T$,

$$\frac{\partial}{\partial s} T_{t,s}x = -T_{t,s}\mathcal{L}_s x\,. \tag{15}$$

We say that $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))_{0\leq s\leq T}$ generates $T(t,s)$ if it is uniquely determined by the above relation. For a continuously embedded Banach space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ in $\mathcal{X}$, we call a continuously differentiable function $t \mapsto x(t) \in \mathcal{Y}$ for $(t,s) \in I_T$ a $\mathcal{Y}$-solution if it satisfies the differential equation

$$\frac{\partial}{\partial t} x(t) = \mathcal{L}_t(x(t)) \quad \text{and} \quad x(s) = x_s \in \mathcal{Y} \quad \text{for } 0 \leq s \leq t \leq T \tag{16}$$

for a given family $(\mathcal{L}_t, \mathcal{Y} \subset \mathcal{D}(\mathcal{L}_t))$ with $(t,s) \in I_T$. Conversely, we introduce sufficient assumptions on $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))$ such that the differential Equation 16 admits a unique solution operator [43, Chap. 5, 44, pp. 127 ff.]. For that, we introduce the notion of admissible subspaces:

Definition 2.2 (Stable and admissible subspaces) Let $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ be a continuously embedded subspace of $\mathcal{X}$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ the generator of a C_0-semigroup on $\mathcal{X}$. Then, the subspace $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is $\mathcal{L}$-stable, if it is invariant under the semigroup, i.e. $e^{t\mathcal{L}}\mathcal{Y} \subset \mathcal{Y}$ for all $t \geq 0$, and the reduced vector-valued function $e^{t\mathcal{L}}|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}$ is stable, i.e. there are $\tilde{c}, \tilde{\omega} \geq 0$ such that

$$\|e^{t\mathcal{L}}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{t\tilde{\omega}}\,.$$

Moreover, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is $\mathcal{L}$-admissible if it is $\mathcal{L}$-stable and the reduced function is strongly continuous. The same definitions translates to C_0-evolution systems with time-independent constants.

Note that the additional continuity assumption in the definition of $\mathcal{L}$-admissible subspaces induces, that the reduced vector-valued function, i.e. $e^{t\mathcal{L}}|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}$, defines a semigroup again. Here, the stability of the reduced vector-valued map is not required and even implied by [12, Prop. I.5.5]. However, in the following, we consider families of generators over $[0,T]$ for which we assume to have uniform stability constants $\tilde{c}, \tilde{\omega}$ for all $s \in [0,T]$. The describing properties, which refer to evolution systems of hyperbolic type, are:

(1) For every $0 \leq s \leq T$, $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))$ generates a C_0-semigroup and admits uniform stability constants $c, \omega \geq 0$, i.e. $\|e^{t\mathcal{L}_s}\|_{\mathcal{X}\to\mathcal{X}} \leq ce^{t\omega}$ for all $t \geq 0$ and $s \in [0, T]$;

(2) There exists a $\mathcal{L}_s$-admissible subspace $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ and uniform stability coefficients $\tilde{c}, \tilde{\omega}$ for all $0 \leq s \leq T$;

(3) The map $s \mapsto \mathcal{L}_s$ is $\mathcal{Y}$-bounded (see Eq. 5) and continuous in $\mathcal{B}((\mathcal{Y}, \|.\|_{\mathcal{Y}}), (\mathcal{X}, \|.\|_{\mathcal{X}}))$.

To achieve a $\mathcal{Y}$-solution of the differential equation (16), one can additionally assume

(4) $T_{t,s}\mathcal{Y} \subseteq \mathcal{Y}$ for $(t, s) \in I_T$;

(5) $(s, t) \mapsto T_{s,t}$ is strongly continuous on $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$.

Under these assumptions, one can show:

Theorem 2.3 (Hyperbolic evolutions systems [43, Thm. 5.3.1/4.3]) *Let $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))_{s\geq 0}$ be a family of operators satisfying assumptions $(1-3)$. Then, there is a unique hyperbolic C_0-evolution system on $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ satisfying*

$$\|T_{t,s}\|_{\mathcal{X}\to\mathcal{X}} \leq c\, e^{(t-s)\omega}\,, \quad \lim_{t\downarrow s} \frac{T_{t,s}x - x}{t-s} = \mathcal{L}_s x\,, \quad \textit{and} \quad \frac{\partial}{\partial s} T_{t,s}x = -T_{t,s}\mathcal{L}_s x$$

for all $(t, s) \in I_T$ and $x \in \mathcal{Y}$. If further (4) and (5) hold, then for every $x_s \in \mathcal{Y}$, $T_{t,s}x_s$ is a unique solution for the initial value problem in (16).

Proof. To apply the result [43, Thm. 5.3.1/4.3], one can directly translate the stability assumptions given in $(1)-(5)$ via the Feller-Miyadera-Phillips generation theorem (see [12, Thm. III.3.8]) and [12, Prop. I.5.5] to the assumptions $H_1, ..., H_3$ and E_4, E_5 given in [43, p. 135, 141] as well as the definition of stability in [43, Def. 5.2.1]. $\square$

2.2 Bosonic Sobolev preserving semigroups

One of the main examples in our work is the class of bosonic Sobolev preserving semigroups, which are discussed in detail in [20]. For those, we shortly introduce bosonic (or continuous variable) systems in the second quantization formalism [7, Chap. 5.2]. Let $\mathcal{H} = L^2(\mathbb{R}^m)$ be the Fock space of m-modes endowed with an orthonormal (Fock-)basis $\{|\mathbf{n}\rangle\}_{\mathbf{n}\in\mathbb{N}^m}$, where n_j labels the number of particles in the j^{th}-mode. Then, the annihilation and creation operators are defined by

$$a_j\,|\mathbf{n}\rangle = \sqrt{n_j}\,|n_1, ..., n_j - 1, ..., n_m\rangle\,, \quad a_j^\dagger\,|\mathbf{n}\rangle = \sqrt{n_j + 1}\,|n_1, ..., n_j + 1, ..., n_m\rangle$$

and satisfy

$$[a_i, a_j^\dagger]\,|\mathbf{n}\rangle = \delta_{i,j}\,|\mathbf{n}\rangle\;, \quad [a_i, a_j]\,|\mathbf{n}\rangle = 0$$

for all $\mathbf{n} \in \mathbb{N}^m$ and $i, j \in \{1, ..., m\}$. Moreover, the number operator is defined by

$$N_j\,|\mathbf{n}\rangle = a_j^\dagger a_j\,|\mathbf{n}\rangle = n_j\,|\mathbf{n}\rangle \tag{17}$$

and counts the number of particles in the j^{th}-mode. Another important class are the coherent states, which are eigenvectors of a_j, i.e. $a_j\,|\alpha\rangle = \alpha_j\,|\alpha\rangle$ for all $j \in \{1, ..., m\}$. Here, the j^{th}-mode is defined by $|\alpha_j\rangle = \exp\left(-\frac{|\alpha_j|^2}{2}\right)\sum_{j=0}^{\infty}\frac{\alpha_j^n}{\sqrt{n!}}\,|n_j\rangle$ for any $\alpha_j \in \mathbb{C}$. In most of our proofs, we use the finitely supported subspace $\mathcal{H}_f = \{|\psi\rangle = \sum_{\mathbf{n}=0}^{M}\lambda_{\mathbf{n}}\,|\mathbf{n}\rangle \mid M \in \mathbb{N}\,,\,\mathbf{n} \in \mathbb{N}^m\,,\,\lambda_{\mathbf{n}} \in \mathbb{C}\}$ as a domain for a_j, $a_j^\dagger$, and polynomials in both and extent the bound afterwards by continuity in an appropriate topology.

For $A \in \mathcal{B}(\mathcal{H})$, the adjoint is uniquely defined by $\langle A\phi\,,\varphi\rangle = \langle \phi\,, A^\dagger\varphi\rangle$ for all $\phi,\varphi \in \mathcal{H}$ [48, Sec. 2.1] and the space of self-adjoint operators denoted by $\mathcal{B}_{\mathrm{sa}}(\mathcal{H}) := \{A \in \mathcal{B}(\mathcal{H}) \,|\, A = A^\dagger\}$. Then, we define the trace by $\mathrm{tr}[A] = \sum_{\mathbf{n}\in\mathbb{N}^m} \langle \mathbf{n}|\, A\, |\mathbf{n}\rangle$ for a positive $A \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$, i.e. there is a $B \in \mathcal{B}(\mathcal{H})$ such that $A = B^\dagger B$, and the trace norm by $\|A\|_1 = \mathrm{tr}\Big[\sqrt{A^\dagger A}\Big]$. Finally, the space of self-adjoint trace-class operators is denoted by $\mathcal{T}_{1,\mathrm{sa}} := \{A \in \mathcal{B}_{\mathrm{sa}}(\mathcal{H}) \,|\, \|A\|_1 < \infty\}$. Using the convention $\mathbf{n} \leq M$ iff $n_j \leq M$ for all $j \in \{1,...,m\}$, we define

$$\mathcal{T}_f := \{A = \sum_{\mathbf{n},\mathbf{m}=0}^{M} a_{\mathbf{nm}}\, |\mathbf{n}\rangle\langle\mathbf{m}| \mid M \in \mathbb{N}\,,\ \mathbf{n},\mathbf{m} \in \mathbb{N}^m\,,\ a_{\mathbf{n},\mathbf{m}} = \bar{a}_{\mathbf{m},\mathbf{n}} \in \mathbb{C}\} \tag{18}$$

as before, which is dense in $\mathcal{T}_{1,\mathrm{sa}}$ [48, Sec. 3.1]. Then, we define the bosonic Sobolev spaces by

$$\mathcal{D}(\mathcal{W}^{\mathbf{k}}) = \mathcal{W}^{-\mathbf{k}}(\mathcal{T}_{1,\mathrm{sa}}) \quad \text{via} \quad \mathcal{W}^{-\mathbf{k}}(x) := \prod_{j=1}^{m}(\mathbb{1}+N_j)^{-k_j/4}\, x \prod_{j=1}^{m}(\mathbb{1}+N_j)^{-k_j/4}\,. \tag{19}$$

Since $\mathcal{W}^{-\mathbf{k}}$ is an injective bounded operator, its inverse $\mathcal{W}^{\mathbf{k}}$ is a closed operator on $\mathcal{D}(\mathcal{W}^{\mathbf{k}})$ by the closed graph theorem and $(\mathcal{D}(\mathcal{W}^{\mathbf{k}}), \|\cdot\|_{\mathcal{W}^{\mathbf{k}}})$ a Banach space. For the sake of notation, we define

$$W^{\mathbf{k},1} := \mathcal{D}(\mathcal{W}^{\mathbf{k}}) \quad \text{and} \quad \|\cdot\|_{W^{\mathbf{k},1}} := \|\cdot\|_{\mathcal{W}^{\mathbf{k}}} \tag{20}$$

with the special case $(W^{0,1}, \|\cdot\|_{W^{0,1}}) = (\mathcal{T}_{1,\mathrm{sa}}, \|\cdot\|_1)$. Interestingly, one can show that if $0 \leq k_j < k_j'$ for all $j \in \{1,...,m\}$, $W^{\mathbf{k}',1}$ is compactly embedded in $W^{\mathbf{k},1}$ [20].

Next, we discuss a sufficient condition so that an operator $(\mathcal{L}, \mathcal{T}_f)$ is a generator. Note that the considered operators are often defined on the domain $\mathcal{T}_f$ (see Eq. 18) and then extended to their closure if possible. First, we assume that the operator is in GKSL form, i.e.

$$\mathcal{L} : \mathcal{T}_f \to \mathcal{T}_f \quad x \mapsto \mathcal{L}(x) = -i[H, x] + \sum_{j=1}^{K} L_j x L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, x\} \tag{21}$$

with $K \in \mathbb{N}$, $L_j := p_j(a_1, a_1^\dagger, ...)$ defined by polynomials of a_j and $a_j^\dagger$ for all $j \in \{1,...,m\}$, and symmetric $H := p_H(a_1, a_1^\dagger, ...)$, i.e. $\langle A\psi, \phi\rangle = \langle \psi, A\phi\rangle$ for all $\psi,\phi \in \mathcal{H}_f$. Here, we assume that the monomials in each mode are ordered like $(a_j^\dagger)^k a_j^l$. Second, we assume that there are non-negative sequences $(\mathbf{k}_r)_{r\in\mathbb{N}} \subset \mathbb{N}^m$ diverging coordinate-vice to infinity and $(\omega_{\mathbf{k}_r})_{r\in\mathbb{N}}$ so that

$$\mathrm{tr}\Big[\mathcal{W}^{\mathbf{k}_r}\mathcal{L}(x)\Big] \leq \omega_{\mathbf{k}_r}\, \mathrm{tr}\Big[\mathcal{W}^{\mathbf{k}_r}(x)\Big] \tag{22}$$

for all positive semi-definite $x \in \mathcal{T}_f$. Then, the following theorem holds:

Theorem 2.4 ([20, Thm. 4.4]) *Let $(\mathcal{L}, \mathcal{T}_f)$ be an operator of the form (21) satisfying Equation 22, then the closure $\overline{\mathcal{L}}$ generates a positivity preserving C_0-semigroup $(T_t)_{t\geq 0}$ on $W^{\boldsymbol{k}_r,1}$ for $r \in \mathbb{N}$ and*

$$\|T_t\|_{W^{\boldsymbol{k}_r,1}\to W^{\boldsymbol{k}_r,1}} \leq e^{\omega_{\boldsymbol{k}_r} t} \tag{23}$$

for all $t \geq 0$. In the special case $\boldsymbol{k} = 0$, the semigroup is contractive and trace-preserving.

