

Tensor network decompositions for absolutely maximally entangled states

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Absolutely maximally entangled (AME) states of k qudits are quantum states that have maximal entanglement for all possible bipartitions of the sites/parties, and they can also be interpreted as perfect tensors. We consider the problem of whether such states can be decomposed into a tensor network with a small number of tensors, such that all physical and all auxiliary spaces have the same dimension D . We find that certain AME states with $k = 6$ can be decomposed into a network with only three 4-leg tensors; we provide concrete solutions for local dimension $D = 5$ and higher. Our result implies that certain AME states with six parties can be created with only three two-site unitaries from a product state of three Bell pairs, or equivalently, with six two-site unitaries acting on a product state on six qudits. We also consider the problem for $k = 8$, where we find similar tensor network decompositions with six 4-leg tensors.

1 Introduction

Tensor networks play an important role in many branches of theoretical physics [1]. They were originally developed in the study of quantum many body systems [2, 3], where they can effectively simulate quantum states and operators with low entanglement. Ground states of gapped Hamiltonians have area law entanglement, therefore physical properties of such

states can be computed with tensor networks. In one dimensional systems these tensor networks are called Matrix Product States (MPS) and Matrix Product Operators (MPO).

In this work we investigate a question which is in a certain sense opposite to the typical scenario: We ask whether tensor networks can produce *states with maximal entanglement* in certain selected situations. There are a number of motivations to study this question, which we explain below.

We are interested in special states of finite quantum systems, which are called Absolutely Maximally Entangled states (AME). We consider homogeneous cases, where the physical system is composed of k sites, and each site (or spin) is described by a D -dimensional Hilbert space. A state in the tensor product Hilbert space is absolutely maximally entangled if it has maximal entanglement for all possible bipartitions [4, 5, 6, 7, 8]. AME’s are intimately related to quantum error correcting codes [9]. They can be seen as a quantum mechanical version of Orthogonal Arrays (OA’s) [7, 10, 11] known from combinatorial design theory in mathematics [12]. AME’s can also be interpreted as “perfect tensors” [13].

There are two central questions regarding AME’s. First, for which D, k do they exist, and how many inequivalent states or families of states can exist for a given D, k ? Second, given a known AME, how can we actually construct it, either in an analogue computer or in a real world quantum computer?

Regarding the first question only partial results are known, and they are summarized in the online Table [14] (see also [15]). A famous

example is the quantum version of the problem of Euler’s 36 officers, which concerns finding an AME with $D = 6$ and $k = 4$. This problem has recently been solved in [16], see also [17]. The uniqueness of the AME for $k = 4$ and $D \geq 3$ was investigated recently in [18]. In this work we do not provide completely new AME’s, but we contribute to the understanding of their internal structure, by developing tensor network decompositions in certain selected cases.

This means that we contribute to the second question above: we contribute to understanding the structure of certain selected AME’s by finding tensor network decompositions for them. These tensor networks can be interpreted alternatively as quantum circuits which create these AME’s from uncorrelated states. Similar questions were considered in the works [19, 20, 10]. Very often AME’s are constructed as “graph states”, and such a construction can be seen as the action of a quantum circuit on initial product states [19]. Our methods are closely related, but different in certain technical details. Most importantly, we manage to create selected AME’s with a small number of two-site gates. The number of the gates applied is smaller than in the case of the graph states considered in [19, 20].

It is known that in large systems almost all states are close to maximally entangled [21], but their actual constructions require an exponentially large number of quantum gates. (see also [22]). In contrast, we find selected states in smaller systems, which have maximal entanglement, but which can be created with few two-site gates.

The idea to use tensor networks to create larger objects from smaller building blocks also appeared recently in the study of quantum error correcting codes, see [23] and later [24, 25]. A few years ago AME’s also appeared as building blocks of the so-called “holographic error correcting codes” [26], which can be seen as discrete models of the AdS/CFT duality [27, 28]. Our work can contribute to this line of research, by showing that in certain cases the fundamental multi-leg tensors in such codes can actually be decomposed into a network of smaller tensors.

The idea of such decompositions actually appeared much earlier, although in a slightly different context. Instead of an AME one can also consider states which have maximal entanglement for certain selected bipartitions, such that the choice for the allowed bipartitions is dictated by geometrical principles. This leads to the so-called planar maximally entangled states, also known as perfect tangles, or block perfect tensors [29, 30, 31, 32]. If the number k of constituents is even, then such states can be interpreted as operators that act unitarily in multiple directions in a planar arrangement of the sites. In the simplest case of $k = 4$ one obtains the so-called dual unitary operators [33], which appeared earlier in pure mathematics as “biunitary operators” [34, 35], and in the holographic code literature as “doubly unitary operators” [36, 37]. For a treatment of bi-unitarity and its applications to quantum information theory and quantum combinatorial objects, see [38, 39]. Generalization of these objects to more constituents and also higher dimensional lattices have been treated in multiple works [40, 41, 42, 43, 44]. In certain cases the relevant multi-leg tensors admit a factorization (tensor network decomposition) using 4-leg tensors. The main idea goes back to Jones (see [29]), but it was independently rediscovered in various different situations [40, 41, 43]. Our work can be seen as a further step along this direction, with the speciality that we consider the decomposition of actual AME’s.

We should also note the recent work [45], which considered the entanglement pattern of a tensor network in the geometry of a dodecahedron. The constituting tensors were AME’s and a certain local projector, and the resulting quantum state has close to maximal entanglement, but it is not an actual AME. In our work we focus on AME’s, and we do not investigate states with big, but not maximal entanglement.

We focus on the cases $k = 6$ and $k = 8$. Our main idea is to use perfect 4-leg tensors in the decomposition. This immediately guarantees maximal entanglement for many bipar-

titions of the resulting state. Afterwards, requiring maximal entanglement for the remaining bipartitions will pose conditions for the constituent tensors. In our concrete computations we will work with tensors related to linear maps over finite fields. For $k = 6$ we find decomposable AME's for selected local dimensions $D \geq 5$.

2 Definitions

We consider finite Hilbert spaces of k parties (or spins), each of them carrying a Hilbert space of dimension $D \geq 2$. The full Hilbert space is thus the tensor product

$$\mathcal{H}^{(k)} = \otimes_{j=1}^k \mathbb{C}^D.$$

We will denote the set of sites by $S = \{1, 2, \dots, k\}$.

2.1 Maximal entanglement

For a given pure state $|\Psi\rangle \in \mathcal{H}$, the density matrix is given by $\rho = |\Psi\rangle\langle\Psi|$. For a bipartition $S = A \cup B$ with $B = S \setminus A$, the reduced density matrix of subsystem A is defined as the partial trace over subsystem B :

$$\rho_A = \text{Tr}_B(\rho).$$

The von Neumann entanglement is then defined as

$$s = -\text{Tr}(\rho_A \log(\rho_A)).$$

The entanglement is symmetric with respect to the exchange $A \leftrightarrow B$.

The entanglement is zero if $|\Psi\rangle$ is a product state of the form $|\Psi_A\rangle \otimes |\Psi_B\rangle$, where $|\Psi_{A,B}\rangle$ are states of the Hilbert spaces of the two subsystems A and B , respectively. Let us assume that $|A| \leq |B|$. The maximal value of the entanglement is reached when ρ_A is proportional to the identity matrix. More concretely

$$\rho_A = 1/D^a, \quad (2.1)$$

where $a = |A|$, leading to the maximal value $s = a \log(D)$. In such a case we say that the state $|\Psi\rangle$ is maximally entangled for the given bipartition.

A state $|\Psi\rangle$ is said to be ℓ -uniform, for some positive integer ℓ , if it has maximal entanglement for all bipartitions with $|A| = \ell$. If a state is ℓ -uniform then it is also ℓ' -uniform for any $0 < \ell' < \ell$. A state $|\Psi\rangle$ is absolutely maximally entangled (AME state), if it is ℓ -uniform with $\ell = \lfloor k/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part.

Of particular interest are the cases when k is even. In such a case the state can be interpreted as a “multiunitary matrix” via a state-operator correspondence [10]. Let us spell out this connection, by working in a concrete basis. The tensor components of the vector $|\Psi\rangle$ can be denoted as

$$\Psi_{a_1, a_2, \dots, a_k}.$$

Let us choose a bipartition, where $A = \{1, 2, \dots, k/2\}$. Then we construct a linear operator U acting on the $k/2$ -fold tensor products by assigning the matrix elements as

$$U_{a_1, a_2, \dots, a_{k/2}}^{b_1, b_2, \dots, b_{k/2}} = D^{k/4} \Psi_{a_1, a_2, \dots, a_{k/2}, b_1, b_2, \dots, b_{k/2}}. \quad (2.2)$$

Then it is immediately seen that the condition (2.1) of maximal entanglement is equivalent to unitarity of U :

$$U^\dagger U = 1.$$

Different bipartitions into two equal subsets lead to different unitary operators, acting from one half of the system to the other half. However, all of the different U matrices can be obtained from each other by reshuffling their indices, corresponding to the permutation of sites in the original vector $|\Psi\rangle$.

2.2 4-leg tensors

The smallest non-trivial case with an even number of tensor legs is $k = 4$. If a state $|\Psi\rangle \in \mathcal{H}^{(4)}$ is an AME, then via relations (2.2) one obtains unitary operators that act on two sites. It is useful to discuss this case in more detail.

A pictorial representation of such a state/operator is given in Fig. 1. Here the numbers stand for the spaces 1-4, which are arranged in this representation anti-clockwise.