The same holds true if the coefficients of the operator are time-dependent but continuous and if the right-hand side in Equation 22 does not depend on t [37, Thm. 4.5].

3 Main results

This section presents the main results of the paper and discusses its assumptions. The paper is split into two parts: The first is about the Trotter product formula and the second about the quantum Zeno effect. We start with the former and a generalization Trotter published in 1959 of Lie's product formula for matrices to contractive C_0-semigroups on a Banach space $\mathcal{X}$ generated by $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ and $(\mathcal{K},\mathcal{D}(\mathcal{K}))$. In this section, we assume that $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ and $(\mathcal{K},\mathcal{D}(\mathcal{K}))$ generates a contractive C_0-semigroup, without explicitly mentioning. Moreover, assume that the closure of $\mathcal{L}+\mathcal{K}$ generates a contractive C_0-semigroup. Then, [57] shows that for all $x\in\mathcal{X}$

$$\left(e^{\frac{\mathcal{L}}{n}}e^{\frac{\mathcal{K}}{n}}\right)^n x \to e^{\mathcal{L}+\mathcal{K}}x \qquad \text{for} \qquad n\to\infty\,. \tag{24}$$

For our results, we need two main assumptions. First, we assume that there is a (reference) Banach space $(\mathcal{Y}\subset\mathcal{X},\|\cdot\|_{\mathcal{Y}})$ on which the operators $\mathcal{L}$ and $\mathcal{K}$ as well as all products are not only well-defined but also relatively $\mathcal{Y}$-bounded. Repeat Equation 5: an operator $(A,\mathcal{D}(A))$ is relatively $\mathcal{Y}$-bounded if $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$ is continuously embedded in $(\mathcal{X},\|\cdot\|_{\mathcal{X}})$ and

$$\|A\|_{\mathcal{Y}\to\mathcal{X}}<\infty\,. \tag{25}$$

Second, we assume that the closure of $(\mathcal{L}+\mathcal{K})$ generates a contractive C_0-semigroup on $\mathcal{X}$ and a stable vector-valued map on $\mathcal{Y}$ — $\mathcal{Y}$ is $(\mathcal{L}+\mathcal{K})$-stable — which means $e^{t(\mathcal{L}+\mathcal{K})}(\mathcal{Y})\subset\mathcal{Y}$ and there are a $\tilde{c},\omega\geq 0$ so that

$$\|e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{Y}}\leq\tilde{c}e^{t\omega}\,. \tag{26}$$

Before diving into the first result, we briefly discuss the motivation behind the above assumptions. These assumptions embody the essential steps used in the current proof strategy. For instance, they cover the following examples. First, we consider an operator-theoretical example assuming certain relative boundedness assumption between the involved generators.

Example 1. Assume that the closure of $(\mathcal{L}+\mathcal{K},\mathcal{D}(\mathcal{L}+\mathcal{K}))$ defines a contractive C_0-semigroup again and that $\mathcal{K}^2$, $\mathcal{L}\mathcal{K}$, $\mathcal{K}\mathcal{L}$, and $\mathcal{L}^2$ are relatively $(\overline{\mathcal{L}+\mathcal{K}})^2$-bounded. Then, we equip $\mathcal{Y}=\mathcal{D}((\overline{\mathcal{L}+\mathcal{K}})^2)$ with the norm

$$\|x\|_{\mathcal{Y}}\coloneqq\|x\|_{\mathcal{X}}+\|(\overline{\mathcal{L}+\mathcal{K}})x\|_{\mathcal{X}}+\|(\overline{\mathcal{L}+\mathcal{K}})^2x\|_{\mathcal{X}}$$

for $x\in\mathcal{Y}$, which defines a Banach space by repeating the reasoning why $\mathcal{D}(\mathcal{L})$ with the graph norm defines a Banach space twice [29, Sec. 5.2]. Specially, it defines an $(\mathcal{L}+\mathcal{K})$-admissible subspace. Therefore, it satisfies the assumptions Assumption 25 and 26 with $\omega=0$ so that the bound does not depend exponentially on t. Note that the above construction is equivalent to the so-called Sobolev Tower of order 2 [12, Sec. 5] if $\overline{\mathcal{L}+\mathcal{K}}$ is invertible.

Second, we consider a bosonic example of decoupled quadratic generators:

Example 2. Let $(H_1,\mathcal{H}_f)$ and $(H_2,\mathcal{H}_f)$ be two decoupled quadratic Hamiltonian in annihilation and creation operators defined on an m-mode bosonic quantum system admitting the structure

$$H=\sum_{i\in\{1,..,m\}}\sum_{k_i+\ell_i\leq 2}\lambda_{k_i\ell_i}(a_i^\dagger)^{k_i}a_i^{\ell_i}+p(N_1,...,N_m) \tag{27}$$

Then, it is shown in [36, Lem. 30] that there is a constant $\tilde{c}$ such that for all $x\in\mathcal{D}(W^{\mathbf{k},1})$

$$\|-i[H_j,x]\|_1\leq\tilde{c}\|\mathcal{W}^4(x)\|_1 \qquad \text{and} \qquad \|-[H_i[H_j,x]]\|_1\leq\tilde{c}\|\mathcal{W}^8(x)\|_1\,,$$

where $W^{k,1} = \prod_{j=1}^m (\mathbb{1} + N_j)^{k/4}\, x\, \prod_{j=1}^m (\mathbb{1} + N_j)^{k/4}$ and $i, j \in \{1,2\}$. Following the proof of Lemma 5.6 and applying Theorem 4.4 (restated in Thm. 2.4) in [20], the closures of the Hamiltonians define quantum Markov semigroups with the property

$$\|e^{-it[H_j,\cdot]}x\|_{W^{k,1}} \leq e^{\omega t}\|x\|_{W^{k,1}}$$

for a constant $\tilde{\omega} \geq 0$, $k \in \{4,8\}$, $j \in \{1,2\}$. Here, we used the notation for Sobolev spaces introduced in Equation 20. Therefore, the above assumptions 25 and 26 are satisfied for $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}}) = (W^{4,1}, \|\cdot\|_{W^{4,1}})$. More details and open bosonic quantum systems are given in Section 4.4.

Keeping these two examples in mind, we prove an explicit convergence rate for the Trotter product formula in the strong-operator topology.

Theorem (See Theorem 4.2) *Let $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ be a Banach space such that $\mathcal{Y} \subset \mathcal{D}(\mathcal{L}+\mathcal{K})$ and the closure of $(\mathcal{L}+\mathcal{K}, \mathcal{Y})$ generates a contractive C_0-semigroup on $\mathcal{X}$. If additionally $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}^2$, $\mathcal{L}\mathcal{K}$, $\mathcal{K}\mathcal{L}$, $\mathcal{L}^2$ are relatively $\mathcal{Y}$-bounded (see Eq. (5)) and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is $(\mathcal{L}+\mathcal{K})$-stable, then there are $c, \omega \geq 0$ such that for all $x \in \mathcal{Y}$, $t \geq 0$, and $n \in \mathbb{N}$*

$$\left\| \left(e^{\frac{t}{n}\mathcal{L}} e^{\frac{t}{n}\mathcal{K}} \right)^n x - e^{t(\mathcal{L}+\mathcal{K})}x \right\|_{\mathcal{X}} \leq c\frac{t^2}{n} e^{t\omega}\|x\|_{\mathcal{Y}}\,.$$

With respect to Example 2, we verified that two decoupled bosonic quadratic Hamiltonian satisfy Trotter's product formula with the convergence rate n^{-1} in the strong operator topology.

Remark (Variation of time-steps). Note that the proof of the above theorem (also the following theorems) holds even for time-steps that are not equally distributed, thanks to the generality of Proposition 4.1 and the semigroup property of the limiting semigroup. In this context, it suffices for the largest time-step to converge to zero asymptotically faster than order $\frac{1}{\sqrt{n}}$ in the strong topology. This directly affects the overall convergence rate of the product formula.

In the next step, we improve the convergence rate similar to the matrix, in which the symmetrized Lie product formula demonstrates

$$\| \left(e^{\frac{\mathcal{L}}{2n}} e^{\frac{\mathcal{K}}{n}} e^{\frac{\mathcal{L}}{2n}} \right)^n - e^{\mathcal{L}+\mathcal{K}} \|_{\mathcal{X}\to\mathcal{X}} = \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{28}$$

for matrices $\mathcal{L}, \mathcal{K} \in \mathcal{B}(\mathcal{X})$ on a finite dimensional Banach space $\mathcal{X}$. Here, the analysis to achieve a convergence rate of order n^m for an $m \in \mathbb{N}$ can be reduced to the bound

$$\|F(t) - e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{X}} \leq bt^{m+1}\,, \tag{29}$$

where $F(t) := e^{tp_1\mathcal{L}} e^{tp_2\mathcal{K}} e^{tp_3\mathcal{L}} \ldots e^{tp_{m-1}\mathcal{L}} e^{tp_m\mathcal{K}}$ with coefficients $p_1, ..., p_m \in \mathbb{R}$ (see Lem. 4.3). In the special case of the symmetrized Trotter product formula, explicit upper bounds on Equation 29 are proven to apply Lemma 4.3:

Theorem (See Theorem 4.4) *Let $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ be a Banach space such that the closure of $(\mathcal{Y}, \mathcal{L}+\mathcal{K})$ generates a contractive C_0-semigroup in $\mathcal{X}$. If additionally all products of $\mathcal{K}$ and $\mathcal{L}$ up to the third power[2] are relatively $\mathcal{Y}$-bounded and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is $(\mathcal{L}+\mathcal{K})$-stable, then for all $x \in \mathcal{Y}$ and $t \geq 0$ there are $c, \omega \geq 0$ so that*

$$\left\| \left(e^{\frac{t}{2n}\mathcal{K}} e^{\frac{t}{n}\mathcal{L}} e^{\frac{t}{2n}\mathcal{K}} \right)^n x - e^{t(\mathcal{L}+\mathcal{K})}x \right\|_{\mathcal{X}} \leq c\frac{t^3}{n^2} e^{t\omega}\|x\|_{\mathcal{Y}}\,.$$

[2] $\mathcal{L}$, $\mathcal{K}$, $\mathcal{L}^2$, $\mathcal{K}^2$, $\mathcal{L}\mathcal{K}$, $\mathcal{K}\mathcal{L}$, $\mathcal{K}^3$, $\mathcal{K}^2\mathcal{L}$, $\mathcal{K}\mathcal{L}^2$, $\mathcal{L}^3$, $\mathcal{L}^2\mathcal{K}$, $\mathcal{K}\mathcal{L}\mathcal{K}$, $\mathcal{L}\mathcal{K}^2$, $\mathcal{L}\mathcal{K}\mathcal{L}$

One observation of the proof is that higher-order convergence rates need higher order derivatives and thereby a higher regularity of the input state. Utilizing the fact that in [36, Lem. 30] any polynomial in annihilation and creation operator can be bounded by a $\mathcal{W}^k$ for $k \in \mathbb{N}$, the Trotter-Suzuki formula [54] can be applied assuming that the input state is in a regular enough Sobolev space. However, one has to keep in mind that the evolution needs to be time-reversible for $m \geq 3$ (see Eq. 34) as shown in [51].

Finally, we extend the product formula to the time-dependent case. Here, the ideas are roughly the same by exploiting the assumption of a subspace on which the evolution defines a stable function. The limit under consideration is

$$\prod_{j=1}^{n} U(s_j, s_{j-1}) V(s_j, s_{j-1}) \to T(t, s) \qquad \text{for} \qquad n \to \infty\,, \tag{30}$$

where $0 \leq s = s_0 < ... < s_n = t$ is a partition of $[s,t]$, $U(t,s)$ and $V(t,s)$ are two C_0-evolution systems with generator families $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))$ and $(\mathcal{K}_s, \mathcal{D}(\mathcal{K}_s))$, and appropriate assumptions.

Theorem (See Theorem 4.5) *Let $U(t,s)$ and $V(t,s)$ be contractive C_0-evolution systems for $(t,s) \in I_T$ and with generator families $(\mathcal{L}_t, \mathcal{D}(\mathcal{L}_t))_{t\in[0,T]}$ and $(\mathcal{K}_t, \mathcal{D}(\mathcal{K}_t))_{t\in[0,T]}$. Moreover, assume that there is a Banach space $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ so that the closure of $(\mathcal{L}_t + \mathcal{K}_t, \mathcal{Y})_{t\in[0,T]}$ generates a unique contractive C_0-evolution system $T(t,s)$ on $\mathcal{X}$ for $(t,s) \in I_T$. If $\mathcal{L}_s$, $\mathcal{K}_s$, $\mathcal{L}_s\mathcal{L}_t$, $\mathcal{K}_s\mathcal{L}_t$, $\mathcal{L}_s\mathcal{K}_t$, $\mathcal{K}_s\mathcal{K}_t$ are well-defined on $\mathcal{Y}$, relatively $\mathcal{Y}$-bounded, and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is $(\mathcal{L}_t + \mathcal{K}_t)$-stable uniformly for all $(s,t) \in I_T$, in particular $\|T(t,s)\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{(t-s)\omega}$, then there is a $c \geq 0$ so that*

$$\left\|\left(\prod_{j=1}^{n} U(s_j, s_{j-1}) V(s_j, s_{j-1})\right)x - T(t,0)x\right\|_{\mathcal{X}} \leq c\frac{t^2}{n}e^{t\omega}\|x\|_{\mathcal{Y}}$$

for all $x \in \mathcal{Y}$, $t \in [0,T]$, and $s_0 = 0 < \frac{t}{n} < ... < \frac{(n-1)t}{n} < t = s_n$.

Example 2 can be extended to time-dependent Hamiltonians defined by decoupled quadratic Hamiltonians with coefficients depending continuously on time:

Example 3. Let $s \geq 0$ and $(H_1(t), \mathcal{H}_f)_{t\in[s,T]}$ and $(H_2(t), \mathcal{H}_f)_{[s,T]}$ be two decoupled quadratic Hamiltonians of the form Equation 27 with time-dependent coefficients on an m-mode bosonic quantum system. Then, the same bounds as in Example 2 hold true by [36, Lem. 30] and the closures of $(-i[H_1(t), \cdot], \mathcal{T}_f)_{t\in[s,T]}$, $(-i[H_2(t), \cdot], \mathcal{T}_f)_{t\in[s,T]}$ define a Sobolev preserving quantum evolution system [20, Lem. 5.6, Thm. 4.5]. Moreover, the assumptions for the above theorem is satisfied (more details and open bosonic quantum systems are given in Section 4.4).

Similar to the Trotter product formula, the quantum Zeno effect describes an operator product of a contractive projection $P \in \mathcal{B}(\mathcal{X})$, i.e. $P^2 = P$, and a C_0-semigroup $e^{t\mathcal{L}}$:

$$(Pe^{\frac{\mathcal{L}}{n}})^n \to e^{P\mathcal{L}P}P \qquad \text{for} \quad n \to \infty\,. \tag{31}$$

Under certain assumptions the convergence as well as the convergence rate can be achieved in the strong operator topology [37, 4]. We start with the projective Zeno effect without assuming the asymptotic Zeno condition [37, Eq. 6]. Again, we assume in the following that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is a generator of a contractive C_0-semigroup and P a projective contraction.

Proposition 3.1 (See Proposition 5.1) *Assume there is a Banach space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ invariant under P such that the closure of $(P\mathcal{L}P, \mathcal{Y})$ defines a contractive C_0-semigroup, $\mathcal{L}P$ and $\mathcal{L}^2P$ are relatively $\mathcal{Y}$-bounded, and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a $P\mathcal{L}P$-stable subspace, in particular $\|e^{tP\mathcal{L}P}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{t\omega}$. Then, there is a $c_1 \geq 0$ such that for all $t \geq 0$, $x \in \mathcal{Y}$, and $n \in \mathbb{N}$*

$$\|\left(Pe^{t\frac{\mathcal{L}}{n}}P\right)^n Px - e^{tP\mathcal{L}P}Px\|_{\mathcal{X}} \leq \frac{c_1t^2}{n}\left(e^{t\omega}\|Px\|_{\mathcal{Y}} + \|(P\mathcal{L}P)^2x\|_{\mathcal{X}}\right).$$

If we additionally assume that $(P\mathcal{L}P)^2$ is relatively $\mathcal{Y}$-bounded, then there is a c_2 such that

$$\| \left(Pe^{\frac{\mathcal{L}}{n}}P\right)^n Px - e^{P\mathcal{L}P}Px\|_{\mathcal{X}} \leq \frac{c_2(1+e^{t\omega})}{n}\|Px\|_{\mathcal{Y}}\,.$$

This abstract result can be applied to verify the construction of the $X(\theta)$-gate of the bosonic CAT code [34].

Example 4. In 1-mode, we define the Schrödinger CAT-states by

$$|CAT_\alpha^+\rangle \coloneqq \frac{1}{\sqrt{2(1+e^{2|\alpha|^2})}}\left(|\alpha\rangle + |-\alpha\rangle\right) \quad \text{and} \quad |CAT_\alpha^-\rangle \coloneqq \frac{1}{\sqrt{2(1-e^{2|\alpha|^2})}}\left(|\alpha\rangle - |-\alpha\rangle\right)\,,$$

which are orthonormal and play the role of the logical qubits in the bosonic CAT code. Then, the projections onto the code space spanned by $|CAT_\alpha^+\rangle$ and $|CAT_\alpha^-\rangle$ is denoted by $P_\alpha = |CAT_\alpha^+\rangle\langle CAT_\alpha^+| + |CAT_\alpha^-\rangle\langle CAT_\alpha^-|$. Moreover, the logical X gate is $X_\alpha = |CAT_\alpha^+\rangle\langle CAT_\alpha^-| + |CAT_\alpha^-\rangle\langle CAT_\alpha^+|$ so that the rotation around the X-axis is described by

$$X(\theta) = \cos\left(\frac{\theta}{2}\right)(P_\alpha^+ + P_\alpha^-) + i\sin\left(\frac{\theta}{2}\right)X_\alpha\,.$$

Due to the following identity for $H = a + a^\dagger$

$$P_\alpha H P_\alpha = (\alpha + \alpha^\dagger)X_\alpha\,, \tag{32}$$

the above rotation can be achieved by the Zeno effect with $\mathcal{P}_\alpha(\cdot) = P_\alpha \cdot P_\alpha$

$$\| \left(\mathcal{P}_\alpha e^{-i\frac{t}{n}[H,\cdot]}\mathcal{P}_\alpha\right)^n x - e^{-it[P_\alpha H P_\alpha,\cdot]}\mathcal{P}_\alpha x\|_{1\to 1} = \mathcal{O}\left(\frac{t^2}{n}\right)$$

because

$$e^{tiP_\alpha H P_\alpha}P_\alpha = \cos\left(t(\alpha+\alpha^\dagger)\right)P_\alpha + i\sin\left(t(\alpha+\alpha^\dagger)\right)X_\alpha\,.$$

Finally, we extend the result to more general contractions M as well as contractive evolution systems similarly to the case of bounded generators considered in [38]. This extends the result in [37] to evolution systems.

Theorem (See Theorem 5.3) *Let $I_T \ni (t,s) \mapsto V(t,s) \in \mathcal{B}(\mathcal{X})$ be a C_0-evolution system with generator family $(\mathcal{L}_t, \mathcal{D}(\mathcal{L}_t))_{t\in[0,T]}$, $M \in \mathcal{B}(\mathcal{X})$ a contraction, and P a projection satisfying*

$$\|M^n - P\|_{\mathcal{X}\to\mathcal{X}} \leq \delta^n$$

for $\delta \in (0,1)$ and all $n \in \mathbb{N}$. Moreover, we assume the asymptotic Zeno condition:

$$\|PV(t,s)(\mathbb{1}-P)\|_{\mathcal{X}\to\mathcal{X}} \leq (t-s)b \quad \textit{and} \quad \|(\mathbb{1}-P)V(t,s)P\|_{\mathcal{X}\to\mathcal{X}} \leq (t-s)b$$

for $b \geq 0$ and all $t \in [0,T]$. Furthermore, there is a Banach space $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ invariant under P so that the closure of $(P\mathcal{L}_tP, \mathcal{Y})_{t\in[0,T]}$ generates a unique contractive C_0-evolution system $T(t,s)$ for $(t,s) \in I_T$ commuting with P. Moreover, assume that $\mathcal{L}_tP$, $P\mathcal{L}_tP$, and $P\mathcal{L}_tP\mathcal{L}_sP$ are well-defined on $\mathcal{Y}$ and relatively $\mathcal{Y}$-bounded for all $(t,s) \in I_T$, and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a $P\mathcal{L}_tP$-stable subspace, in particular $\|T(t,s)\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{(t-s)\omega}$. Then, there is a $c \geq 0$ so that

$$\Big\|\prod_{j=1}^n MV(s_j, s_{j-1})x - T(t,0)Px\Big\|_{\mathcal{X}} \leq c\frac{t^2}{n}\left(\|x\|_{\mathcal{X}} + e^{t\omega}\|Px\|_{\mathcal{Y}}\right)$$

for all $t \geq 0$, $s_0 = 0 < \frac{t}{n} < \ldots < \frac{t(n-1)}{n} < t = s_n$, $x \in \mathcal{Y}$ and $n \in \mathbb{N}$.

4 Trotter-like product formulas

In this section, we discuss quantitative bounds for Trotter-like product formulas. We begin with the fundamental Trotter product formula [57] and enhance the convergence rate through the Suzuki approach [53]. Additionally, we generalize the involved semigroups to evolution systems. Finally, we delve into bosonic examples, in particular form bosonic Hamiltonian learning, as well as error correction. Most of our proofs rely on a telescopic sum approach known for example from proving the Lie-Trotter product formula [45, 59] and generalized in the following.

Lemma 4.1 *Let $I_T \ni (t,s) \mapsto F(t,s) \in \mathcal{B}(\mathcal{X})$ be a vector-valued map with $\|F(t,s)\|_{\mathcal{X}\to\mathcal{X}} \leq 1$ and $F(s,s) = \mathbb{1}$ for all $(t,s) \in I_T := \{(t,s)\,|\,T \geq t \geq s \geq 0\}$. Moreover, let $T(t,s)$ be an evolution system, then for all $(t,s) \in I_T$*

$$\|\prod_{j=1}^{n} F(s_j,s_{j-1})x - T(t,s)x\|_{\mathcal{X}} \leq n \max_{j\in\{1,\dots,n\}} \{\|\left(F(s_j,s_{j-1}) - T(s_j,s_{j-1})\right) T(s_{j-1},s_0)x\|_{\mathcal{X}}\},$$

for any increasing sequence $s = s_0 < s_1 < \dots < s_{n-1} < s_n = t$. Note that the product over $F(s_j,s_{j-1})$ is in decreasing order from the left to the right.

Proof. The statement directly follows by the following telescopic sum [45, 59]:

$$\prod_{j=1}^{n} F_j x - \prod_{j=1}^{n} T_j x = \sum_{k=1}^{n} \left(\prod_{j=k+1}^{n} F_j\right) (F_k - T_k) \left(\prod_{j=1}^{k-1} T_j\right) x\,, \tag{33}$$

where the products are in decreasing order from the left to the right. Then, the definition $T_j = T(s_j,s_{j-1})$, the semigroup property, i.e.

$$T(t,s) = T(s_n,s_{n-1}) \circ T(s_{n-1},s_{n-2}) \circ \cdots \circ T(s_1,s_0) =: \prod_{j=1}^{n} T_j\,,$$

and the contractivity of $F(t,s)$ combined with the submultiplicativity of the norm finish the proof. □

4.1 Trotter product formula

In the following, we consider a set of assumptions on the generators $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ and $(\mathcal{K},\mathcal{D}(\mathcal{K}))$ such that a quantitative error bound on Equation 24 can be found.

Theorem 4.2 *Let $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ and $(\mathcal{K},\mathcal{D}(\mathcal{K}))$ be generators of two contractive C_0-semigroups and $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ a Banach space such that $\mathcal{Y} \subset \mathcal{D}(\mathcal{L}+\mathcal{K})$ and[3] the closure of $(\mathcal{L}+\mathcal{K},\mathcal{Y})$ generates a contractive C_0-semigroup on $\mathcal{X}$. If additionally $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}^2$, $\mathcal{L}\mathcal{K}$, $\mathcal{K}\mathcal{L}$, $\mathcal{L}^2$ are relatively $\mathcal{Y}$-bounded (see Eq. (5)) and $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$ is $(\mathcal{L}+\mathcal{K})$-stable, in particular $\|e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{t\omega}$, then there is a $c \geq 0$ such that for all $x \in \mathcal{Y}$, $t \geq 0$, and $n \in \mathbb{N}$*

$$\left\|\left(e^{\frac{t}{n}\mathcal{L}}e^{\frac{t}{n}\mathcal{K}}\right)^n x - e^{t(\mathcal{L}+\mathcal{K})}x\right\|_{\mathcal{X}} \leq c\frac{t^2}{n}e^{t\omega}\|x\|_{\mathcal{Y}}\,.$$

Proof. By rescaling the generators, assume $0 \leq s \leq t \leq 1$. Then, we apply Proposition 4.1 to

$$F(t,s) = e^{(t-s)\mathcal{L}}e^{(t-s)\mathcal{K}} \quad T(t,s) = e^{(t-s)(\mathcal{L}+\mathcal{K})}$$

[3] In the following, we omit the assumption $\mathcal{Y} \subset \mathcal{D}(\mathcal{L}+\mathcal{K})$ because it is clear by the notation $(\mathcal{L}+\mathcal{K},\mathcal{Y})$

w.r.t. the sequence $s_0 = 0 < \frac{1}{n} < ... < \frac{n-1}{n} < 1$ so that

$$\| \left(e^{\frac{\mathcal{L}}{n}} e^{\frac{\mathcal{K}}{n}}\right)^n x - e^{\mathcal{L}+\mathcal{K}} x\|_{\mathcal{X}} \leq n \max_{s\in[0,1-1/n]} \| \left(e^{\frac{\mathcal{L}}{n}} e^{\frac{\mathcal{K}}{n}} - e^{\frac{\mathcal{L}+\mathcal{K}}{n}}\right) e^{s(\mathcal{L}+\mathcal{K})} x\|_{\mathcal{X}} .$$

Then, the integral equation (13) shows for $x_s \coloneqq e^{s(\mathcal{L}+\mathcal{K})} x$

$$\begin{aligned} e^{\frac{\mathcal{L}}{n}} e^{\frac{\mathcal{K}}{n}} x_s - e^{\frac{\mathcal{L}+\mathcal{K}}{n}} x_s &= (e^{\frac{\mathcal{L}}{n}} - \mathbb{1}) x_s + e^{\frac{\mathcal{L}}{n}} (e^{\frac{\mathcal{K}}{n}} - \mathbb{1}) x_s - (e^{\frac{\mathcal{L}+\mathcal{K}}{n}} - \mathbb{1}) x_s \\ &= \frac{1}{n} \int_0^1 \left(e^{s_1 \frac{\mathcal{L}}{n}} \mathcal{L} + e^{\frac{\mathcal{L}}{n}} e^{s_1 \frac{\mathcal{K}}{n}} \mathcal{K} - e^{s_1 \frac{\mathcal{L}+\mathcal{K}}{n}} (\mathcal{L}+\mathcal{K}) \right) x_s ds_1 \\ &= \frac{1}{n^2} \int_0^1 \int_0^1 \Big(s_1 e^{s_1 s_2 \frac{\mathcal{L}}{n}} \mathcal{L}^2 + e^{s_2 \frac{\mathcal{L}}{n}} \mathcal{L}\mathcal{K} + s_1 e^{\frac{\mathcal{L}}{n}} e^{s_1 s_2 \frac{\mathcal{K}}{n}} \mathcal{K}^2 \\ &\qquad\qquad - s_1 e^{s_1 s_2 \frac{\mathcal{L}+\mathcal{K}}{n}} (\mathcal{L}+\mathcal{K})^2 \Big) x_s ds_2 ds_1 . \end{aligned}$$