There are 3 different bipartitions of the set of 4 sites into two equal subsets. A state $|\Psi\rangle$

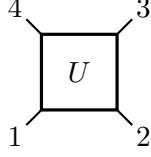


Figure 1: Graphical depiction of a 4-leg tensor. It can be interpreted as an element of the four-fold tensor product space $H^{(4)}$, or alternatively, as a linear operator acting on the product space of two sites.

is an AME if it has maximal entanglement for each of these 3 bipartitions. Correspondingly, one can obtain three unitary two-site operators using the state-operator correspondence. First of all, we obtain a two-site operator U given by the elements

$$U_{a_1 a_2}^{a_3 a_4} = D |\Psi_{a_1 a_2 a_3 a_4}\rangle.$$

This operator is seen in the diagram as acting from the pair of sites (1, 2) to the pair (3, 4), thus in the vertical direction upwards.

Alternatively, we can also construct the operator U^t via the correspondence

$$(U^t)_{a_1 a_4}^{a_3 a_2} = D |\Psi_{a_1 a_2 a_3 a_4}\rangle.$$

This operator is the partial transpose of U with respect to the second space. The operator acts from the pair of sites (1, 4) to the pair of sites (3, 2), which in the diagram is seen as acting horizontally, from left to right.

Finally, we can construct the operator U^R via

$$(U^R)_{a_1 a_3}^{a_2 a_4} = D |\Psi_{a_1 a_2 a_3 a_4}\rangle.$$

This operator acts from the pair of sites (1, 3) to the pair of sites (2, 4), thus on the diagram it is seen to act from one diagonal to the other diagonal.

The state $|\Psi\rangle$ is an AME, if all three operators U, U^t, U^R are unitary. The *dual unitary* operators are those where U and U^t are unitary, but U^R is arbitrary. In such a case we observe unitary action along directions which are natural from a geometric point of view, once the sites are arranged in the plane as the vertices of a square as in Fig. 1. Dual unitarity is then a very natural condition in multiple different situations, where such tensors

are arranged in a planar geometry. The two main examples are holographic error correcting codes [36], and one dimensional quantum circuits (where the extra dimension is given by the time coordinate) [33].

3 Tensor network decompositions

A tensor network is a particular way to construct states and operators of larger Hilbert spaces using states and operators (tensors) of smaller spaces. The idea is to take a product of a number of fundamental tensors (each of them with a small number of indices) and to contract some of the indices to obtain states and operators corresponding to the larger Hilbert space. The contractions can be visualized by the Penrose graphical notation, where the fundamental tensors are denoted by boxes, their indices with lines (wires), and contraction of indices is denoted by joining the lines. The remaining free lines are then interpreted as the indices of the final object (state or operator).

In this work we consider tensor network decompositions of specific vectors. Our main examples will be vectors of the six-fold tensor product $\mathcal{H}^{(6)}$, but we also consider vectors from $\mathcal{H}^{(8)}$. Our goal is to develop tensor networks where the fundamental tensor is an element of $\mathcal{H}^{(4)}$. Then the graphical depiction of the network will take the form of a 4-valent graph, with 6 or 8 outer legs, respectively.

3.1 Decompositions for $k = 6$

In the case of $k = 6$ we are looking for a decomposition into three fundamental tensors, in the way that it is depicted in Figure 2. Here each circle represents a tensor with 4 legs, and we allow the tensors to be different. Therefore, they are denoted as A, B, C .

This tensor network can be formalized most easily once we perform the state-operator correspondence as explained in the previous section. Then we obtain the relation

$$U^{(123)} = C^{(23)} B^{(13)} A^{(12)}. \quad (3.1)$$

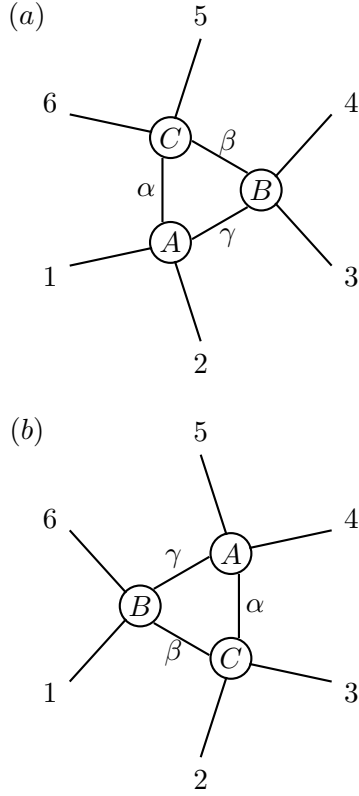


Figure 2: Tensor networks for $k = 6$. (a) $\triangleright$ -decomposition, (b) $\triangleleft$ -decomposition. Here α, β, γ stand for auxiliary spaces that are of the same dimension D as the physical spaces.

Now U is a three site-operator, and A, B and C are two-site operators. The upper indices show the spaces on which the operators act. Note that we paired the spaces 4, 5, and 6 of the original vector with spaces 1, 2, and 3, in this order. For the graphical interpretation of this factorization see Fig. 3. We call this the $\triangleright$ -decomposition, due to the graphical picture of the resulting tensor network.

Note that in the picture the incoming and outgoing spaces are attached to “legs” which are opposite to each other. This is a convention that we applied also in our previous work [44]. This convention is convenient, because in this convention the simple choice $U = I$ with I being the identity matrix satisfies the constraints of dual unitarity and multi-directional unitarity in various cases [44].

We also consider the same type of decomposition, but with a different ordering of the

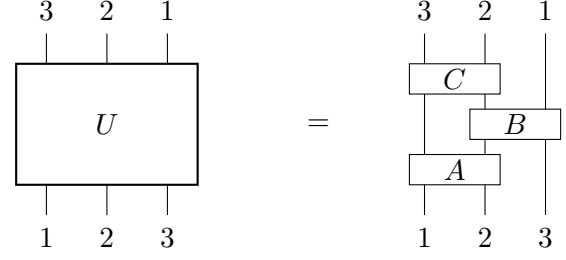


Figure 3: Decomposition of unitary operators, which corresponds to the $\triangleright$ -decomposition from Figure 2. It is understood that the operator U acts from the bottom to the top. We identified the outgoing spaces 1, 2, 3 of the unitary operator U with the sites 4, 5, 6 of the state $|\Psi\rangle$, in the given order.

tensors, corresponding to the factorization

$$U^{(123)} = A^{(12)} B^{(13)} C^{(23)}.$$

We call this the $\triangleleft$ -decomposition. Clearly, such a decomposition can be obtained from (3.1) simply by a permutation of the sites $1 \leftrightarrow 3$, which can be seen in the graphical representation as a space reflection combined with a time reflection. However, it is useful to keep both definitions at the same time. A given unitary $U^{(123)}$ might be factorizable only in one way, or perhaps both ways.

If a factorization is possible both ways, then the two-site operators are typically different. In those cases when the factorization is possible both ways, with the same two-site operators, one obtains the celebrated Yang-Baxter relation with discrete parameters:

$$A^{(12)} B^{(13)} C^{(23)} = C^{(23)} B^{(13)} A^{(12)}. \quad (3.2)$$

This equation is the central algebraic ingredient in the theory of integrable models, with connections to multiple areas of pure mathematics.

In this work we are interested in AME’s. It is then very natural to ask, whether (3.2) has solutions which are perfect tensors. To our knowledge, this question has not yet been investigated in the literature. On the other hand, it is well known that there are dual-unitary solutions (see for example [46], with relations to Yang-Baxter maps). In Example 3 below, we give a solution of (3.2) with perfect tensors, such that the product is also a perfect tensor.

3.2 Decompositions with perfect tensors

So far we did not pose any constraints for the three tensors A, B, C . However, it is a very natural idea (though, as we will see later, not absolutely necessary) that they should be chosen as perfect tensors, in order to reach perfection of the composite object.

In this subsection we show that if A, B, C are perfect tensors, then this guarantees maximal entanglement of the six site state for some, but not all, bipartitions. Then in the remainder of the work we will focus on specific choices for A, B, C such that the remaining conditions of maximal entanglement can also be satisfied.

For six sites there are 10 bipartitions, and each of them has to be treated. We proceed by selecting the bipartitions based on a geometric arrangement according to Figure 2: we associate the six sites with vertices of a regular hexagon. We say that a quantum state on six sites (more precisely, the corresponding operator U) is *hexagonal unitary* or *tri-unitary*, if the state is such that it has maximal entanglement for the three bipartitions that are obtained by cutting the hexagon into two pieces with those symmetry axes that are disjunct from the vertices [40, 44].

Theorem 1. *If A, B, C are dual unitary operators, then the resulting six-site state is hexagonal unitary¹. This holds for both the $\triangleright$ - and the $\triangleleft$ -decomposition.*

Proof 1. *Instead of a formal proof we provide a “graphical argument”, using Figure 2. It is enough to check that one obtains a unitary operator when acting from the triplet $(1, 2, 3)$ to $(4, 5, 6)$, and similarly from $(2, 3, 4)$ to $(5, 6, 1)$ and also from $(6, 1, 2)$ to $(3, 4, 5)$. Note that in each case the resulting three site operator can be written as a product of three two-site operators. Depending on the choice of the triplets one needs to work with the original operators*

¹This theorem is a special case of a general statement about similar planar arrangements, first published in [29]. It is claimed in that article that the idea stems from Vaughan Jones. Concrete cases appeared in many works afterwards, often independently discovered, see for example [43].

A, B, C or their reshuffled versions A^t, B^t, C^t . Dual unitarity means that all six operators are unitary. This implies unitarity of the composite operator for all three choices.

Let us now consider the case when A, B, C are all perfect tensors. In this case it is guaranteed that we find maximal entanglement for three more bipartitions. In the remainder of this section we focus on the $\triangleright$ -decomposition; the $\triangleleft$ -decomposition can be treated analogously.

Theorem 2. *Consider the $\triangleright$ -decomposition. If A, B, C are all perfect tensors, then the resulting six site state has maximal entanglement for the following three bipartitions:*

$$\begin{aligned} &(1, 2, 4) \cup (3, 5, 6), \\ &(1, 2, 5) \cup (3, 4, 6), \\ &(2, 5, 6) \cup (1, 3, 4). \end{aligned}$$

Proof 2. *Each one of these bipartitions is such that it can be transformed into the previous case by permuting one of the pairs $(1, 2)$, $(3, 4)$ or $(5, 6)$. For example, the bipartition $(1, 2, 4) \cup (3, 5, 6)$ can be transformed into $(1, 2, 3) \cup (4, 5, 6)$ by exchanging the pair of sites $(3, 4)$. If A, B, C are all perfect tensors, then after such a permutation they remain dual unitary, therefore we can apply the previous theorem to establish the unitarity of the resulting composite operators.*

An alternative proof can be given by considering a composition of operations, acting from one subset of the sites to the complementary set. It can be seen that there is always a sequence of operations where we take products of two-site unitary operators, given that A, B, C are perfect.

With this we established that perfection of the constituent tensors guarantees maximal entanglement for 6 out of the 10 bipartitions of the state $|\Psi\rangle$ on 6 sites. The remaining 4 bipartitions are those where we select exactly one site from each pair $(1, 2)$, $(3, 4)$, $(5, 6)$ into

one part. To be concrete, the bipartitions are

$$\begin{aligned} &(1, 3, 5) \cup (2, 4, 6), \\ &(1, 4, 5) \cup (2, 3, 6), \\ &(1, 3, 6) \cup (2, 4, 5), \\ &(1, 4, 6) \cup (2, 3, 5). \end{aligned} \quad (3.3)$$

In this case there is no way to express the operator acting from one subset of the sites to the other subset of sites as a product of three two-site operators. Therefore, perfection of A, B, C does not guarantee that $|\Psi\rangle$ is an AME. The unitarity conditions for the bipartitions (3.3) need to be checked.

As it was explained above, in this work we usually consider decompositions into perfect tensors. In Section 7 we will see that other decompositions are sometimes possible. However, in Section 5 we will find that in the linear case when $k = 6$ that the only possible decompositions are those into perfect tensors.

At the same time, we point out that such a simple decomposition of AME states are unlikely to exist for local dimensions $D = 2$ and $D = 3$. In dimension $D = 2$ there is no perfect tensor with four legs, and for $D = 3$ there is only one perfect tensor up to local unitary transformations [18]. In accordance with these simple observation, our methods find decompositions only for bigger local dimension, starting from $D = 5$.

3.3 Creating AME states

Here we show that the tensor network decompositions introduced above can be used to create the AME's from simple initial states. For simplicity we focus on the $\triangleright$ -decomposition.

The task here is to start with an initial state $|\Psi_0\rangle$ with very simple spatial structure, and then to apply a small number of two-site unitary operators on it, in order to produce our selected AME states. To this end, let us start with the case $k = 6$ and a simple product state

$$|\Psi_0\rangle = |v\rangle \otimes |v\rangle \otimes |v\rangle \otimes |v\rangle \otimes |v\rangle \otimes |v\rangle,$$

where $|v\rangle$ is any selected one-site state in the local Hilbert spaces of dimension D . As a first

step we create a state consisting of three Bell pairs. Let $\mathcal{K}$ be the two-site operator that acts as

$$\mathcal{K}(|v\rangle \otimes |v\rangle) = \frac{1}{\sqrt{D}} \sum_a |a\rangle \otimes |a\rangle,$$

where now $a = 1, \dots, D$ labels the states in our computational basis. We create three Bell pairs as ²

$$\mathcal{K}^{(14)} \mathcal{K}^{(25)} \mathcal{K}^{(36)} |\Psi_0\rangle.$$

Afterwards we act with the right hand side of the factorization (3.1), and obtain

$$|\Psi\rangle = C^{(23)} B^{(13)} A^{(12)} \mathcal{K}^{(14)} \mathcal{K}^{(25)} \mathcal{K}^{(36)} |\Psi_0\rangle. \quad (3.4)$$

This means that if we find a desired tensor network decomposition, then the AME can be created from a state of three Bell pairs with only three two-site unitary operators, and from a product state with only six two-site unitary operators.

Later, in Section 7, we also consider AME states with $k = 8$ parties. In that case we find tensor network decompositions with 6 four-leg tensors. In that case we need 4 additional gates to create 4 Bell pairs, therefore in total we need 10 two-site gates to create the AME states from product states.

It is known that AME states can also be created using the so-called “graph state” construction, which is reviewed in Appendix B. There we show that the minimal number of gates required to create the AME states is bigger in the case of the graph states, than with our methods using the tensor network decomposition.

4 States from classical maps

In this work we consider a specific subset of states/operators, namely those that can be obtained from classical maps. Consider the case

²The creation of the Bell pairs can be seen as a special case of the so-called “graph state” construction, if the operator $\mathcal{K}$ is chosen as the so-called controlled Z-gate (see Appendix B). Our final formula (3.4) is also related to the construction of the graph states, but it involves fewer number of gates, because we do not decompose the gates A, B and C into the controlled Z and Fourier gates. For more details see Appendix B.

of an AME with a given k and local dimension D . As explained above, for any bipartition of the sites into two equal subsets we obtain a unitary operator which acts from one subset of the sites to the other subset. We say that the AME is associated with a classical map if these unitary operators are *permutation matrices* in a selected basis. To be more concrete, let us fix a basis for the local Hilbert spaces, and consider the bipartition $S = (1, 2, \dots, k/2) \cup (k/2+1, \dots, k)$. The basis states of the $k/2$ -fold tensor product space can be labeled by tuples $(a_1, a_2, \dots, a_{k/2})$, where $a_j \in X$, with $X = \{0, 1, 2, \dots, D\}$. We consider bijective maps $u : X^{k/2} \rightarrow X^{k/2}$, and use the notation

$$u(a_1, a_2, \dots, a_{k/2}) = (b_1, b_2, \dots, b_{k/2}). \quad (4.1)$$

We say that a state $|\Psi\rangle \in \mathcal{H}^{(k)}$ is associated with the classical map u if it can be written as

$$|\Psi\rangle = \frac{1}{D^{k/4}} \sum_{a_1, \dots, a_{k/2}} |a_1, \dots, a_{k/2}, b_1, \dots, b_{k/2}\rangle,$$

where it is understood that the b -variables are obtained from (4.1).

According to (2.2), the action of the unitary operator U acting between the two subsystems is then given by

$$U|a_1, a_2, \dots, a_{k/2}\rangle = |b_1, b_2, \dots, b_{k/2}\rangle.$$

Therefore, it is indeed a permutation matrix on a set of size $D^{k/2}$.

The state $|\Psi\rangle$ has $D^{k/2}$ components in the given basis. It can be seen that this is the minimum number of non-zero components in order to fulfil the unitarity condition. Therefore we say that the state has *minimal support*.

The connection between the state $|\Psi\rangle$ and the unitary matrices has to hold for all bipartitions, and this leads us to the concept of *orthogonal arrays* (OA). Consider an array of size $r \times c$ consisting of symbols taken from X . Consider a subset of n columns and the resulting subarray. We say that this subarray consists of *orthogonal* columns if every n -tuple $(c_1, c_2, \dots, c_n) \in X^n$ appears exactly once among the rows of the subarray. This notion of combinatorial orthogonality has some

similarities with that from linear algebra, but it is a different concept.

An orthogonal array of *strength* s on D symbols is an array with c columns and D^s rows, such that every subset of s columns is orthogonal in the above sense. An example of an OA can be seen in Table 1.

a_1	a_2	b_1	b_2
0	0	0	0
0	1	1	2
0	2	2	1
1	0	1	1
1	1	2	0
1	2	0	2
2	0	2	2
2	1	0	1
2	2	1	0

Table 1: An example of an orthogonal array. Here, $X = \{0, 1, 2\}$, and the array is of strength 2. This means that within every pair of columns each ordered pair of symbols is present exactly once. The OA is described by a linear map $(b_1, b_2) = (a_1 + a_2, a_1 - a_2)$ over $\mathbb{Z}_3$.

In the case of an AME related to classical maps, we can compile an array with $D^{k/2}$ rows and k columns, such that each row is given by

$$(a_1, a_2, \dots, a_{k/2}, b_1, b_2, \dots, b_{k/2})$$

with some $(a_1, a_2, \dots, a_{k/2}) \in X^{k/2}$, and the b -variables are obtained according to (4.1). This way we obtain an OA of strength $k/2$. It is known that from every OA of strength $k/2$ and k columns one can construct an AME [7, 10, 11].

Let us now discuss the tensor network decomposition for the states obtained from OA's. Focusing again on the case of $k = 6$, we consider an AME of the form

$$|\Psi\rangle = \frac{1}{D^{3/2}} \sum_{a_1, a_2, a_3} |a_1, a_2, a_3, b_1, b_2, b_3\rangle,$$

where $a_j, b_j \in \{0, 1, 2, \dots, D-1\}$, and the b -variables are given by the function $u : X^3 \rightarrow X^3$ as

$$(b_1, b_2, b_3) = u(a_1, a_2, a_3).$$

Then we intend to find a factorization analogous to (3.1):

$$u^{(123)} = \mathcal{C}^{(23)} \mathcal{B}^{(13)} \mathcal{A}^{(12)}, \quad (4.2)$$

where now $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are classical functions acting from X^2 to X^2 . Similar to the factorization into unitary operators, we expect that these functions will typically correspond to AME's, and therefore describe orthogonal arrays themselves. These smaller OA's will be of size $D^2 \times 4$, and strength 2. To our knowledge, this type of decomposition of OA's into smaller OA's has not yet been considered in the mathematical literature.