The integral on the right-hand side is well-defined due to the continuity of the integrand in s_1 and s_2 given by the assumptions on the domain, i.e. $x_s \in \mathcal{Y}$. Moreover, the assumption that $\mathcal{L}^2$ is $\mathcal{Y}$-bounded and $\mathcal{Y}$ $(\mathcal{K}+\mathcal{L})$-admissible induces constants $a_{\mathcal{L}^2}, \tilde{c}, \tilde{\omega} \geq 0$ satisfying

$$\|e^{s\frac{\mathcal{L}}{n}} \mathcal{L}^2 x_s\|_{\mathcal{X}} \leq \|\mathcal{L}^2 x_s\|_{\mathcal{X}} \leq a_{\mathcal{L}^2} \|x_s\|_{\mathcal{Y}} \leq a_{\mathcal{L}^2} \tilde{c} e^{s\tilde{\omega}} \|x\|_{\mathcal{Y}} .$$

The same calculation holds for all other terms of the integrand in the last line with appropriate constants $a_{\mathcal{L}\mathcal{K}}, a_{\mathcal{K}^2}, a_{\mathcal{K}\mathcal{L}} \geq 0$ defined by the $\mathcal{Y}$-boundedness. This implies

$$\| \left(e^{\frac{\mathcal{L}}{n}} e^{\frac{\mathcal{K}}{n}}\right)^n x - e^{\mathcal{L}+\mathcal{K}} x\|_{\mathcal{X}} \leq \frac{1}{n} \left(a_{\mathcal{L}^2} + \frac{3}{2} a_{\mathcal{L}\mathcal{K}} + a_{\mathcal{K}^2} + a_{\mathcal{K}\mathcal{L}} \right) \tilde{c} e^{\tilde{\omega}} \|x\|_{\mathcal{Y}} .$$

and finishes the proof of the theorem by $t^2 c = (a_{\mathcal{L}^2} + \frac{3}{2} a_{\mathcal{L}\mathcal{K}} + a_{\mathcal{K}^2} + a_{\mathcal{K}\mathcal{L}})\tilde{c}$ and $t\omega = \tilde{\omega}$. Here, $\mathcal{L}$, $\mathcal{K}$ were rescaled to $t\mathcal{L}$, $t\mathcal{K}$ so that a_A becomes $t^2 a_A$ for all $A \in \{\mathcal{L}^2, \mathcal{K}^2, \mathcal{K}\mathcal{L}, \mathcal{L}\mathcal{K}\}$. □

Remark. Note that in the above proof the assumption that $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a $(\mathcal{K}+\mathcal{L})$-stable Banach space can be weakened to a normed space invariant under the C_0-semigroup $e^{t(\mathcal{L}+\mathcal{K})}$ satisfying $\|e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c} e^{t\omega}$. Together with the other assumptions, the same poof works.

4.2 Trotter-Suzuki product formula

Reconsidering the proof strategy of Theorem 4.2, improving the convergence rate of the product formula reduces to finding an appropriate $F(t)$ so that

$$\|F\Big(\frac{1}{n}\Big) - e^{\frac{\mathcal{L}+\mathcal{K}}{n}}\|_{\mathcal{X}\to\mathcal{X}} = \mathcal{O}\Big(\frac{1}{n^{m+1}}\Big) . \tag{34}$$

In the case of the symmetrized Lie product formula $m = 2$ and $F(t) \coloneqq e^{t\frac{\mathcal{L}}{2}} e^{t\mathcal{K}} e^{t\frac{\mathcal{L}}{2}}$ (see Thm. 4.4). Following this idea, we assume the structure

$$F(t) \coloneqq e^{tp_1 A} e^{tp_2 B} e^{tp_3 A} \cdots e^{tp_{m-1} A} e^{tp_m B}$$

for $p_1, ..., p_m \in \mathbb{R}$. For that, Suzuki proved different schemes to achieve higher order convergence rates (see [23] for an overview) for bounded generators. In the following, we consider a general framework and explicitly demonstrate the symmetrized product formula afterwards.

Lemma 4.3 *Let* $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ *and* $(\mathcal{K}, \mathcal{D}(\mathcal{K}))$ *be generators of two contractive* C_0*-semigroups,* $m \in \mathbb{N}_{\geq 1}$*, and* $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ *a Banach space such that the closure of* $(\mathcal{L}+\mathcal{K}, \mathcal{Y})$ *generates a* C_0*-semigroup on* $\mathcal{X}$*. Moreover, we assume there exists a contractive and strongly continuous map* $t \mapsto F(t)$ *for* $t \geq 0$ *such that*

$$\|F(t) - e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{X}} \leq bt^{m+1}\,. \tag{35}$$

If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *is* $(\mathcal{L}+\mathcal{K})$*-stable, in particular* $\|e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{t\omega}$*, then for all* $x \in \mathcal{Y}$ *and* $t \geq 0$

$$\left\|\left(F\Big(\frac{t}{n}\Big)\right)^n x - e^{t(\mathcal{L}+\mathcal{K})}x\right\|_{\mathcal{X}} \leq \frac{t^{m+1}}{n^m}\tilde{c}be^{t\omega}\|x\|_{\mathcal{Y}}\,.$$

Proof. Similar to the proof of Theorem 4.2, we apply Proposition 4.1 to

$$F(t,s) = F(t-s) \quad T(t,0s = e^{(t-s)(\mathcal{L}+\mathcal{K})}$$

w.r.t. the sequence $s_0 = 0 < t\frac{1}{n} < \dots < t\frac{n-1}{n} < s_n = t$ so that

$$\begin{aligned}
\|\left(F\Big(\frac{t}{n}\Big)\right)^n x - e^{t(\mathcal{L}+\mathcal{K})}x\|_{\mathcal{X}} &\leq n \max_{s\in[0,t(1-1/n)]} \|\left(F\Big(\frac{t}{n}\Big) - e^{\frac{t}{n}(\mathcal{L}+\mathcal{K})}\right)e^{s(\mathcal{L}+\mathcal{K})}x\|_{\mathcal{X}} \\
&\leq \frac{t^{m+1}}{n^m}\tilde{c}be^{t\omega}\|x\|_{\mathcal{Y}}
\end{aligned}$$

where we used the assumption in Equation 35 and that $\mathcal{Y}$ is $(\mathcal{L}+\mathcal{K})$-stable in the last step. □

On quantum lattice system in the infinite volume limit, [2] shows that Equation 35 is satisfied and the above convergence rate in the strong topology is achieved. Next, we apply the above structure to the symmetrized Trotter product formula (or second order Trotter-Suzuki product formula) and give assumptions such that quadratic convergence is achieved.

Theorem 4.4 *Let* $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ *and* $(\mathcal{K}, \mathcal{D}(\mathcal{K}))$ *be generators of two contractive* C_0*-semigroups and* $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ *a Banach space such that the closure of* $(\mathcal{Y}, \mathcal{L}+\mathcal{K})$ *generates a contractive* C_0*-semigroup in* $\mathcal{X}$*. If additionally all products of* $\mathcal{K}$ *and* $\mathcal{L}$ *up to the third power*[4] *are relatively* $\mathcal{Y}$*-bounded and* $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *is* $(\mathcal{L}+\mathcal{K})$*-stable, in particular* $\|e^{t(\mathcal{L}+\mathcal{K})}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{t\omega}$*, then for all* $x \in \mathcal{Y}$ *and* $t \geq 0$ *there is a* $c \geq 0$ *so that*

$$\left\|\left(e^{\frac{t}{2n}\mathcal{K}}e^{\frac{t}{n}\mathcal{L}}e^{\frac{t}{2n}\mathcal{K}}\right)^n x - e^{t(\mathcal{L}+\mathcal{K})}x\right\|_{\mathcal{X}} \leq c\frac{t^3}{n^2}e^{t\omega}\|x\|_{\mathcal{Y}}\,.$$

Proof. Similarly to the proof of Theorem 4.2, we use the integral equation (13) to prove a

[4] $\mathcal{L}$, $\mathcal{K}$, $\mathcal{L}^2$, $\mathcal{K}^2$, $\mathcal{L}\mathcal{K}$, $\mathcal{K}\mathcal{L}$, $\mathcal{K}^3$, $\mathcal{K}^2\mathcal{L}$, $\mathcal{K}\mathcal{L}^2$, $\mathcal{L}^3$, $\mathcal{L}^2\mathcal{K}$, $\mathcal{K}\mathcal{L}\mathcal{K}$, $\mathcal{L}\mathcal{K}^2$, $\mathcal{L}\mathcal{K}\mathcal{L}$

bound like Equation 35 to apply Lemma 4.3. For $x \in \mathcal{Y}$

$$
\begin{aligned}
&e^{\frac{\mathcal{K}}{2n}} e^{\frac{\mathcal{L}}{n}} e^{\frac{\mathcal{K}}{2n}} x - e^{\frac{\mathcal{L}+\mathcal{K}}{n}} x \\
&= \frac{1}{n} \int_0^1 e^{s_1 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}}{2} x + e^{\frac{\mathcal{K}}{2n}} e^{s_1 \frac{\mathcal{L}}{n}} \mathcal{L} x + e^{\frac{\mathcal{K}}{2n}} e^{\frac{\mathcal{L}}{n}} e^{s_1 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}}{2} x - e^{s_1 \frac{\mathcal{L}+\mathcal{K}}{n}} (\mathcal{L}+\mathcal{K}) x ds_1 \\
&= \frac{1}{n^2} \int_0^1 \int_0^1 s_1 e^{s_1 s_2 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^2}{4} x + e^{s_2 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}}{2} \mathcal{L} x + s_1 e^{\frac{\mathcal{K}}{2n}} e^{s_1 s_2 \frac{\mathcal{L}}{n}} \mathcal{L}^2 x - s_1 e^{s_1 s_2 \frac{\mathcal{L}+\mathcal{K}}{n}} (\mathcal{L}+\mathcal{K})^2 x \\
&\qquad + e^{s_2 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^2}{4} x + e^{\frac{\mathcal{K}}{2n}} e^{s_2 \frac{\mathcal{L}}{n}} \mathcal{L} \frac{\mathcal{K}}{2} x + s_1 e^{\frac{\mathcal{K}}{2n}} e^{\frac{\mathcal{L}}{n}} e^{s_1 s_2 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^2}{4} x ds_2 ds_1 \\
&= \frac{1}{n^3} \int_0^1 \int_0^1 \int_0^1 s_1^2 s_2 e^{s_1 s_2 s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^3}{8} x + s_2 e^{s_2 s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^2}{4} \mathcal{L} x + s_1 e^{s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}}{2} \mathcal{L}^2 x + s_1^2 s_2 e^{\frac{\mathcal{K}}{2n}} e^{s_1 s_2 s_3 \frac{\mathcal{L}}{n}} \mathcal{L}^3 x \\
&\qquad + s_2 e^{s_2 s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^3}{8} x + e^{s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}}{2} \mathcal{L} \frac{\mathcal{K}}{2} x + s_2 e^{\frac{\mathcal{K}}{2n}} e^{s_2 s_3 \frac{\mathcal{L}}{n}} \mathcal{L}^2 \frac{\mathcal{K}}{2} x \\
&\qquad + s_1 e^{s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^3}{8} x + s_1 e^{\frac{\mathcal{K}}{2n}} e^{s_3 \frac{\mathcal{L}}{n}} \mathcal{L} \frac{\mathcal{K}^2}{4} x + s_1^2 s_2 e^{\frac{\mathcal{K}}{2n}} e^{\frac{\mathcal{L}}{n}} e^{s_1 s_2 s_3 \frac{\mathcal{K}}{2n}} \frac{\mathcal{K}^3}{8} x \\
&\qquad - s_1^2 s_2 e^{s_1 s_2 s_3 \frac{\mathcal{L}+\mathcal{K}}{n}} (\mathcal{L}+\mathcal{K})^3 x \, ds_3 ds_2 ds_1 \,.
\end{aligned}
$$