Similar to the arguments presented in Subsection 3.2, we can establish that if we construct an array via the function $u : X^3 \rightarrow X^3$ as given by the product in (4.2), such that the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}$ describe orthogonal arrays, then in the resulting array the orthogonality condition will hold for 12 out of the 20 subsets of size 3.

4.1 Linear maps

Often it is convenient to work with linear maps over finite fields. To this end, let us assume that the set X is identified with a finite field $\mathbb{F}$ of order D . Then for general k the function $u : X^{k/2} \rightarrow X^{k/2}$ can be represented by a matrix G of size $k/2 \times k/2$ over $\mathbb{F}$. For the matrix elements we will use the standard notation

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

and it is understood that the mapping $u : X^3 \rightarrow X^3$ is performed as

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = G \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Matrices over finite fields for which all minors are non-vanishing are called *superregular* (or in some references, *full superregular*) [47, 48]. It is well known that such matrices are directly related to linear MDS codes, and they generate orthogonal arrays [47, 48]. One

family of superregular matrices are the *Cauchy matrices*. For a finite field $\mathbb{F}$ they are given by

$$g_{ij} = \frac{1}{x_i - y_j}, \quad (4.3)$$

where $x_i, y_j \in \mathbb{F}$, for $1 \leq i \leq k$ and $1 \leq j \leq k$, such that all variables are distinct. The determinant of the Cauchy matrix (4.3) is

$$\det(G) = \frac{\prod_{i < j} (x_i - x_j)(y_j - y_i)}{\prod_{i,j} (x_i - y_j)}.$$

It follows that the determinant is non-zero. Every submatrix is also a Cauchy matrix, therefore all minors are non-zero, and G is indeed superregular. A Cauchy matrix of size $k/2 \times k/2$ requires a total number of k distinct elements, therefore this type of superregular matrix can exist only if $D \geq k$.

One can argue that if we keep k fixed, and then increase D , then the probability that a given matrix is superregular is increased, such that in the $D \rightarrow \infty$ limit this probability becomes 1. From the well known formula for the order of the general linear group, we can deduce that the probability of any given $c \times c$ minor of a random matrix being zero is

$$1 - \prod_{i=1}^c (1 - D^{-i}),$$

which is of order $1/D$. For fixed k we have a fixed number of minors that need to be checked. So, by the union bound, the probability that at least one of them is zero is still of order $1/D$. Thus the probability of a random matrix being superregular tends to 1. For small values of D they are comparatively rare.

4.2 Factorization with linear maps

Let us now consider the factorization problem for $k = 6$. We assume that both the function $u : X^3 \rightarrow X^3$ and also the functions $\mathcal{A}, \mathcal{B}, \mathcal{C} : X^2 \rightarrow X^2$ are linear. Therefore, they can be represented by matrices of size 3×3 and 2×2 , respectively.

Starting with the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}$, we represent them as

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ B &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ C &= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \end{aligned}$$

They can be embedded into maps acting on X^3 simply as

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \tilde{B} &= \begin{bmatrix} b_{11} & 0 & b_{12} \\ 0 & 1 & 0 \\ b_{21} & 0 & b_{22} \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{bmatrix}. \end{aligned} \quad (4.4)$$

Finally, the factorization on the level of 3×3 matrices is performed as

$$G = \tilde{C}\tilde{B}\tilde{A}, \quad \text{and} \quad G = \tilde{A}\tilde{B}\tilde{C} \quad (4.5)$$

for $\triangleright$ -decomposition and $\triangleleft$ -decomposition, respectively.

Then the remaining task is to find matrices G, A, B, C , such that one of the decompositions in (4.5) holds, and that all the minors of G are invertible.

It is important that the factorizations in (4.5) are never unique: there is some gauge freedom in the choice of the factors. Consider the diagonal matrices

$$E_1 = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e_3 \end{bmatrix},$$

where $e_1, e_3 \in \mathbb{F}$ are nonzero. Then we can write, for example,

$$G = \tilde{C}E_3(E_3)^{-1}\tilde{B}E_1(E_1)^{-1}\tilde{A},$$

which yields the gauge transformations

$\tilde{C} \rightarrow \tilde{C}E_3$, $\tilde{B} \rightarrow (E_3)^{-1}\tilde{B}E_1$, $\tilde{A} \rightarrow (E_1)^{-1}\tilde{A}$, such that the specific form of the matrices given in (4.4) is retained.

The factorization (4.5) is very natural for matrices over the complex numbers. One can argue that in $GL(3, \mathbb{C})$ almost all matrices G can be factorized this way: it is easiest to see this when every matrix is close to the identity matrix, in which case we can immediately read off the leading terms in the matrices A, B, C (up to the gauge freedom). However, in the case of finite fields it happens with nonvanishing probability that such a factorization is not possible. In the next section we compute the necessary conditions for the factorization for the case of $k = 6$. Finally, in Section 6 we provide a number of examples, with a factorizable G that also leads to an AME.

5 Conditions of factorizability

There are two possible strategies for finding solutions to our problem. One can start with the matrices A, B, C , requiring that they are superregular, and afterwards requiring that the product matrix G is also superregular. Alternatively, one can start with a superregular G , and investigate whether it can be factorized in the way we require. We choose this second path, and we find that whenever the factorization is possible, the factors are perfect in each case.

We start with a $\triangleright$ -decomposition of a gate G into the product (4.5). We assume that A, B, C all correspond to dual unitary gates. Hence each element of $\{a_{11}, a_{22}, b_{11}, b_{22}, c_{11}, c_{22}\}$ must be nonzero in $\mathbb{F}$. Then, by gauge transformation we may assume $a_{11} = a_{22} = b_{22} = 1$. After some algebraic manipulations (see Appendix A), this leads to the following general solution:

$$a_{11} = 1 \quad (5.1)$$

$$a_{12} = \frac{g_{12}}{g_{11}} \quad (5.2)$$

$$a_{21} = \frac{g_{21}g_{33} - g_{23}g_{31}}{g_{22}g_{33} - g_{23}g_{32}} \quad (5.3)$$

$$a_{22} = 1 \quad (5.4)$$

$$b_{11} = g_{11} \quad (5.5)$$

$$b_{12} = g_{13} \quad (5.6)$$

$$b_{21} = \frac{g_{31} - c_{21}a_{21}}{g_{33}} = \frac{g_{11}(g_{22}g_{31} - g_{21}g_{32})}{g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33}} \quad (5.7)$$

$$b_{22} = 1 \quad (5.8)$$

$$c_{11} = \frac{g_{22} - a_{12}g_{21}}{1 - a_{12}a_{21}} = \frac{(g_{11}g_{22} - g_{12}g_{21})(g_{22}g_{33} - g_{23}g_{32})}{g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33}} \quad (5.9)$$

$$c_{12} = g_{23} \quad (5.10)$$

$$c_{21} = \frac{g_{32} - a_{12}g_{31}}{1 - a_{12}a_{21}} = \frac{(g_{11}g_{32} - g_{12}g_{31})(g_{22}g_{33} - g_{23}g_{32})}{g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33}} \quad (5.11)$$

$$c_{22} = g_{33}. \quad (5.12)$$

If G has a $\triangleright$ -decomposition then it must be given by this general solution. Moreover, it is easy to check that this solution will work when

$$g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} \neq 0 \quad (5.13)$$

in $\mathbb{F}$, and not otherwise. Another notable feature of the solution is that all variables in it are necessarily nonzero (assuming G is super-regular and (5.13) holds) and that A, B, C will be nonsingular. In other words, if G represents a linear perfect tensor with 6 legs, and it has

a $\triangleright$ -decomposition into linear tensors with 4 legs, then those factors will themselves be perfect. It is an open question whether this statement is true in the nonlinear case for $k = 6$.

Applying the space reflection and interchanging A with C , we find that the necessary and sufficient condition for the gate G to have a $\triangleleft$ -decomposition is that

$$g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} \neq 0$$

in $\mathbb{F}$. If G has a decomposition $G = \tilde{A}\tilde{B}\tilde{C}$ then it will be given by

$$\begin{aligned}
a_{11} &= \frac{(g_{22}g_{33} - g_{23}g_{32})(g_{11}g_{22} - g_{12}g_{21})}{g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{12}g_{21}g_{33} - g_{11}g_{23}g_{32}} \\
a_{12} &= g_{21} \\
a_{21} &= \frac{(g_{12}g_{33} - g_{13}g_{32})(g_{11}g_{22} - g_{12}g_{21})}{g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{12}g_{21}g_{33} - g_{11}g_{23}g_{32}} \\
a_{22} &= g_{11} \\
b_{11} &= g_{33} \\
b_{12} &= g_{31} \\
b_{21} &= \frac{g_{33}(g_{13}g_{22} - g_{12}g_{23})}{g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{12}g_{21}g_{33} - g_{11}g_{23}g_{32}} \\
b_{22} &= 1 \\
c_{11} &= 1 \\
c_{12} &= \frac{g_{32}}{g_{33}} \\
c_{21} &= \frac{g_{11}g_{23} - g_{13}g_{21}}{g_{11}g_{22} - g_{12}g_{21}} \\
c_{22} &= 1.
\end{aligned}$$

Similar to the question of superregularity, we can give a probabilistic argument that shows that for large D almost all superregular matrices can be factorized. In this simple case, if we keep $k = 3$ and increase D , then the probability of not finding a factorization decreases as $1/D$.

6 Examples

6.1 $D = 2$ and $D = 3$

For the finite fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$ there are no superregular matrices of size 3×3 . Indeed, there are no orthogonal arrays of strength 3 on 6 columns for $D \leq 3$, since they do not satisfy the Bush bound [49]. Therefore, our methods are not applicable in these cases.