Note that the input $x \in \mathcal{Y}$ plays the role of $x_s = e^{s(\mathcal{L}+\mathcal{K})} x$ in the proof of Theorem 4.2. Again, the integral is well-defined (see [12, Lem. II.1.3]) and the relative boundedness

$$
\|A\|_{\mathcal{Y} \to \mathcal{X}} \leq a_A
$$

for $A \in \{\mathcal{K}^3, \mathcal{K}^2\mathcal{L}, \mathcal{K}\mathcal{L}^2, \mathcal{L}^3, \mathcal{K}\mathcal{L}\mathcal{K}, \mathcal{L}\mathcal{K}\mathcal{L}, \mathcal{L}^2\mathcal{K}, \mathcal{L}\mathcal{K}^2\}$ shows

$$
\begin{aligned}
&\|e^{\frac{\mathcal{K}}{2n}} e^{\frac{\mathcal{L}}{n}} e^{\frac{\mathcal{K}}{2n}} x - e^{\frac{\mathcal{L}+\mathcal{K}}{n}} x\|_{\mathcal{X}} \\
&\qquad \leq \frac{1}{n^3} \int_0^1 \int_0^1 \int_0^1 \Big(\frac{2 s_1^2 s_2 + s_2 + s_1}{8} \|\mathcal{K}^3 x\|_{\mathcal{X}} + \frac{s_2}{4} \|\mathcal{K}^2 \mathcal{L} x\|_{\mathcal{X}} + \frac{s_1}{2} \|\mathcal{K}\mathcal{L}^2 x\|_{\mathcal{X}} + s_1^2 s_2 \|\mathcal{L}^3 x\|_{\mathcal{X}} \\
&\qquad\qquad + \frac{1}{4} \|\mathcal{K}\mathcal{L}\mathcal{K} x\|_{\mathcal{X}} + \frac{s_2}{2} \|\mathcal{L}^2 \mathcal{K} x\|_{\mathcal{X}} + \frac{s_1}{4} \|\mathcal{L}\mathcal{K}^2 x\|_{\mathcal{X}} + s_1^2 s_2 \|(\mathcal{L}+\mathcal{K})^3 x\|_{\mathcal{X}} \Big) ds_3 ds_2 ds_1 \\
&\qquad \leq \frac{1}{6n^3} \Big(a_{\mathcal{K}^3} + \frac{3}{4} a_{\mathcal{K}^2\mathcal{L}} + \frac{3}{2} a_{\mathcal{K}\mathcal{L}^2} + a_{\mathcal{L}^3} + \frac{3}{2} a_{\mathcal{K}\mathcal{L}\mathcal{K}} + \frac{3}{2} a_{\mathcal{L}^2\mathcal{K}} + \frac{3}{4} a_{\mathcal{L}\mathcal{K}^2} \Big) \|x\|_{\mathcal{Y}} \\
&\qquad\quad + \frac{1}{6n^3} \Big(a_{\mathcal{L}^3} + a_{\mathcal{L}^2\mathcal{K}} + a_{\mathcal{L}\mathcal{K}\mathcal{L}} + a_{\mathcal{L}\mathcal{K}^2} + a_{\mathcal{K}\mathcal{L}^2} + a_{\mathcal{K}\mathcal{L}\mathcal{K}} + a_{\mathcal{K}^2\mathcal{L}} + a_{\mathcal{K}^3} \Big) \|x\|_{\mathcal{Y}} \\
&\qquad \leq \frac{1}{6n^3} \Big(2a_{\mathcal{K}^3} + \frac{7}{4} a_{\mathcal{K}^2\mathcal{L}} + \frac{5}{2} a_{\mathcal{K}\mathcal{L}^2} + 2a_{\mathcal{L}^3} + \frac{5}{2} a_{\mathcal{K}\mathcal{L}\mathcal{K}} + \frac{5}{2} a_{\mathcal{L}^2\mathcal{K}} + \frac{7}{4} a_{\mathcal{L}\mathcal{K}^2} \Big) \|x\|_{\mathcal{Y}} \,.
\end{aligned}
$$

By rescaling the time and the above calculation, there is a $b \geq 0$

$$
\|e^{\frac{t}{2n}\mathcal{K}} e^{\frac{t}{n}\mathcal{L}} e^{\frac{t}{2n}\mathcal{K}} x - e^{\frac{t}{n}(\mathcal{L}+\mathcal{K})} x\|_{\mathcal{X}} \leq b \frac{t^3}{n^3} \|x\|_{\mathcal{Y}}
$$

so that Lemma 4.3 finishes the proof of the statement. □

4.3 Time-dependent Trotter product formula

Finally, we prove a Trotter product formula for evolution systems of the form

$$
\prod_{j=1}^{n} U(s_j, s_{j-1}) V(s_j, s_{j-1}) \to T(t,s) \qquad \text{for} \qquad n \to \infty \,,
$$

where $0 \leq s = s_0 < \dots < s_n = t$ is a partition of $[s,t]$, $U(t,s)$ and $V(t,s)$ are two differentiable C_0-evolution systems with generator families $(\mathcal{L}_s, \mathcal{D}(\mathcal{L}_s))$ and $(\mathcal{K}_s, \mathcal{D}(\mathcal{K}_s))$, and appropriate assumptions.

Theorem 4.5 *Let $U(t,s)$ and $V(t,s)$ be contractive C_0-evolution system for $(t,s)\in I_T$ and with generator families $(\mathcal{L}_t,\mathcal{D}(\mathcal{L}_t))_{t\in[0,T]}$ and $(\mathcal{K}_t,\mathcal{D}(\mathcal{K}_t))_{t\in[0,T]}$. Moreover, assume that there is a Banach space $(\mathcal{Y}\subset\mathcal{X},\|\cdot\|_{\mathcal{Y}})$ so that the closure of $(\mathcal{L}_t+\mathcal{K}_t,\mathcal{Y})_{t\in[0,T]}$ generates a unique contractive C_0-evolution systems $T(t,s)$ on $\mathcal{X}$ for $(t,s)\in I_T$. If $\mathcal{L}_s$, $\mathcal{K}_s$, $\mathcal{L}_s\mathcal{L}_t$, $\mathcal{K}_s\mathcal{L}_t$, $\mathcal{L}_s\mathcal{K}_t$, $\mathcal{K}_s\mathcal{K}_t$ are well-defined on $\mathcal{Y}$, relatively $\mathcal{Y}$-bounded, and $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$ is $(\mathcal{L}_t+\mathcal{K}_t)$-stable uniformly for all $(s,t)\in I_T$, in particular $\|T(t,s)\|_{\mathcal{Y}\to\mathcal{Y}}\leq\tilde{c}e^{(t-s)\omega}$, then there is a $c\geq 0$ so that*

$$\left\|\Big(\prod_{j=1}^{n}U(s_j,s_{j-1})V(s_j,s_{j-1})\Big)x-T(t,0)x\right\|_{\mathcal{X}}\leq c\frac{t^2}{n}e^{t\omega}\|x\|_{\mathcal{Y}}$$

for all $x\in\mathcal{Y}$, $t\in[0,T]$, and $s_0=0<\frac{t}{n}<\ldots<\frac{(n-1)t}{n}<t=s_n$.

Proof. First, we rescale the time parameter so that $0\leq s\leq t\leq 1$ and apply Proposition 4.1 to

$$F(t,s)=U(t,s)V(t,s)$$

with respect to the sequence $s_0=0<\frac{1}{n}<\ldots<\frac{n-1}{n}<1=s_n$. Then, we achieve

$$\begin{aligned}&\|\Big(\prod_{j=1}^{n}U(s_j,s_{j-1})V(s_j,s_{j-1})\Big)x-T(1,0)x\|_{\mathcal{X}}\\&\qquad\leq n\max_{j\in\{1,\ldots,n\}}\|\Big(U(s_j,s_{j-1})V(s_j,s_{j-1})-T(s_j,s_{j-1})\Big)T(s_{j-1},s_0)x\|_{\mathcal{X}}\,.\end{aligned}$$

Then, the integral equation (13) as well as Lemma 2.1 show with $s_j(\tau)=s_j+\frac{1-\tau}{n}$

$$\begin{aligned}&U(s_j,s_{j-1})V(s_j,s_{j-1})\tilde{x}-T(s_j,s_{j-1})\tilde{x}\\&=(U(s_j,s_{j-1})-\mathbb{1})\tilde{x}+U(s_j,s_{j-1})(V(s_j,s_{j-1})-\mathbb{1})\tilde{x}-(T(s_j,s_{j-1})-\mathbb{1})\tilde{x}\\&=\frac{1}{n}\int_0^1\frac{\partial}{\partial\tau_1}\Big(U(s_j,s_{j-1}(\tau_1))\tilde{x}+U(s_j,s_{j-1})V(s_j,s_{j-1}(\tau_1))\tilde{x}-T(s_j,s_{j-1}(\tau_1))\tilde{x}\Big)d\tau_1\\&=\frac{1}{n}\int_0^1\Big(U(s_j,s_{j-1}(\tau_1))\mathcal{L}_{s_{j-1}(\tau_1)}\tilde{x}+U(s_j,s_{j-1})V(s_j,s_{j-1}(\tau_1))\mathcal{K}_{s_{j-1}(\tau_1)}\tilde{x}\\&\qquad\qquad-T(s_j,s_{j-1}(\tau_1))(\mathcal{L}_{s_{j-1}(\tau_1)}+\mathcal{K}_{s_{j-1}(\tau_1)})\tilde{x}\Big)d\tau_1\\&=\frac{1}{n^2}\iint_0^1\Big(\tau_1U(s_j,s_{j-1}(\tau_1\tau_2))\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}\mathcal{L}_{s_{j-1}(\tau_1)}\tilde{x}+U(s_j,s_{j-1}(\tau_2))\mathcal{L}_{s_{j-1}(\tau_2)}\mathcal{K}_{s_{j-1}(\tau_1)}\tilde{x}\\&\qquad\qquad+\tau_1U(s_j,s_{j-1})V(s_j,s_{j-1}(\tau_1\tau_2))\mathcal{K}_{s_{j-1}(\tau_1\tau_2)}\mathcal{K}_{s_{j-1}(\tau_1)}\tilde{x}\\&\qquad\qquad-\tau_1T(s_j,s_{j-1}(\tau_1\tau_2))(\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}+\mathcal{K}_{s_{j-1}(\tau_1\tau_2)})(\mathcal{L}_{s_{j-1}(\tau_1)}+\mathcal{K}_{s_{j-1}(\tau_1)})\tilde{x}\Big)d\tau_2d\tau_1\,,\end{aligned}\tag{36}$$

where $\tilde{x}:=T(s_{j-1},s_0)x$. Next, we use the relative boundedness assumptions as well as the property that $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$ defines an $(\mathcal{L}_s+\mathcal{K}_s)$-stable subspace to show

$$\|\mathcal{L}_{s_{j-1}(\tau_1)}\tilde{x}\|_{\mathcal{X}}\leq b\|T(s_{j-1},s_0)x\|_{\mathcal{Y}}\leq b\,\tilde{c}\,e^{(s_{j-1}-s_0)\omega}\|x\|_{\mathcal{Y}}$$

and

$$\begin{aligned}\|U(s_j,s_{j-1}(\tau_1\tau_2))\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}\mathcal{L}_{s_{j-1}(\tau_1)}\tilde{x}\|_{\mathcal{X}}&\leq\|\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}\mathcal{L}_{s_{j-1}(\tau_1)}T(s_{j-1},s_0)x\|_{\mathcal{X}}\\&\leq b\|T(s_{j-1},s_0)x\|_{\mathcal{Y}}\\&\leq b\,\tilde{c}\,e^{(s_{j-1}-s_0)\omega}\|x\|_{\mathcal{Y}}\end{aligned}$$

as well as similar bounds for all the other terms coming up in Equation 36. These two bounds finish the proof of the error bound of the statement. □

4.4 Application: Bosonic strongly continuous semigroups

In this section, we focus on bosonic C_0-semigroups defined on the space of self-adjoint trace-class operators on the Fock space (see Sec. 2.2). Specially, we consider generators admitting the following GKSL structure

$$\mathcal{L} : \mathcal{T}_f \to \mathcal{T}_f \quad x \mapsto \mathcal{L}(x) = -i[H, x] + \sum_{j=1}^{K} L_j x L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, x\} \tag{37}$$

with $K \in \mathbb{N}$, $L_j := p_j(a_1, a_1^\dagger, ...)$ for all $j \in \{1, ..., m\}$, and a symmetric $H := p_H(a_1, a_1^\dagger, ...)$. Note that in the time-dependent case the coefficients would additionally depend continuously on time. Assuming Equation 22, i.e. there are sequences $(\mathbf{k}_r)_{r\in\mathbb{N}} \subset \mathbb{N}^m$ diverging coordinate-wise to infinity and $(\omega_{\mathbf{k}_r})_{r\in\mathbb{N}}$ so that

$$\operatorname{tr}\Big[\mathcal{W}^{\mathbf{k}_r}\mathcal{L}(x)\Big] \leq \omega_{\mathbf{k}_r} \operatorname{tr}\Big[\mathcal{W}^{\mathbf{k}_r}(x)\Big]$$

for all positive semi-definite $x \in \mathcal{T}_f$, not only shows that the closure of $(\mathcal{L}, \mathcal{T}_f)$ generates a quantum dynamical C_0-semigroup, but also that it is Sobolev preserving, i.e.

$$\|T_t\|_{W^{\mathbf{k}_r,1} \to W^{\mathbf{k}_r,1}} \leq e^{\omega_{\mathbf{k}_r} t} \tag{38}$$

for all $t \geq 0$. Moreover, the above inequality holds not only for the points of $\mathbf{k}_r$ but can be extended by interpolation (for more details [20, Thm. 4.4, Lem. E.4]). Note that $W^{\mathbf{k}_r,1}$ is a $\mathcal{L}$-admissible subspace and plays exactly the role of the Banach space $\mathcal{Y}$ discussed in the statements before. One interesting class of examples for Sobolev-preserving semigroups is this of decoupled quadratic Hamiltonians with an additional polynomial in the local number operators, i.e.

$$H = \sum_{i\in\{1,..,m\}} \sum_{k_i+\ell_i\leq 2} \lambda_{k_i\ell_i}(a_i^\dagger)^{k_i} a_i^{\ell_i} + p(N_1, ..., N_m) \tag{39}$$

defined on $\mathcal{H}_f$ and with $\lambda_{k_i\ell_i} = \overline{\lambda}_{\ell_i k_i} \in \mathbb{C}$ as well as $p \in \mathbb{R}[x_1, ..., x_m]$. This example can be eassily extended to C_0-evolution system by assuming that $\lambda_{k_i\ell_i}$ depends continuously on time for $t \in [0, T]$. Then, the operator family $(-i[H(t), \cdot], \mathcal{T}_f)_{t\in[0,T]}$ generates a unique C_0-evolution system (see proof of[20, Lem. 5.6] and [20, Thm. 4.4-4.5]).

Another example is the generator of the Ornstein-Uhlenbeck semigroup which is defined as

$$\mathcal{L}_{\mathrm{qOU}} = \lambda^2\mathcal{L}[a] + \mu^2\mathcal{L}[a^\dagger] \tag{40}$$

for $\mu, \lambda \geq 0$. In [20, Sec. 5.1], we showed that this semigroup is Sobolev-preserving as well.

Generalizing the first term in the above generator, one achieves the so called l-photon driven dissipation generated by

$$\mathcal{L}[a^l - \alpha^l] \tag{41}$$

for $\alpha \in \mathbb{C}$ and $l \in \mathbb{N}$. This operator is used in bosonic error correction codes and satisfies the moment stability bound (22) as well. Therefore, it generates a Sobolev preserving quantum Markov semigroup (see [20, Sec. 5.2]).

Besides the property of the semigroups to be Sobolev-preserving, the finite degree assumption on $\mathcal{L}$ in Equation 37 implies by [20, Lem. E3] the existence of a $\mathbf{k} \in \mathbb{N}^m$ and $b \geq 0$ such that

$$\|\mathcal{L}(x)\|_1 \leq b\|x\|_{W^{\mathbf{k},1}} . \tag{42}$$

With the help of these two properties, we can apply the above theorems to bosonic Sobolev preserving C_0-semigroups. For that, let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ and $(\mathcal{K}, \mathcal{D}(\mathcal{K}))$ be generators of contractive

C_0-semigroups on $\mathcal{T}_{1,\mathrm{sa}}$ satisfying Equation 37. Additionally, assume that the closure of $(\mathcal{L}+\mathcal{K},\mathcal{T}_f)$ generates a Sobolev preserving, which satisfies Equation 38 in particular. This follows directly from [20] if Equation 22 is satisfied. Then, there is a $\mathbf{k}$ and $b\geq 0$ such that Theorem 4.5 and 4.4 show

$$\begin{aligned}\left\|\left(e^{\frac{t}{n}\mathcal{L}}e^{\frac{t}{n}\mathcal{K}}\right)^n x-e^{t(\mathcal{L}+\mathcal{K})}x\right\|_{\mathcal{X}}&\leq b\frac{t^2}{n}e^{t\omega_{2\mathbf{k}}}\|x\|_{\mathcal{W}^{2\mathbf{k},1}}\\ \left\|\left(e^{\frac{t}{2n}\mathcal{K}}e^{\frac{t}{n}\mathcal{L}}e^{\frac{t}{2n}\mathcal{K}}\right)^n x-e^{t(\mathcal{L}+\mathcal{K})}x\right\|_{\mathcal{X}}&\leq b\frac{t^3}{n^2}e^{t\omega_{3\mathbf{k}}}\|x\|_{\mathcal{W}^{3\mathbf{k},1}}\,.\end{aligned}$$

Here $\mathcal{K}$ and $\mathcal{L}$ can be for example quadratic Hamiltonian of the form Equation 39 or one of the other examples mentioned above. The constant $\mathbf{k}$ coming up in the above bound can be understood as the component-wise maximum of $\mathbf{k}$'s satisfying Equation 42 for $\mathcal{L}$ and $\mathcal{K}$. Then, concatenations of $\mathcal{L}$ and $\mathcal{K}$ simply accumulate in the regularity $\mathbf{k}$ of the Sobolev space — we get $2\mathbf{k}$ or $3\mathbf{k}$.

An interesting observation is that only the limiting semigroup generated by $\overline{\mathcal{L}+\mathcal{K}}$ needs to be Sobolev preserving, which enables the scheme to be used as a "regularization" method in the sense of Sobolev preserving semigroups. More explicitly, many semigroups even satisfy a stronger assumption than Equation 22, which is

$$\operatorname{tr}\left[\mathcal{W}^{\mathbf{k}_r}\mathcal{L}(x)\right]\leq-\mu_{\mathbf{k}_r}\operatorname{tr}\left[\mathcal{W}^{\mathbf{k}_r+\mathbf{l}}(x)\right]+c_{\mathbf{k}_r} \tag{43}$$

for a sequence $\mathbf{k}_r$ converging component-wise to infinity, $\mathbf{l}\in\mathbb{N}^m$ depending highly on the structure of the generator $\mathcal{L}$, and constants $\mu_{\mathbf{k}_r},c_{\mathbf{k}_r}\geq 0$. This stronger assumption not only allows us to improve the bound Equation 38 for states to a constant scaling in t (see [20, Prop. 5.1]), but also to bound all other terms which are not of leading order by a constant. For example, we can show that for any Hamiltonian defined by $H=p_H(a_1,a_1^\dagger,...,a_m,a_m^\dagger)$ there is a $l\in\mathbb{N}$ so that

$$-i[H,\cdot]+\mathcal{L}[a^l-\alpha^l](\cdot) \tag{44}$$

generates a Sobolev preserving semigroup and, in particular, it satisfies Equation 43. This idea is for example used in the recent bosonic Hamiltonian Learning scheme [36], in which a similar method is used to derive Lieb-Robinson bounds.

The same reasoning translates to the time-dependent case as shortly mentioned before in the case of decoupled quadratic Hamiltonians. Here, the coefficients of the polynomials in Equation 37 depend continuously on time $t\in[0,T]$. Then, a similar statement as Theorem 2.4 can be proven if Equation 22 is satisfied with a time-independent right-hand side (see [20, Thm. 4.5]). Assuming that all involved evolution systems are well-defined and of GKSL form (37), the assumption that the Trotter-limit is again Sobolev preserving implies

$$\left\|\prod_{j=1}^{n}U(s_{j+1},s_j)V(s_{j+1},s_j)x-T(t,s)x\right\|_{\mathcal{X}}\leq c\frac{t^2}{n}e^{t\omega}\|x\|_{W^{2\mathbf{k},1}}$$

for a $c,\omega\geq 0$, $\mathbf{k}\in\mathbb{N}^m$ and all $t\in[0,T]$, $x\in W^{2\mathbf{k},1}$ by Theorem 4.5. Trotter's product formula for time-independent and time-dependent quantum Markov semigroups play a key role in the simulation of certain Hamiltonian. For example, it was shown that local Hamiltonians can be simulated efficiently [32] by using a Trotter product formula. As mentioned in the introduction, the idea is to decompose a Hamiltonian or Lindbladian in simpler for example local terms. It is important to mention that it is not clear yet how to utilize information propagation bounds to achieve a bound dependent of the system size (see [36]). In the case of evolutions systems, the question of efficient simulation is directly related to adiabatic quantum computation proposed by Feynman [19] and developed by [32] and many other works like [18, 17]. In bosonic systems, similar protocols are constructed and the present work can be utilized to find explicit error bounds on the Trotter product formula (see for example [50]).

5 Quantum Zeno effect

As mentioned, the quantum Zeno effect is an operator product formula of a contractive projection $P \in \mathcal{B}(\mathcal{X})$ and a C_0-semigroup $e^{t\mathcal{L}}$:

$$(Pe^{\frac{\mathcal{L}}{n}})^n \to e^{P\mathcal{L}P}P \qquad \text{for} \quad n \to \infty\,.$$

We start with the projective Zeno effect without assuming the asymptotic Zeno condition [37, Eq. 6]. Then, we extend the result to more general contractions M as in [37, 4]. Finally, we apply the abstract theory again to Sobolev preserving semigroups, in particular, to an example from bosonic continuous error correction.

5.1 Projective Zeno effect

We start with the simplest case presented in Equation 31 where a C_0-semigroup is interrupted by a contractive projection P.

Proposition 5.1 *Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ be a generator of a contractive C_0-semigroup and P a projective contraction. Assume there is a Banach space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ invariant under P such that the closure of $(P\mathcal{L}P, \mathcal{Y})$ defines a contractive C_0-semigroup, $\mathcal{L}P$ and $\mathcal{L}^2P$ are relatively $\mathcal{Y}$-bounded, and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a $P\mathcal{L}P$-stable subspace, in particular $\|e^{tP\mathcal{L}P}\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{t\omega}$. Then, there is a $c_1 \geq 0$ such that for all $t \geq 0$, $x \in \mathcal{Y}$, and $n \in \mathbb{N}$*

$$\| \left(Pe^{t\frac{\mathcal{L}}{n}}P\right)^n Px - e^{tP\mathcal{L}P}Px\|_{\mathcal{X}} \leq \frac{c_1 t^2}{n}\left(e^{t\omega}\|Px\|_{\mathcal{Y}} + \|(P\mathcal{L}P)^2 x\|_{\mathcal{X}}\right).$$

If we additionally assume that $(P\mathcal{L}P)^2$ is relatively $\mathcal{Y}$-bounded, then there is a c_2 such that

$$\| \left(Pe^{\frac{\mathcal{L}}{n}}P\right)^n Px - e^{P\mathcal{L}P}Px\|_{\mathcal{X}} \leq \frac{c_2(1+e^{t\omega})}{n}\|Px\|_{\mathcal{Y}}\,.$$

Proof. By rescaling the generator $\mathcal{L}$, we assume $t \in [0,1]$. Then, we apply Proposition 4.1 to

$$F(t,s) = Pe^{(t-s)\mathcal{L}}P \quad T(t,s) = e^{(t-s)P\mathcal{L}P}$$

defined on the space $P\mathcal{X}$ on which P is the identity operator and with respect to the sequence $s_0 = 0 < \frac{1}{n} < \ldots < \frac{n-1}{n} < 1 = s_n$ so that

$$\| \left(Pe^{\frac{\mathcal{L}}{n}}P\right)^n Px - e^{P\mathcal{L}P}Px\|_{\mathcal{X}} \leq n \max_{s\in[0,1-1/n]} \| \left(Pe^{\frac{\mathcal{L}}{n}}P - e^{\frac{P\mathcal{L}P}{n}}\right) e^{sP\mathcal{L}P}Px\|_{\mathcal{X}}\,.$$

Then, we use $e^{P\mathcal{L}P}Px = Pe^{P\mathcal{L}P}Px$ and the integral equation (13) shows

$$Pe^{\frac{\mathcal{L}}{n}}Px_s - e^{\frac{P\mathcal{L}P}{n}}Px_s = \frac{1}{n^2}\int_0^1\int_0^1 \left(s_1Pe^{s_1s_2\frac{\mathcal{L}}{n}}\mathcal{L}^2P - s_1e^{s_1s_2P\frac{P\mathcal{L}P}{n}}(P\mathcal{L}P)^2\right)x_s ds_2 ds_1\,,$$

where $x_s := e^{sP\mathcal{L}P}Px$. In a first step, we use the contractivity of the involved semigroups and of P to show

$$\|\int_0^1\int_0^1 s_1Pe^{s_1s_2\frac{\mathcal{L}}{n}}\mathcal{L}^2Px_s ds_2 ds_1\|_{\mathcal{X}} \leq \int_0^1\int_0^1 s_1\|\mathcal{L}^2Px_s\|_{\mathcal{X}} ds_2 ds_1 = \frac{1}{2}\|\mathcal{L}^2Px_s\|_{\mathcal{X}}\,.$$

Then, the assumptions that $\mathcal{L}^2P$ is relatively $\mathcal{Y}$-bounded, i.e. $\|\mathcal{L}^2P\|_{\mathcal{Y}\to\mathcal{X}} < a_1$ for an $a_1 \geq 0$ and $\mathcal{Y}$ is a $P\mathcal{L}P$-stable subspace, in particular $\|e^{t\mathcal{L}}\|_{\mathcal{Y}\to\mathcal{Y}} \leq b_1 e^{t\omega}$ for a $b_1, \omega \geq 0$ are utilized to show

$$\|\int_0^1\int_0^1 s_1 P e^{s_1 s_2 \frac{\mathcal{L}}{n}} \mathcal{L}^2 P x_s ds_2 ds_1\|_{\mathcal{X}} \leq \frac{1}{2}\|\mathcal{L}^2 P x_s\|_{\mathcal{X}} \leq \frac{a_1}{2}\|e^{sP\mathcal{L}P}Px\|_{\mathcal{Y}} \leq \frac{a_1 b_1}{2} e^{\omega}\|Px\|_{\mathcal{Y}}\,. \tag{45}$$

Therefore, $c_1 = \frac{a_1 b_1}{2}$ and rescaling of t implies

$$\|\left(Pe^{t\frac{\mathcal{L}}{n}}P\right)^n Px - e^{tP\mathcal{L}P}Px\|_{\mathcal{X}} \leq \frac{c_1 t^2}{n}\left(e^{t\omega}\|Px\|_{\mathcal{Y}} + \|(P\mathcal{L}P)^2 x\|_{\mathcal{X}}\right).$$

If $(P\mathcal{L}P)^2$ is additionally relatively $\mathcal{Y}$-bounded with constants a_2, we directly achieve

$$\|\left(Pe^{\frac{\mathcal{L}}{n}}P\right)^n Px - e^{P\mathcal{L}P}Px\|_{\mathcal{X}} \leq \frac{c_2(1+e^{t\omega})}{n}\|Px\|_{\mathcal{Y}}\,,$$

with $c_2 = \max\{c_1, \frac{a_2}{2}\}$. This finishes the proof of the theorem. □