6.2 $D = 4$

In the case of $D = 4$ we have the finite field $GF(4)$, which is given by the four elements $\{0, 1, \omega, \omega^2\}$, together with the algebraic relations

$$\omega + \omega = \omega^2 + \omega^2 = 0, \quad \omega^2 = \omega + 1.$$

It is well known [49] that the following Vandermonde matrix is superregular:

$$G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \quad (6.1)$$

and hence produces an orthogonal array with 6 columns, 4 symbols, and strength 3. Therefore, it leads to an AME with $D = 4$ and $k = 6$. It is known [48] that up to equivalence relations this is the only superregular matrix for $D = 4$ and $k = 6$.

However, this G does not have a $\triangleright$ -decomposition since

$$g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} = 2\omega^2 - 2\omega = 0.$$

It also does not have a $\triangleleft$ -decomposition since

$$g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} = 2\omega^2 - 2\omega = 0.$$

6.3 $D \geq 5$

Now we find examples for all finite fields with dimension $D \geq 5$.

Example 1. Consider using the matrix (6.1) over some field, with some choice of ω . Provided $\omega \notin \{0, \pm 1, -2\}$, the result will be superregular. This can be seen by considering its minors and its determinant; the determinant is

$$\det(G) = -\omega(\omega - 1)^2(\omega + 2).$$

Furthermore, we find

$$\begin{aligned} & g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} \\ &= g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} \\ &= 2\omega^2 - \omega^4 - \omega = \omega(1 - \omega)(\omega^2 + \omega - 1). \end{aligned}$$

In any field of order strictly larger than 5, there will be a choice of ω such that this quantity is nonzero. In that case we will have a $\triangleright$ -decomposition and a $\triangleleft$ -decomposition of a perfect tensor.

Example 2. Let

$$G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}.$$

Then G is a perfect tensor with a $\triangleright$ -decomposition over fields of characteristic 5, because

$$\begin{aligned} & g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} \\ &= -2 - 2 - 4 + 1 = -7. \end{aligned}$$

It also has a $\triangleleft$ -decomposition because

$$\begin{aligned} & g_{11}g_{22}g_{33} + g_{13}g_{21}g_{32} - g_{11}g_{23}g_{32} - g_{12}g_{21}g_{33} \\ &= -7. \end{aligned}$$

Therefore we obtain solutions for any $D = 5^n$ with $n \geq 1$. This fills the gap left by the previous example.

Example 3. It is easy to write down conditions under which we can factorize a perfect tensor into factors that satisfy the Yang-Baxter equation (3.2). Using (4.4) and equating coefficients we find that necessary and suf-

ficient conditions for $\tilde{A}\tilde{B}\tilde{C} = \tilde{C}\tilde{B}\tilde{A}$ are that

$$\begin{aligned} a_{12}c_{11} + a_{11}b_{12}c_{21} &= b_{11}a_{12} \\ a_{12}c_{12} + a_{11}b_{12}c_{22} &= b_{12} \\ a_{21}b_{11} &= a_{21}c_{11} + a_{11}b_{21}c_{12} \\ a_{21}b_{12}c_{21} &= a_{12}b_{21}c_{12} \\ a_{22}c_{12} + a_{21}b_{12}c_{22} &= b_{22}c_{12} \\ b_{21} &= a_{21}c_{21} + a_{11}b_{21}c_{22} \\ b_{22}c_{21} &= a_{22}c_{21} + a_{12}b_{21}c_{22}. \end{aligned}$$

If we assume that a_{21} and c_{12} are nonzero (which is true when A, B, C are perfect), then this system of equations is equivalent to

$$\begin{aligned} a_{12} &= \frac{b_{12}}{c_{12}}(1 - a_{11}c_{22}) \\ c_{21} &= \frac{b_{21}}{a_{21}}(1 - a_{11}c_{22}) \\ a_{22} &= b_{22} - \frac{a_{21}b_{12}c_{22}}{c_{12}} \\ c_{11} &= b_{11} - \frac{a_{11}b_{21}c_{12}}{a_{21}}, \end{aligned}$$

with $a_{11}, a_{21}, b_{11}, b_{12}, b_{21}, b_{22}, c_{12}, c_{22}$ arbitrary. Adding a requirement for A, B, C and $G = ABC$ to be perfect adds restrictions that all minors are nonzero. But these conditions are possible to satisfy simultaneously. As a concrete example, consider again $D = 5$ and the superregular matrix

$$G = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix}.$$

This can be factorized in both ways, with the 2×2 matrices given by

$$A = \begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}.$$

7 Extension to $k = 8$

In this Section we extend the previous results to AME states with 8 parties. In this case the AME is equivalent to unitary operators U that act on $\mathcal{H}^{(4)}$. We are looking for a tensor network decomposition of the AME using 4-leg tensors, which is equivalent to a factorization of the operator U into two-site unitary

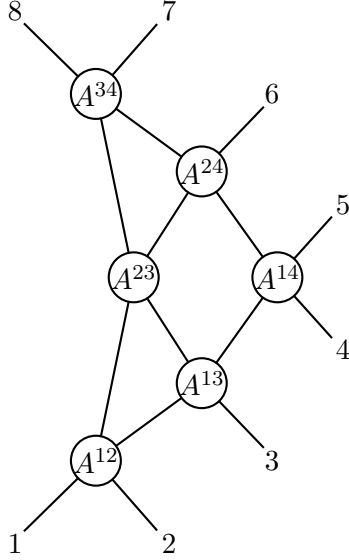


Figure 4: Tensor network for an AME with $k = 8$ parties. The labeling of the four-leg tensors $A^{(ab)}$ with $1 \leq a < b \leq 4$ follows from the operator/state correspondence described in the text, see eq. (7.1).

operators. There are multiple ways to perform such a factorization. Given a selected ordering of sites, we focus on the factorization

$$U = A^{(34)} A^{(24)} A^{(23)} A^{(14)} A^{(13)} A^{(12)}, \quad (7.1)$$

where now the operators $A^{(jk)}$ act only on sites j and k . This factorization, which is illustrated in Fig. 4, includes one two-site operator for each pair (j, k) of sites.

Once again we only consider those AME which are related to orthogonal arrays obtained from linear maps. Similar to the case of $k = 6$, this means that eventually we will be dealing with matrices of small sizes over finite fields. To be more concrete, the 4-site unitary operator U will be described by a 4×4 matrix G . Then we are looking for a factorization of the form

$$G = \tilde{A}^{(34)} \tilde{A}^{(24)} \tilde{A}^{(23)} \tilde{A}^{(14)} \tilde{A}^{(13)} \tilde{A}^{(12)}, \quad (7.2)$$

where now $\tilde{A}^{(j,k)}$ are matrices of size 4×4 , such that they are equal to the identity matrix except for the rows and columns j and k , where they are given by the elements of a 2×2 matrix $A^{(j,k)}$.

Direct computation and using the gauge freedom solves the factorization as follows. Matrix elements of the factors will be denoted as $A_{lm}^{(jk)}$, where the pairs of indices (jk) stand for the choice of the matrix, and lm stand for the row and column indices with $l, m = 1, 2$. We find

$$\begin{aligned} A_{12}^{(12)} &= g_{12}/g_{11}, & A_{21}^{(12)} &= m_{12}/m_{11}, \\ A_{12}^{(13)} &= g_{13}/g_{11}, & A_{21}^{(13)} &= \frac{m_{13}g_{11}}{M_{12} - M_{11}}, \\ A_{11}^{(14)} &= g_{11}, & A_{12}^{(14)} &= g_{14}, \\ A_{21}^{(14)} &= g_{11}m_{14}/(M_{24} - M_{34} + M_{44}), \\ A_{12}^{(23)} &= \frac{(M_{11} - M_{12})(g_{23}(M_{24} - M_{34} + M_{44}) - g_{13}g_{24}m_{14})}{N_{12}(M_{24} - M_{34} + M_{44})m_{11}}, \\ A_{21}^{(23)} &= \frac{m_{11}(g_{11}g_{32}g_{44} - g_{12}g_{31}g_{44} + g_{12}g_{34}g_{41} - g_{11}g_{34}g_{42})}{N_{34}(M_{11} - M_{12})}, \\ A_{11}^{(24)} &= m_{11}N_{12}/(M_{11} - M_{12}), & A_{12}^{(24)} &= g_{24}, \\ A_{21}^{(24)} &= \frac{-m_{11}m_{24}N_{12}}{(M_{11} - M_{12})(M_{44} - M_{34})}, \\ A_{11}^{(34)} &= N_{34}m_{44}/(M_{44} - M_{34}), & A_{12}^{(34)} &= g_{34}, \\ A_{21}^{(34)} &= N_{34}m_{34}/(M_{44} - M_{34}), & A_{22}^{(34)} &= g_{44}. \end{aligned}$$

Here $G = [g_{ij}]$ is the final matrix, $N_{ij} = \det G[i, j|i, j]$ is the determinant of the submatrix induced by rows and columns i and j , and $m_{ij} = \det G(i, j)$ is the determinant of the 3×3 submatrix of G with row i and column j deleted, and $M_{ij} = g_{ij}m_{ij}$. Every single matrix element that is not included in the list above was set to unity by gauge freedom. The conditions which are needed to prevent division by zero are:

$$M_{11} \neq M_{12}, M_{34} \neq M_{44}, M_{24} - M_{34} + M_{44} \neq 0, N_{12} \neq 0, N_{34} \neq 0, m_{11} \neq 0.$$

The last three of these conditions are automatically satisfied when G is superregular.

7.1 Examples

By the Bush bound [49] there are no orthogonal arrays on 8 columns with strength 4 unless $D \geq 6$. Famously, there is no orthogonal array on 4 columns with strength 2 with $D = 6$ symbols (since that would provide a pair of orthogonal Latin squares of order 6). It follows that for $D = 6$ there is also no orthogonal array of strength 3 on 6 columns or of strength 4 on 8 columns. Hence the smallest D for which a 4×4 superregular matrix might occur is $D = 7$. Our first two examples meet this bound.

Example 4. We take $\mathbb{F} = GF(7)$ and let G be the superregular matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 2 & 6 \\ 1 & 6 & 5 & 4 \end{bmatrix}.$$

By our solution it factorizes with

$$\begin{aligned} A^{(12)} &= \begin{bmatrix} 1 & 1 \\ 6 & 1 \end{bmatrix} & A^{(13)} &= \begin{bmatrix} 1 & 1 \\ 6 & 1 \end{bmatrix} \\ A^{(14)} &= \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix} & A^{(23)} &= \begin{bmatrix} 1 & 5 \\ 4 & 1 \end{bmatrix} \\ A^{(24)} &= \begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix} & A^{(34)} &= \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}. \end{aligned}$$

Example 5. We take $\mathbb{F} = GF(7)$ and let G be the superregular matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 \\ 1 & 5 & 3 & 4 \end{bmatrix}.$$

By our solution it factorizes with

$$\begin{aligned} A^{(12)} &= \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} & A^{(13)} &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ A^{(14)} &= \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix} & A^{(23)} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A^{(24)} &= \begin{bmatrix} 6 & 5 \\ 3 & 1 \end{bmatrix} & A^{(34)} &= \begin{bmatrix} 4 & 3 \\ 4 & 4 \end{bmatrix}. \end{aligned}$$

Notably, $A^{(23)}$ is not perfect in this case, even though G is superregular. Such behaviour was impossible in the $k = 6$ case. In fact, since $A^{(23)}$ is the identity matrix, we see that this particular G can be factorized using just five 4-leg tensors.

Our next example is for $D = 8$. A local state of dimension 8 can be understood as the tensor product space of three qubits, therefore such a solution could be interesting quantum computation based on qubits.

Example 6. Let $\mathbb{F} = GF(8)$ be the field of order 8. It can be understood as $\mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$, or more formally as the algebra defined by the basis elements $1, x, x^2$ with coefficients in $\mathbb{Z}_2$ and the additional relation $x^3 = x + 1$. Now let us take G to be

$$\begin{bmatrix} x & x^2 & x^2 + x & x^2 + 1 \\ x^2 + x & 1 & x & x^2 + x + 1 \\ x^2 + x + 1 & x + 1 & x^2 + 1 & x^2 + x \\ 1 & x^2 + x & x^2 & x + 1 \end{bmatrix}.$$

This matrix is a Cauchy matrix with $(x_1, x_2, x_3, x_4) = (0, x^2 + x, 1, x^2)$ and $(y_1, y_2, y_3, y_4) = (x^2 + 1, x^2 + x + 1, x + 1, x)$ in (4.3), therefore all of its minors are nonzero, so we have a perfect tensor. Also, by our so-

lution above it factorizes with

$$\begin{aligned} A^{(12)} &= \begin{bmatrix} 1 & x \\ x+1 & 1 \end{bmatrix} \\ A^{(13)} &= \begin{bmatrix} 1 & x+1 \\ x^2+1 & 1 \end{bmatrix} \\ A^{(14)} &= \begin{bmatrix} x & x^2+1 \\ x^2+1 & 1 \end{bmatrix} \\ A^{(23)} &= \begin{bmatrix} 1 & x^2+x \\ x^2+x & 1 \end{bmatrix} \\ A^{(24)} &= \begin{bmatrix} x^2+1 & x^2+x+1 \\ x^2+1 & 1 \end{bmatrix} \\ A^{(34)} &= \begin{bmatrix} 1 & x^2+x \\ x^2+x & x+1 \end{bmatrix}. \end{aligned}$$

Example 7. We take $\mathbb{F} = GF(11)$ and let G be the superregular matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 9 \\ 1 & 7 & 8 & 5 \end{bmatrix}.$$

By our solution it factorizes with

$$\begin{aligned} A^{(12)} &= \begin{bmatrix} 1 & 1 \\ 8 & 1 \end{bmatrix} & A^{(13)} &= \begin{bmatrix} 1 & 1 \\ 10 & 1 \end{bmatrix} \\ A^{(14)} &= \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} & A^{(23)} &= \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} \\ A^{(24)} &= \begin{bmatrix} 3 & 4 \\ 8 & 1 \end{bmatrix} & A^{(34)} &= \begin{bmatrix} 4 & 9 \\ 3 & 5 \end{bmatrix}. \end{aligned}$$

Again, $A^{(23)}$ is not perfect, although in this example it is not the identity. Note that by inspecting the general solution we can see that $A^{(23)}$ is the only one of the 6 factors which might contain a zero entry.

Permuting the rows of G we get another superregular matrix

$$G' = \begin{bmatrix} 1 & 7 & 8 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

For this G' there is no decomposition of the form (7.2), since it satisfies $M_{34} = M_{44} = 10$.

8 Discussion

In this work we obtained tensor network decompositions for selected AME states. Our results are based on the classical linear maps. A drawback of our method is that we could obtain solutions only for relatively large local Hilbert spaces. For $k = 6$ we found solutions starting from $D \geq 5$, and for $k = 8$ our smallest solution is for $D = 7$. It would be interesting to develop alternative methods that could settle the problem for smaller local dimension. It would be interesting to establish decomposability or perhaps indecomposability of selected known AME's with $D = 2, 3, 4$.

Interestingly, we find that for large D it becomes easier to satisfy the conditions of factorizability. We gave a probabilistic argument that keeping k fixed and letting $D \rightarrow \infty$ we will find that almost all perfect tensors built from linear maps can be factorized. This is parallel to the statement that keeping k fixed and letting $D \rightarrow \infty$ it is always possible to find an AME [6], and that in larger and larger Hilbert spaces almost all states are close to maximally entangled [21].

Returning to small local dimensions, we remark that there are no perfect 4-leg tensors for $D = 2$ [50]. Therefore we conjectured that AME states with qubits cannot be decomposed in the way that we formulated here. It is known that for qubits all AME states are essentially graph states, which means that they can be obtained by the action of a larger number of two-site unitary operators on product states, but not with the small number of gates that we find for higher dimensions (see the discussion in Appendix B).

Alternatively, one can also consider a weaker set of constraints, by requiring maximal entanglement only for a selected set of bipartitions. As was mentioned in the Introduction, for such cases decompositions were already presented in the literature in multiple independent works [29, 40, 41, 43]. It would be useful to develop a general theory for such decompositions/factorizations.

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References

- [1] R. Orús, “Tensor networks for complex quantum systems,” *Nat. Rev. Phys.* **1** (2019) no. 9, 538–550, [arXiv:1812.04011](#) [[cond-mat.str-el](#)].
- [2] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, “Matrix Product State Representations,” *Quantum Info. Comput.* **7** (2007) no. 5, 401–430, [quant-ph/0608197](#).
- [3] J. I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, “Matrix product states and projected entangled pair states: Concepts, symmetries, theorems,” *Rev. Mod. Phys.* **93** (2021) 045003, [arXiv:2011.12127](#) [[quant-ph](#)].
- [4] P. Facchi, G. Florio, G. Parisi, and S. Pascazio, “Maximally multipartite entangled states,” *Phys. Rev. A* **77** (2008) no. 6, 060304, [arXiv:0710.2868](#) [[quant-ph](#)].
- [5] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H.-K. Lo, “Absolute maximal entanglement and quantum secret sharing,” *Phys. Rev. A* **86** (2012) no. 5, 052335, [arXiv:1204.2289](#) [[quant-ph](#)].
- [6] W. Helwig and W. Cui, “Absolutely Maximally Entangled States: Existence and Applications,” *arXiv e-prints* (2013), [arXiv:1306.2536](#) [[quant-ph](#)].
- [7] D. Goyeneche and K. Życzkowski, “Genuinely multipartite entangled states and orthogonal arrays,” *Phys. Rev. A* **90** (2014) no. 2, 022316, [arXiv:1404.3586](#) [[quant-ph](#)].
- [8] M. Gaeta, A. Klimov, and J. Lawrence, “Maximally entangled states of four nonbinary particles,” *Phys. Rev. A* **91** (2015) no. 1, 012332, [arXiv:1411.6178](#) [[quant-ph](#)].
- [9] Z. Raissi, C. Gogolin, A. Riera, and A. Acín, “Optimal quantum error correcting codes from absolutely maximally entangled states,” *J. Phys. A* **51** (2018) no. 7, 075301, [arXiv:1701.03359](#) [[quant-ph](#)].
- [10] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, “Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices,” *Phys. Rev. A* **92** (2015) no. 3, 032316, [arXiv:1506.08857](#) [[quant-ph](#)].
- [11] D. Goyeneche, Z. Raissi, S. Di Martino, and K. Życzkowski, “Entanglement and quantum combinatorial designs,” *Phys. Rev. A* **97** (2018) no. 6, 062326, [arXiv:1708.05946](#) [[quant-ph](#)].
- [12] A. S. Hedayat, N. J. Sloane, and J. Stufken, *Orthogonal Arrays: Theory and Applications*. Springer, 1999.
- [13] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, “Chaos in quantum channels,” *J. High Energy Phys.* **2016** (2016) 4, [arXiv:1511.04021](#) [[hep-th](#)].
- [14] F. Huber and N. Wyderka, “Table of AME states.” Online available, 2021. <https://www.tp.nt.uni-siegen.de/~fhuber/ame.html>.
- [15] F. Huber, C. Eltschka, J. Siewert, and O. Gühne, “Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity,” *J. Phys. A* **51** (2018) no. 17, 175301, [arXiv:1708.06298](#) [[quant-ph](#)].
- [16] S. A. Rather, A. Burchardt, W. Bruzda, G. Rajchel-Mieldzióć, A. Lakshminarayan, and K. Życzkowski, “Thirty-six Entangled Officers of Euler: Quantum Solution to a Classically