Remark 5. To get a better understanding of the above result, we compare it to Theorem I in [37]. Assume that $(P\mathcal{L}P, \mathcal{D}(P\mathcal{L}P))$ generates a contractive C_0-semigroup. Then, $\mathcal{Y} = \mathcal{D}((P\mathcal{L}P)^2) \cap \mathcal{D}(P\mathcal{L}P)$ defines a Banach space with the norm $\|x\|_{\mathcal{Y}} \coloneqq \|x\|_{\mathcal{X}} + \|P\mathcal{L}Px\|_{\mathcal{X}} + \|(P\mathcal{L}P)^2x\|_{\mathcal{X}}$. Clearly, $\mathcal{Y}$ is invariant under P and an even $P\mathcal{L}P$-admissible subspace, in particular, $\|e^{tP\mathcal{L}P}\|_{\mathcal{X}\to\mathcal{Y}} \leq 1$. Next, the asymptotic Zeno condition could be used for the following relative bound

$$\begin{aligned}
&\frac{1}{n^2}\|\int_0^1\int_0^1 s_1 P e^{s_1 s_2\frac{\mathcal{L}}{n}}\mathcal{L}^2 P x_s ds_2 ds_1\|_{\mathcal{X}} \\
&\qquad = \frac{1}{n}\|\int_0^1 P e^{s_1\frac{\mathcal{L}}{n}}(P + \mathbb{1} - P)\mathcal{L}Px_s ds_1 - P\mathcal{L}Px_s\|_{\mathcal{X}} \\
&\qquad \leq \frac{1}{n}\|\int_0^1 P e^{s_1\frac{\mathcal{L}}{n}}(\mathbb{1} - P)\mathcal{L}Px_s ds_1 + \frac{1}{n}P e^{s_1\frac{\mathcal{L}}{n}}\mathcal{L}P\mathcal{L}Px_s ds_1\|_{\mathcal{X}} \\
&\qquad \leq \frac{1}{n^2}\left(b^2\|x_s\|_{\mathcal{X}} + b\|P\mathcal{L}Px_s\|_{\mathcal{X}} + \|(P\mathcal{L}P)^2 x_s\|_{\mathcal{X}}\right) \\
&\qquad \leq \frac{1}{n^2}\max\{b^2, b, 1\}\|x_s\|_{\mathcal{Y}}\,,
\end{aligned}$$

which is slightly different from the relative bound used in Equation 45. Here, b is defined in Theorem I in [37].

In the next step, we generalize the above result to C_0-evolution systems:

Theorem 5.2 *Let $I_T \ni (t,s) \mapsto V(t,s) \in \mathcal{B}(\mathcal{X})$ be a C_0-evolution system with generator family $(\mathcal{L}_t, \mathcal{D}(\mathcal{L}_t))_{t\in[0,T]}$ and $P \in \mathcal{B}(\mathcal{X})$ a projective contraction. Assume that there is a Banach space $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ invariant under P so that the closure of $(P\mathcal{L}_tP, \mathcal{Y})_{t\in[0,T]}$ generates a unique contractive C_0-evolution systems $T(t,s)$ for $(t,s) \in I_T$ commuting with P. Moreover, assume that $\mathcal{L}_tP$, $\mathcal{L}_t\mathcal{L}_sP$, $P\mathcal{L}_tP$, and $P\mathcal{L}_tP\mathcal{L}_sP$ are well-defined on $\mathcal{Y}$ and relatively $\mathcal{Y}$-bounded uniformly for all $(t,s) \in I_T$. If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a $P\mathcal{L}_tP$-stable subspace, in particular $\|T(t,s)\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{(t-s)\omega}$, then there is a $c \geq 0$ so that*

$$\Big\|\prod_{j=1}^n PV(s_j, s_{j-1})Px - T(t,0)Px\Big\|_{\mathcal{X}} \leq c\frac{t^2}{n}e^{t\omega}\|Px\|_{\mathcal{Y}}$$

for all $t \geq 0$, $s_0 = 0 < \frac{t}{n} < \ldots < \frac{t(n-1)}{n} < t = s_n$, $x \in \mathcal{Y}$, and $n \in \mathbb{N}$.

Note that the assumption about the existence of the Banach space $\mathcal{Y}$ is motivated by hyperbolic evolution systems for which by definition an admissible subspace exists (see Thm. 2.3).

Proof. As before, we rescale $t \in [0,1]$ and apply Proposition 4.1 to

$$F(t,s) = PV(t,s)P$$

defined on $P\mathcal{X}$ w.r.t. the sequence $s_0 = 0 < \frac{1}{n} < \ldots < \frac{n-1}{n} < 1 = s_n$ so that

$$\begin{aligned}
&\Big\| \prod_{j=1}^{n} PV(s_j, s_{j-1})Px - T(t,0)Px \Big\|_{\mathcal{X}} \\
&\qquad\qquad \leq n \max_{j\in\{1,\ldots,n\}} \| \left(PV(s_j,s_{j-1})P - T(s_j,s_{j-1})\right) T(s_{j-1},s_0)Px\|_{\mathcal{X}} \,.
\end{aligned}$$

Then, we use $T(s_{j-1},s_0)Px = PT(s_{j-1},s_0)Px$ and the integral equation (13) to show with $s_j(\tau) = s_j + \frac{1-\tau_1}{n}$

$$\begin{aligned}
&PV(s_j,s_{j-1})Px_j - T(s_j,s_{j-1})Px_j \\
&\quad = \frac{1}{n}\int_0^1 \Big(PV(s_j,s_{j-1}(\tau_1))\mathcal{L}_{s_{j-1}(\tau_1)}P - T(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1)}P \Big) x_j d\tau_1 \\
&\quad = \frac{1}{n^2}\int_0^1\int_0^1 \Big(\tau_1 PV(s_j,s_{j-1}(\tau_1\tau_2))\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}\mathcal{L}_{s_{j-1}(\tau_1)}P \\
&\qquad\qquad - \tau_1 T(s_j,s_{j-1}(\tau_1\tau_2))P\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}P\mathcal{L}_{s_{j-1}(\tau_1)}P \Big) x_j d\tau_2 d\tau_1 \,,
\end{aligned}$$

where $x_j := PT(s_{j-1},s_0)Px$. Note that the integral above is again well-defined by Lemma 2.1 as well as [12, Lem. B.15]. As in the proof of Proposition 5.1, we use the assumption to show

$$\begin{aligned}
\| \int_0^1\int_0^1 \tau_1 PV(s_j,s_{j-1}(\tau_1))\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}\mathcal{L}_{s_{j-1}(\tau_1)}Px_j d\tau_2 d\tau_1\|_{\mathcal{X}} &\leq \frac{a_1}{2}\|T(s_{j-1},s_0)Px\|_{\mathcal{Y}} \\
&\leq c_1 e^{\omega}\|Px\|_{\mathcal{Y}}
\end{aligned}$$

for $c_1 = \frac{a_1,b_1}{2}$ with $a_1, b_1 \geq 0$ and similarly

$$\| \int_0^1\int_0^1 \tau_1 T(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}P\mathcal{L}_{s_{j-1}(\tau_1)}Px_j d\tau_2 d\tau_1\|_{\mathcal{X}} \leq c_2 e^{\omega}\|x_j\|_{\mathcal{Y}}$$

for $c_2 = \frac{a_2,b_2}{2}$ with $a_2, b_2 \geq 0$. Rescaling and combining both bounds show

$$\Big\| \prod_{j=1}^{n} PV(s_j,s_{j-1})Px - T(t,0)Px \Big\|_{\mathcal{X}} \leq \frac{ct^2}{n} e^{t\omega}\|Px\|_{\mathcal{Y}}$$

with $c = 2\max\{c_1, c_2\}$. □

5.2 Time-dependent Zeno with general measurement

In the next step, we generalize Theorem I in [37] to C_0-evolution systems. For that, we use again the help of a reference subspace $\mathcal{Y}$:

Theorem 5.3 *Let $I_T \ni (t,s) \mapsto V(t,s) \in \mathcal{B}(\mathcal{X})$ be a C_0-evolution system with generator family $(\mathcal{L}_t, \mathcal{D}(\mathcal{L}_t))_{t\in[0,T]}$, $M \in \mathcal{B}(\mathcal{X})$ a contraction, and P a projection satisfying*

$$\|M^n - P\|_{\mathcal{X}\to\mathcal{X}} \leq \delta^n \tag{46}$$

for $\delta \in (0,1)$ and all $n \in \mathbb{N}$. Moreover, we assume the asymptotic Zeno condition:

$$\|PV(t,s)(\mathbb{1}-P)\|_{\mathcal{X}\to\mathcal{X}} \leq (t-s)b \quad \text{and} \quad \|(\mathbb{1}-P)V(t,s)P\|_{\mathcal{X}\to\mathcal{X}} \leq (t-s)b \tag{47}$$

for $b \geq 0$ and all $t \in [0,T]$. Furthermore, there is a Banach space $(\mathcal{Y} \subset \mathcal{X}, \|\cdot\|_{\mathcal{Y}})$ invariant under P so that the closure of $(P\mathcal{L}_tP, \mathcal{Y})_{t\in[0,T]}$ generates a unique contractive C_0-evolution system $T(t,s)$ for $(t,s) \in I_T$ commuting with P. Moreover, assume that $\mathcal{L}_tP$, $P\mathcal{L}_tP$, and $P\mathcal{L}_tP\mathcal{L}_sP$ are well-defined on $\mathcal{Y}$ and relatively $\mathcal{Y}$-bounded for all $(t,s) \in I_T$, and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a $P\mathcal{L}_tP$-stable subspace, in particular $\|T(t,s)\|_{\mathcal{Y}\to\mathcal{Y}} \leq \tilde{c}e^{(t-s)\omega}$. Then, there are $c_1, c_2 \geq 0$ so that

$$\Big\|\prod_{j=1}^{n} MV(s_j, s_{j-1})x - T(t,0)Px\Big\|_{\mathcal{X}} \leq \left(\delta^n + \frac{c_1(t+t^2)}{n}\right)\|x\|_{\mathcal{X}} + c_2\frac{t^2}{n}e^{t\omega}\|Px\|_{\mathcal{Y}}$$

for all $t \geq 0$, $s_0 = 0 < \frac{t}{n} < ... < \frac{t(n-1)}{n} < t = s_n$, $x \in \mathcal{Y}$ and $n \in \mathbb{N}$.

Proof. As in the proof of [37, Thm. 5.1], we approximate M by P in the first step. To do so, we follow the proof of Lemma 5.2 in [37]. For that

$$\begin{aligned}\Big\|\prod_{j=1}^{n}(P+(\mathbb{1}-P)M)V(s_j,s_{j-1})x - \prod_{j=1}^{n} PV(s_j,s_{j-1})x\Big\|_{\mathcal{X}}\\ = \Big\|\sum_{\tau\in\{0,1\}^n\setminus(1,\dots,1)}\prod_{j=1}^{n}(\tau_jP+(1-\tau_j)(\mathbb{1}-P)M)V(s_j,s_{j-1})x\Big\|_{\mathcal{X}}.\end{aligned}$$

Next, we analyse how many transitions in terms of $PV(s_j, s_{j-1})(1-P)$ exist in the above sum. This is specified in Lemma 5.4 in [37]. Next, we use the bounds

$$\|M(\mathbb{1}-P)V(s_j,s_{j-1})\|_{\mathcal{X}\to\mathcal{X}} \leq \delta \qquad \text{and} \qquad \|M(\mathbb{1}-P)V(s_j,s_{j-1})PMV(s_{j+1},s_j)\| \leq \delta\frac{b}{n}$$

and follow the calculations on [37, p. 12] to show

$$\left\|\prod_{j=1}^{n} MV(s_j,s_{j-1})x - \prod_{j=1}^{n} PV(s_j,s_{j-1})Px\right\|_{\mathcal{X}} \leq \left(\delta^n + \frac{tb}{n} + \frac{1}{n}\frac{tb(2+tb)(\delta-\delta^n)}{1-\delta}e^{2b}\right)\|x\|_{\mathcal{X}} \tag{48}$$

for all $x \in \mathcal{X}$. Then, we follow the proof of Theorem 5.2 and apply Proposition 4.1 to

$$F(t,s) = PV(t,s)P$$

defined on $P\mathcal{X}$ w.r.t. the sequence $s_0 = 0 < \frac{t}{n} < ... < \frac{t(n-1)}{n} < t = s_n$ so that

$$\begin{aligned}\Big\|\prod_{j=1}^{n} PV(s_j,s_{j-1})Px - T(t,0)Px\Big\|_{\mathcal{X}}\\ \leq n \max_{j\in\{1,\dots,n\}} \|\left(PV(s_j,s_{j-1})P - T(s_j,s_{j-1})\right)T(s_{j-1},s_0)Px\|_{\mathcal{X}}.\end{aligned}$$