- Impossible Problem,” *Phys. Rev. Lett.* **128** (2022) no. 8, 080507, [arXiv:2104.05122 \[quant-ph\]](#).
- [17] K. Życzkowski, W. Bruzda, G. Rajchel-Mieldzioć, A. Burchardt, S. A. Rather, and A. Lakshminarayan, “ $9 \times 4 = 6 \times 6$: Understanding the quantum solution to the Euler’s problem of 36 officers,” *J. Phys.: Conf. Series* **2448** (2023) no. 1, 012003, [arXiv:2204.06800 \[quant-ph\]](#).
- [18] S. A. Rather, N. Ramadas, V. Kodyalam, and A. Lakshminarayan, “Absolutely maximally entangled state equivalence and the construction of infinite quantum solutions to the problem of 36 officers of Euler,” *Phys. Rev. A* **108** (2023) no. 3, 032412, [arXiv:2212.06737 \[quant-ph\]](#).
- [19] L. Chen and D. L. Zhou, “Graph states of prime-power dimension from generalized CNOT quantum circuit,” *Sci. Rep.* **6** (2016) 27135, [arXiv:1507.05386 \[quant-ph\]](#).
- [20] A. Cervera-Lierta, J. I. Latorre, and D. Goyeneche, “Quantum circuits for maximally entangled states,” *Phys. Rev. A* **100** (2019) no. 2, 022342, [arXiv:1904.07955 \[quant-ph\]](#).
- [21] D. N. Page, “Average entropy of a subsystem,” *Phys. Rev. Lett.* **71** (1993) no. 9, 1291–1294, [arXiv:gr-qc/9305007 \[gr-qc\]](#).
- [22] J. Haferkamp, P. Faist, N. B. T. Kothakonda, J. Eisert, and N. Yunger Halpern, “Linear growth of quantum circuit complexity,” *Nat. Phys.* **18** (2022) no. 5, 528–532, [arXiv:2106.05305 \[quant-ph\]](#).
- [23] T. Farrelly, R. J. Harris, N. A. McMahon, and T. M. Stace, “Tensor-Network Codes,” *Phys. Rev. Lett.* **127** (2021) no. 4, 040507, [arXiv:2009.10329 \[quant-ph\]](#).
- [24] C. Cao and B. Lackey, “Quantum Lego: Building Quantum Error Correction Codes from Tensor Networks,” *PRX Quantum* **3** (2022) no. 2, 020332, [arXiv:2109.08158 \[quant-ph\]](#).
- [25] T. Farrelly, D. K. Tuckett, and T. M. Stace, “Local tensor-network codes,” *New J. Phys.* **24** (2022) no. 4, 043015, [arXiv:2109.11996 \[quant-ph\]](#).
- [26] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, “Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence,” *J. High Energy Phys.* **2015** (2015) 149, [arXiv:1503.06237 \[hep-th\]](#).
- [27] T. Kibe, P. Mandayam, and A. Mukhopadhyay, “Holographic spacetime, black holes and quantum error correcting codes: A review,” *Eu. Phys. J. C* **82** (2022) no. 5, 463, [arXiv:2110.14669 \[hep-th\]](#).
- [28] A. Jahn and J. Eisert, “Holographic tensor network models and quantum error correction: a topical review,” *Quantum Sci. Tech.* **6** (2021) no. 3, 033002, [arXiv:2102.02619 \[quant-ph\]](#).
- [29] J. Berger and T. J. Osborne, “Perfect tangles,” *arXiv e-prints* (2018) , [arXiv:1804.03199 \[quant-ph\]](#).
- [30] M. Doroudiani and V. Karimipour, “Planar maximally entangled states,” *Phys. Rev. A* **102** (2020) no. 1, 012427, [arXiv:2004.00906 \[quant-ph\]](#).
- [31] R. J. Harris, N. A. McMahon, G. K. Brennen, and T. M. Stace, “Calderbank-Shor-Steane holographic quantum error-correcting codes,” *Phys. Rev. A* **98** (2018) 052301, [arXiv:1806.06472 \[quant-ph\]](#).
- [32] Y.-L. Wang, “Planar k-uniform states: a generalization of planar maximally entangled states,” *Quant. Inf. Proc.* **20** (2021) no. 8, 271, [arXiv:2106.12209 \[quant-ph\]](#).
- [33] B. Bertini, P. Kos, and T. Prosen, “Exact Correlation Functions for Dual-Unitary Lattice Models in 1+1

- Dimensions,” *Phys. Rev. Lett.* **123** (2019) no. 21, , [arXiv:1904.02140](#) [[cond-mat.stat-mech](#)].
- [34] V. F. R. Jones, “Planar algebras, I,” *New Zeal. J. Math.* **52** (2021) 1–107, [arXiv:math/9909027](#) [[math.QA](#)].
- [35] U. Krishnan and V. S. Sunder, “On Biunitary Permutation Matrices and Some Subfactors of Index 9,” *Trans. Amer. Math. Soc.* **348** (1996) no. 12, 4691–4736.
- [36] G. Evenbly, “Hyperinvariant Tensor Networks and Holography,” *Phys. Rev. Lett.* **119** (2017) no. 14, 141602, [arXiv:1704.04229](#) [[quant-ph](#)].
- [37] M. Steinberg and J. Prior, “Conformal Properties of Hyperinvariant Tensor Networks,” *Sci. Rep.* **12** (2022) 532, [arXiv:2012.09591](#) [[quant-ph](#)].
- [38] D. J. Reutter and J. Vicary, “Biunitary constructions in quantum information,” *Higher Structures* **3** (2019) no. 1, 109–154, [arXiv:1609.07775](#) [[quant-ph](#)].
- [39] P. W. Claeys, A. Lamacraft, and J. Vicary, “From dual-unitary to biunitary: a 2-categorical model for exactly-solvable many-body quantum dynamics,” [arXiv:2302.07280](#) [[quant-ph](#)].
- [40] C. Jonay, V. Khemani, and M. Ippoliti, “Triunitary quantum circuits,” *Phys. Rev. Res.* **3** (2021) no. 4, 043046, [arXiv:2106.07686](#) [[quant-ph](#)].
- [41] R. Milbradt, L. Scheller, C. Ałmus, and C. B. Mendl, “Ternary unitary quantum lattice models and circuits in $2 + 1$ dimensions,” *Phys. Rev. Lett.* **130** (2023) 090601, [arXiv:2206.01499](#) [[cond-mat.stat-mech](#)].
- [42] Y. Kasim and T. Prosen, “Dual unitary circuits in random geometries,” *J. Phys. A* **56** (2022) no. 2, 025003, [arXiv:2206.09665](#) [[cond-mat.stat-mech](#)].
- [43] G. M. Sommers, D. A. Huse, and M. J. Gullans, “Crystalline Quantum Circuits,” *PRX Quantum* **4** (2023) 030313, [arXiv:2210.10808](#) [[quant-ph](#)].
- [44] M. Mestyán, B. Pozsgay, and I. M. Wanless, “Multi-directional unitarity and maximal entanglement in spatially symmetric quantum states,” *SciPost Phys.* **16** (2024) no. 1, 010, [arXiv:2210.13017](#) [[quant-ph](#)].
- [45] J. I. Latorre and G. Sierra, “Platonic Entanglement,” *Quantum Inf. Comput.* **21** (13 & 14) (2021) 1081–1090, [arXiv:2107.04329](#) [[quant-ph](#)].
- [46] T. Gombor and B. Pozsgay, “Superintegrable cellular automata and dual unitary gates from Yang-Baxter maps,” *SciPost Phys.* **12** (2022) 102, [arXiv:2112.01854](#) [[cond-mat.stat-mech](#)].
- [47] R. Roth and A. Lempel, “On MDS codes via Cauchy matrices,” *IEEE Trans. Inf. Theory* **35** (1989) no. 6, 1314–1319.
- [48] G. Kéri, “Types of superregular matrices and the number of n -arcs and complete n -arcs in $PG(r, q)$,” *J. Comb. Des.* **14** (2006) no. 5, 363–390.
- [49] K. A. Bush, “Orthogonal arrays of index unity,” *Ann. Math. Statistics* **23** (1952) 426–434.
- [50] G. Gour and N. R. Wallach, “All maximally entangled four-qubit states,” *J. Math. Phys.* **51** (2010) no. 11, 112201–112201, [arXiv:1006.0036](#) [[quant-ph](#)].
- [51] W. Helwig, “Absolutely Maximally Entangled Qudit Graph States,” *arxiv e-prints* (2013) , [arXiv:1306.2879](#) [[quant-ph](#)].
- [52] M. van den Nest, J. Dehaene, and B. de Moor, “Graphical description of the action of local Clifford transformations on graph states,” *Phys. Rev. A* **69** (2004) no. 2, 022316, [arXiv:quant-ph/0308151](#) [[quant-ph](#)].

[53] M. Bahramgiri and S. Beigi, “Graph States Under the Action of Local Clifford Group in Non-Binary Case,” *arXiv e-prints* (2006), [arXiv:quant-ph/0610267](https://arxiv.org/abs/quant-ph/0610267) [quant-ph].

A Details of the decomposition for $N = 6$

Here we give the derivation of (5.1)–(5.12). Equating coefficients in $G = \tilde{C}\tilde{B}\tilde{A}$ we get

$$g_{11} = b_{11} \quad (\text{A.1})$$

$$g_{12} = b_{11}a_{12} \quad (\text{A.2})$$

$$g_{13} = b_{12} \quad (\text{A.3})$$

$$g_{21} = c_{11}a_{21} + c_{12}b_{21} \quad (\text{A.4})$$

$$g_{22} = c_{11} + c_{12}b_{21}a_{12} \quad (\text{A.5})$$

$$g_{23} = c_{12} \quad (\text{A.6})$$

$$g_{31} = c_{21}a_{21} + c_{22}b_{21} \quad (\text{A.7})$$

$$g_{32} = c_{21} + c_{22}b_{21}a_{12} \quad (\text{A.8})$$

$$g_{33} = c_{22}. \quad (\text{A.9})$$

Since $b_{11} \neq 0$, we see from (A.1) and (A.2) that

$$a_{12} = \frac{g_{12}}{g_{11}}. \quad (\text{A.10})$$

Substituting (A.5) and (A.6) into (A.4) gives

$$g_{21} = (g_{22} - g_{23}b_{21}a_{12})a_{21} + g_{23}b_{21}. \quad (\text{A.11})$$

Similarly, substituting (A.8) and (A.9) into (A.7) gives

$$g_{31} = (g_{32} - g_{33}b_{21}a_{12})a_{21} + g_{33}b_{21}. \quad (\text{A.12})$$

Multiplying (A.11) by g_{33} and subtracting g_{23} times (A.12), we find that

$$g_{33}g_{21} - g_{23}g_{31} = (g_{33}g_{22} - g_{23}g_{32})a_{21}.$$

Now G is superregular, so its order 2 minor $g_{33}g_{22} - g_{23}g_{32}$ is nonzero in $\mathbb{F}$. Hence

$$a_{21} = \frac{g_{33}g_{21} - g_{23}g_{31}}{g_{33}g_{22} - g_{23}g_{32}}.$$

Next, substituting (A.4) into (A.5) we find that

$$g_{22} = c_{11}(1 - a_{12}a_{21}) + a_{12}g_{21}.$$

From this we conclude that either $a_{12}a_{21} = 1$ and $g_{22} = a_{12}g_{21}$, or $a_{12}a_{21} \neq 1$ and

$$c_{11} = \frac{g_{22} - a_{12}g_{21}}{1 - a_{12}a_{21}}. \quad (\text{A.13})$$

Suppose that the former situation occurs. Then, by (A.10) we get that $g_{12}g_{21} = g_{11}g_{22}$. This says that the leading 2×2 minor of G is 0, contradicting superregularity. Hence, (A.13) must hold.

Applying similar reasoning, we find that

$$c_{21} = \frac{g_{32} - a_{12}g_{31}}{1 - a_{12}a_{21}}.$$

Then from (A.7) and the fact that $c_{22} \neq 0$ we find that

$$b_{21} = \frac{g_{31} - c_{21}a_{21}}{g_{33}}.$$

Equations (5.1)–(5.12) now follow.

B Comparison with graph states

It is known that the so-called graph states can be used to obtain AME’s for various values of the number of sites k and local dimension D [51]. Here we review this construction, and we consider selected examples. We compare this construction of AME’s with our results. We assume again that k is even, and for simplicity we consider the case when D is a prime [51]; for D being a prime power, see [19].

Let us choose a basis in the local spaces, denoted by $|a\rangle$, $a = 0, \dots, D-1$ and let us introduce the so-called controlled Z -operator, acting on the tensor product $\mathbb{C}^D \otimes \mathbb{C}^D$ as

$$\mathcal{Z} = \sum_{m,n=0}^{D-1} \omega^{mn} (|m\rangle\langle m| \otimes |n\rangle\langle n|),$$

where

$$\omega = e^{2\pi i/D}.$$

We extend this operator to the two-site operators $\mathcal{Z}_{i,j}$ that act on the N -fold tensor product of the local spaces $\mathbb{C}^D$, such that $\mathcal{Z}_{i,j}$ acts on the site i and j non-trivially. Note the exchange symmetry $i \leftrightarrow j$.

Graph states can be constructed as follows. We define an “initial state”, which is a tensor product of identical local states:

$$|\Psi_0\rangle = \otimes_{j=1}^N \left(\frac{1}{\sqrt{D}} \sum_{a=0}^{D-1} |a\rangle \right).$$

To each pair of non-identical sites (j, k) we associate an integer $\ell = 0, 1, \dots, D$. The graph states are then created as

$$|\Psi\rangle = \prod_{i < j} (\mathcal{Z}_{i,j})^{\ell_{i,j}} |\Psi_0\rangle.$$

The $\mathcal{Z}_{i,j}$ operators commute with each other, because they are all diagonal in the selected basis. Therefore, the ordering of the operator product is irrelevant. Such states are uniquely characterized by the numbers $\ell_{i,j}$ with $i < j$. These numbers encode a weighted graph on N vertices, hence the name “graph states”. In the following we denote by L the matrix of size $N \times N$ obtained via $L_{i,j} = L_{j,i} = \ell_{i,j}$ for any $i < j$ and $L_{j,j} = 0$ for any j .

The total number of $\mathcal{Z}$ gates applied for a selected state is

$$\sum_{i < j} \ell_{i,j}.$$

However, for any $\ell_{i,j} \neq 0$ one could count the operator power $(\mathcal{Z}_{i,j})^{\ell_{i,j}}$ as a single two-site gate. Then the number of the two-site gates applied is seen as the number of non-zero $\ell_{i,j}$.

If one starts from the product state $|\Psi_0\rangle$, then the application of a two-site gate $\mathcal{Z}_{i,j}$ will create “one unit of entanglement” (a Bell pair) in the state between sites i and j . However, it is not obvious what is the entanglement pattern in the final state, once many gates are applied. This problem was solved in the work [51]. It was shown that the final state is an AME if the symmetric matrix L satisfies the following property:

Property 1: For any bipartition of the numbers $(1, 2, \dots, k)$ into two equal sized subsets A and B , the minor of size $k/2 \times k/2$ of L corresponding to the rows taken from subset A and columns from subset B is non-zero in the finite field $\mathbb{F}$.

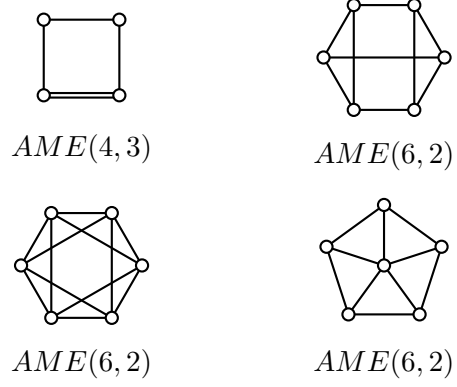


Figure 5: Examples for weighted graphs for $\text{AME}(k, D)$ with different values of the number of parties k and local dimension D . The number of single lines stands for the weight $\ell_{i,j}$ between sites i and j . The number of two-site gates needed to create these AME using the controlled Z -gates is 4, 9, 10 and 12, respectively.

Examples for graphs for which the graph states are AME are shown in Fig 5.

It is important that a selected AME can have multiple representations as a graph state. The different representations can be transformed into each other via local unitary operations, which can be represented also on the graphs themselves. This problem was investigated, for example, in [52, 53].

We next argue that the minimal number of gates required to create an AME via its graph state representation is $(k/2)^2$. The equivalent mathematical problem is to find a symmetric matrix L over the finite field $\mathbb{F}$, satisfying Property 1, such that the number of non-zero elements is minimal. In any column i of L there cannot be $k/2$ zeros in rows other than row i . If there were, then choosing those rows to form A would result in a submatrix with a column of zeros, leading to a minor that is zero. Thus at most $k/2$ entries in column i are zero. As this is true for all k columns, there are at most $k^2/2$ zeros in L , corresponding to $k^2/4$ gates in the AME. Equation (B.2) below shows one way to achieve this many zeros.

Now we show that the states that we treated in Section 4, which are obtained from classical linear maps, are in fact equivalent to graph states of a special form. Let us start with the

formula for this type of states

$$|\Psi\rangle = \frac{1}{D^{k/4}} \sum_{a_1, \dots, a_{k/2}=1}^D \left| a_1, \dots, a_{k/2}, b_1, \dots, b_{k/2} \right\rangle, \quad (\text{B.1})$$

where it is now understood that for each tuple $(a_1, a_2, \dots, a_{k/2})$ we have

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k/2} \end{bmatrix} = G \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k/2} \end{bmatrix},$$

where G is a matrix with size $k/2 \times k/2$ over

$\mathbb{F}$.

Let F be the Fourier matrix of size $D \times D$ with its matrix elements given by

$$F_{ab} = \frac{1}{\sqrt{D}} \omega^{ab}.$$

Let us now act with the Fourier transform on the last $k/2$ spaces. Formally, we write

$$|\Psi'\rangle = (1 \otimes 1 \otimes \dots \otimes 1 \otimes F \otimes F \otimes \dots \otimes F) |\Psi\rangle,$$

where $|\Psi\rangle$ is a state of the form (B.1). We get

$$|\Psi'\rangle = \frac{1}{D^{k/2}} \sum_{a_m, c_n=1}^D \omega^{\sum_{j,m=1}^{k/2} G_{jm} c_j a_m} \left| a_1, a_2, \dots, a_{k/2}, c_1, c_2, \dots, c_{k/2} \right\rangle.$$

This is a graph state with an incidence matrix of size $k \times k$ given by the block matrix

$$L = \begin{pmatrix} 0 & G^T \\ G & 0 \end{pmatrix}, \quad (\text{B.2})$$

where each block is of size $k/2 \times k/2$. If G is super-regular, then L satisfies Property 1.

The special graph states of the above form are created with a total number of $(k/2)^2$ gates from the product states $|\Psi_0\rangle$. In the cases of $k = 6$ and $k = 8$ this means a total number of 9 and 16 gates. As was shown above, this is the minimal number of gates with the graph state construction.

This is to be contrasted with our construction for the tensor network decompositions, where we create the same states with a total number of 6 and 10 gates, respectively (see Section 3.3).