Then, we use $T(s_{j-1},s_0)Px = PT(s_{j-1},s_0)Px$ and the integral equation (13) to show with $s_j(\tau) = s_j + \frac{(1-\tau_1)t}{n}$

$$
\begin{aligned}
&PV(s_j,s_{j-1})Px_j - T(s_j,s_{j-1})Px_j \\
&\quad = \frac{t}{n}\int_0^1 \Big(PV(s_j,s_{j-1}(\tau_1))\mathcal{L}_{s_{j-1}(\tau_1)}P - T(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1)}P \Big) x_j d\tau_1 \\
&\quad = \frac{t}{n}\int_0^1 \Big(PV(s_j,s_{j-1}(\tau_1))(\mathbb{1}-P)\mathcal{L}_{s_{j-1}(\tau_1)}P \\
&\qquad\qquad + PV(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1)}P - T(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1)}P \Big) x_j d\tau_1
\end{aligned}
$$

and for the last two terms

$$
\begin{aligned}
&\frac{t}{n}\int_0^1 \Big(PV(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1)}P - T(s_j,s_{j-1}(\tau_1))P\mathcal{L}_{s_{j-1}(\tau_1)}P \Big) x_j d\tau_1 \\
&\quad = \frac{t^2}{n^2}\int_0^1\int_0^1 \Big(\tau_1 PV(s_j,s_{j-1}(\tau_1\tau_2))(P+\mathbb{1}-P)\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}P\mathcal{L}_{s_{j-1}(\tau_1)}P \\
&\qquad\qquad - \tau_1 T(s_j,s_{j-1}(\tau_1\tau_2))P\mathcal{L}_{s_{j-1}(\tau_1\tau_2)}P\mathcal{L}_{s_{j-1}(\tau_1)}P \Big) x_j d\tau_2 d\tau_1 ,
\end{aligned}
$$

where $x_j \coloneqq PT(s_{j-1},s_0)Px$. Note that the integral above is again well-defined by Lemma 2.1 or [12, Lem. B.15]. Next, we use the relative $\mathcal{Y}$-boundedness as in Proposition 5.1, but also the asymptotic Zeno condition, which implies boundedness for all $t\in[0,T]$

$$
\|P\mathcal{L}_t(\mathbb{1}-P)\|_{\mathcal{X}\to\mathcal{X}}, \|(\mathbb{1}-P)\mathcal{L}_tP\|_{\mathcal{X}\to\mathcal{X}} \le b\,.
$$

Then, $\|P\mathcal{L}_tPx\|_{\mathcal{X}} \le a_1\|x\|_{\mathcal{Y}}$ and $\|P\mathcal{L}_tP\mathcal{L}_sPx\|_{\mathcal{X}} \le a_2\|x\|_{\mathcal{Y}}$ for all $(t,s)\in I_T$ show

$$
\|PV(s_j,s_{j-1})Px_j - T(s_j,s_{j-1})Px_j\|_{\mathcal{X}} \le \frac{t^2}{n^2}\left(b^2\|x_j\|_{\mathcal{X}} + \left(\frac{a_1 b}{2}+a_2\right)\|x_j\|_{\mathcal{Y}}\right).
$$

Applying $\|PT(s_{j-1},s_0)Px\|_{\mathcal{Y}} \le a_3 e^{\omega}\|Px\|_{\mathcal{Y}}$ to the above bound shows

$$
\|PV(s_j,s_{j-1})Px_j - T(s_j,s_{j-1})Px_j\|_{\mathcal{X}} \le \frac{t^2}{n^2}\left(b^2\|Px\|_{\mathcal{X}} + \left(\frac{a_1 b}{2}+a_2\right)e^{\omega}\|Px\|_{\mathcal{Y}}\right).
$$

Combining the bound finishes the proof of the result. □

Remark. If we assume $\mathcal{L}_t\mathcal{L}_sP$ (instead of $P\mathcal{L}_tP$) is relatively $\mathcal{Y}$-bounded for all $(t,s)\in I_T$, then, the proof directly follows by Theorem 5.2 after Equation 48. However, then Theorem 5.3 does not reduce directly to Theorem I in [37].

5.3 Application: The l-photon driven dissipation in error correction

We continue with the example in Section 4.4 about time-(in)dependent C_0-semigroups generated by GKSL-operators (37), in particular, Sobolev preserving semigroups that satisfy Equation 22 (see Sec. 4.4 and 2.2 for more details). In the following, we shortly repeat the proof of these two properties for the l-photon driven dissipation for simplicity in one-mode [20, Lem. 5.3] and give two specific examples for projections satisfying the assumptions of the statements before (5.1, Theorem 5.2, 5.3). The l-photon driven dissipation (44) defined for $x \in \mathcal{T}_f$ by

$$\mathcal{L}[a^l - \alpha^l][x] = (a^l - \alpha^l)x((a^\dagger)^l - \bar{\alpha}^l) - \frac{1}{2}\left\{((a^\dagger)^l - \bar{\alpha}^l)(a^l - \alpha^l), x\right\} \tag{49}$$

is the fundamental dynamics for the bosonic error correction code introduced in [34, 22, 21]. Using the CCR and the simple commutation relation $af(N + j\mathbb{1}) = f(N + (j+1)\mathbb{1})a$ for any real-valued function $f : \mathbb{R} \to \mathbb{R}$, we have for $f(x) = (x+1)^{k/2}1_{x \geq -1}$ and $g_l(x) :=$ $f(x) - f(x-l)$:

$$\mathcal{L}[a_1^l - \alpha^l]^\dagger(f(N)) = -(N - (l-1)\mathbb{1}) \cdots N g_l(N) + \frac{1}{2}\left(\overline{\alpha}^l a^l g_l(N) + \alpha^l g_l(N)(a^\dagger)^l\right).$$

Note that the first term contributes a diagonal and the others, two off-diagonals with respect to the Fock basis. The operator can be decomposed in blocks defined via $a = \frac{h_l(n)g_l(n)}{2}$, $b = \frac{h_l(n+l)g_l(n+l)}{2}$, and $c = \frac{1}{2}\sqrt{h_l(n+l)}\bar{\alpha}^l g_l(n+l)$ by

$$\begin{aligned}
&\frac{1}{2}\Big(-a\,|n\rangle\langle n| - b\,|n+l\rangle\langle n+l| + c^*\,|n\rangle\langle n+l| + c\,|n+l\rangle\langle n|\Big) \\
&\quad := \frac{1}{2}\Big(-h_l(n)g_l(n)\,|n\rangle\langle n| - h_l(n+l)g_l(n+l)\,|n+l\rangle\langle n+l| \\
&\qquad\qquad + \sqrt{h_l(n+l)}\left(\bar{\alpha}^l g_l(n+l)\,|n\rangle\langle n+l| + \alpha^l g_l(n+l)\,|n+l\rangle\langle n|\right)\Big)
\end{aligned}$$

for $n \in \mathbb{N}$ and the real valued function $h_l : \mathbb{R} \to \mathbb{R}, h_l(x) := (x - (l-1)\mathbb{1}) \cdots x 1_{x \geq l}$. By considering the eigenvalues of the above operator, it can be upper bounded by

$$\begin{aligned}
&\frac{1}{2}\Big(-a\,|n\rangle\langle n| - b\,|n+l\rangle\langle n+l| + c^*\,|n\rangle\langle n+l| + c\,|n+l\rangle\langle n|\Big) \\
&\quad \leq \frac{1}{4}\left(-a - b + \sqrt{(a-b)^2 + 4|c|^2}\right)\Big(|n\rangle\langle n| + |n+l\rangle\langle n+l|\Big) \\
&\quad \leq \frac{1}{2}\left(-\min\{a,b\} + |c|\right)\Big(|n\rangle\langle n| + |n+l\rangle\langle n+l|\Big) \\
&\quad = \frac{1}{2}\left(-h_l(n)g_l(n) + \sqrt{h_l(n+l)}|\alpha|^l g_l(n+l)\right)\Big(|n\rangle\langle n| + |n+l\rangle\langle n+l|\Big) \\
&\quad \leq \frac{1}{2}\left(-(n+1)^{k/2-1+l} + c_k^{(l)}\right)\Big(|n\rangle\langle n| + |n+l\rangle\langle n+l|\Big),
\end{aligned}$$

where the constant $c_k^{(l)} \geq 0$ exists because the first term is of negative leading order (a more detailed discussion can be found in [20, Lem. 5.3]). All together, we achieve

$$\mathcal{L}[a^l - \alpha^l]^\dagger(f(N)) \leq -\frac{l}{2}(N + \mathbb{1})^{k/2-1+l} + c_k^{(l)}\mathbb{1},$$

which implies Assumption 22 and a better Sobolev preserving constant [20, Prop. 5.1]:

$$\operatorname{tr}\left[\mathcal{L}[a^l - \alpha^l](\rho)(N + \mathbb{1})^{k/2}\right] \leq -\frac{l}{2}\operatorname{tr}\left[\rho\,(N + \mathbb{1})^{k/2}\right] + c_k^{(l)}.$$

However, we are interested in the Sobolev preserving property of $(P\mathcal{L}P, \mathcal{D}(P\mathcal{L}P))$ because this is one way to find the right Banach space $\mathcal{Y} = W^{k,1}$ in the statements above. Here, the idea is to choose k large enough such that the relative boundedness assumption is achieved (for comparison see Equation 42). However, this strongly depends on P. To analyze this, we will consider two cases: First, $P(\cdot) = K \cdot K^\perp$ for a projection K diagonal in the Fock basis. Then, the same reasoning as above proves the well-definedness as well as the Sobolev preserving property of the semigroup defined by $P\mathcal{L}P$. The same holds true for the time-dependent analogues generated by (Eq. 16). Another example is the $Z(\theta)$ gate of the bosonic error correction code discussed in [34, 22, 21]. The physical codespace to which the evolution is exponentially converging (see Eq. 53) and [1]) is defined by

$$\mathcal{C}_l \coloneqq \operatorname{span}\left\{|\alpha_1\rangle\langle\alpha_2| \,:\, \alpha_1,\alpha_2 \in \left\{\alpha e^{\frac{i2\pi j}{l}} \,|\, j \in \{0,...,l-1\}\right\}\right\},$$

where $|\alpha\rangle$ denotes the coherent state. In particular, we are interested in the case $l = 2$. Next, we define the Schrödinger CAT-states, which represent the logical qubits:

$$|CAT_\alpha^+\rangle \coloneqq \frac{1}{\sqrt{2(1+e^{2|\alpha|^2})}}\left(|\alpha\rangle + |{-\alpha}\rangle\right) \quad \text{and} \quad |CAT_\alpha^-\rangle \coloneqq \frac{1}{\sqrt{2(1-e^{2|\alpha|^2})}}\left(|\alpha\rangle - |{-\alpha}\rangle\right).$$

Note that the above CAT-states are orthonormal. Then, a possible rotation around the x-axis is described in [34] and experimentally realized in [56]. The gate is defined by

$$X(\theta) = \cos\left(\frac{\theta}{2}\right)(P_\alpha^+ + P_\alpha^-) + i\sin\left(\frac{\theta}{2}\right)X_\alpha\,, \tag{50}$$

where $P^+ = |CAT_\alpha^+\rangle\langle CAT_\alpha^+|$, $P^- = |CAT_\alpha^-\rangle\langle CAT_\alpha^-|$ are the projection of the CAT-states and $X_\alpha = |CAT_\alpha^+\rangle\langle CAT_\alpha^-| + |CAT_\alpha^-\rangle\langle CAT_\alpha^+|$. Specially, here we discuss a construction of a $X(\theta)$-gate by a discrete projective Zeno effect w.r.t. the projection $P_\alpha = P_\alpha^+ + P_\alpha^-$ and the driving Hamiltonian $H = a + a^\dagger$. Then, the Zeno limit describes Equation 50:

$$P_\alpha H P_\alpha = (\alpha + \alpha^\dagger)X_\alpha\,. \tag{51}$$

Since $P_\alpha H P_\alpha$ acts only non-trivially on a 2-dimensional subspace, the Zeno semigroup is

$$e^{tiP_\alpha HP_\alpha}P_\alpha = \cos\Big(t(\alpha+\alpha^\dagger)\Big)P_\alpha + i\sin\Big(t(\alpha+\alpha^\dagger)\Big)X_\alpha \tag{52}$$

and constructs the rotation (50) for appropriate t. The convergence

$$\|\left(P_\alpha e^{i\frac{t}{n}[H,\cdot]}\right)^n x - e^{ti[P_\alpha HP_\alpha, P_\alpha\cdot P_\alpha]}P_\alpha x\|_{1\to 1} = \mathcal{O}\Big(\frac{t^2}{n}\Big)$$

is a consequence of Proposition 5.1 or a simple case of Theorem 5.3. Here, $\mathcal{Y}$ is just defined as $\mathcal{P}_\alpha\mathcal{T}_1$ with $\mathcal{P}_\alpha = P_\alpha \cdot P_\alpha$. By the boundedness of $P_\alpha H$ and HP_α all assumptions are satisfied and the bound holds true in the uniform topology. The same would hold true for more complex and time-dependent H defined by a polynomial in annihilation and creation operator. The case $l = 4$ described in [34] can be equally treated.

Remark 6. In practice the projection is implemented by the l-photon driven dissipation semigroup, which converges in the following sense to P_α [1, Theorem 2]

$$\operatorname{tr}\Big[(a^l-\alpha^l)e^{t\mathcal{L}_l}(\rho)(a^l-\alpha^l)^\dagger\Big] \leq e^{-l!t}\operatorname{tr}\Big[(a^l-\alpha^l)\rho(a^l-\alpha^l)^\dagger\Big]\,. \tag{53}$$

This convergence does not satisfy Theorem 5.3. Therefore, a future line of research could be to improve Equation 53 or Assumption 46 in Theorem 5.3 in the direction of Theorem 2 in [4].

6 Conclusion

To summarize, we have seen a range of quantitative error bounds for generalizations of Trotter's product formula and the quantum Zeno effect in the case of C_0-semigroups. Our primary contribution lies not only in the extension of the Trotter product formula and the quantum Zeno effect to evolution systems (time-dependent semigroups) but also in providing Suzuki-higher order error bounds for the Trotter-Suzuki product formula in the strong operator topology and a large class of bosonic examples. Moreover, we discussed some examples to explore possible definitions of the abstract reference space $\mathcal{Y}$.

For applications, a desirable line of research would entail a detailed discussion of possible choices of $\mathcal{Y}$ as well as examples which does not satisfy the assumption. For the quantum Zeno effect, a generalization of the convergence assumption of the contraction M, as discussed in Remark 6, would be fruitful. Here, an interesting interplay of the convergence assumption as well as the regularity of the semigroup quantified by the stable subspace could come up.

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7 References

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