Covariant catalysis requires correlations and good quantum reference frames degrade little

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Catalysts are quantum systems that open up dynamical pathways between quantum states which are otherwise inaccessible under a given set of operational restrictions while, at the same time, they do not change their quantum state. We here consider the restrictions imposed by symmetries and conservation laws, where any quantum channel has to be covariant with respect to the unitary representation of a symmetry group, and present two results. First, for an exact catalyst to be useful, it has to build up correlations to either the system of interest or the degrees of freedom dilating the given process to covariant unitary dynamics. This explains why catalysts in pure states are useless. Second, if a quantum system ("reference frame") is used to simulate to high precision unitary dynamics (which possibly violates the conservation law) on another system via a global, covariant quantum channel, then this channel can be chosen so that the reference frame is approximately catalytic. In other words, a reference frame that simulates unitary dynamics to high precision degrades only very little.

1 Introduction

When a quantum system C takes part in a physical interaction with a different system S, its quantum state typically changes. Moreover, the system C typically becomes correlated to system S (if the two weren't correlated, to begin with). Indeed, suppose contrarily that the joint time evolution of SC is given by a unitary operator U and the two systems end up uncorrelated, and the state of C did not change:

$$U\rho_S \otimes \sigma_C U^\dagger = \rho'_S \otimes \sigma_C. \quad (1)$$

Then there also exists a unitary V such that $V\rho_S V^\dagger = \rho'_S$. Thus, the system C is not required to realize the state-transition $\rho_S \to \rho'_S$ on S.

Once the operations we can implement in an experiment are restricted to some strict subset $\mathcal{O}$ of all conceivable quantum operations, however, there may be situations where the state-transition $\rho_S \to \rho'_S$ is *impossible* via operations in $\mathcal{O}$, but the state-transition $\rho_S \otimes \sigma_C \to \rho'_S \otimes \sigma_C$ *is possible* via operations in $\mathcal{O}$. Thus, the mere presence of system C seems to allow for larger sets of state transitions on S. The prototypical example is given by the set $\mathcal{O}_{\mathrm{LOCC}}$ of *local operations and classical communication* (LOCC), where $S = AB$ and $C = C_A C_B$ each consist of two parts distributed between two players, Alice and Bob. Alice and Bob each can only perform quantum operations on their local systems but can communicate classically, for example, by a phone call. It was observed in Ref. [30] that for some pure quantum states $|\psi\rangle_{AB}, |\psi'\rangle_{AB}$ and $|\phi\rangle_C$ the state-transition $|\psi\rangle_{AB} \otimes |\phi\rangle_C \to |\psi'\rangle_{AB} \otimes |\phi\rangle_C$ is possible even though $|\psi\rangle_{AB} \to |\psi'\rangle_{AB}$ is impossible via LOCC protocols. Ref. [16] generalized this observation to mixed states. In such a situation, system C plays the role of a *catalyst* since it enables a state transition without changing its state itself. In particular, it may be re-used to implement the same state transition on a new system $S' = A'B'$.

In this work, we consider the role of catalysts when the operational restrictions are due to a set of conservation laws that need to be obeyed [9, 22, 28, 60]. As a consequence (see below), every quantum system S carries a unitary representation $g \mapsto W_S(g)$ of a connected Lie group G. The set of implementable operations, which we denote by $\mathcal{O}_{\mathrm{symm.}}$, corresponds to *covariant* quantum channels T. A quantum channel from system S to system S' is said to be covariant if it fulfills

$$T[W_S(g)\rho W_S(g)^\dagger] = W_{S'}(g)T[\rho]W_{S'}(g)^\dagger. \quad (2)$$

A quantum state ρ on S is called *symmetric* if $\rho = W_S(g)\rho W_S(g)^\dagger$ for all $g \in G$. A covariant quantum channel always maps symmetric states to symmetric states. Since all the operations we can implement are covariant, we can only prepare symmetric states. In such a setting, states that are not symmetric are valuable resources, as they can be used to implement quantum channels that are not covariant: If ω_E is an asymmetric state on system E, and T is a covariant quantum channel on SE, then the channel

$$\rho \mapsto \mathrm{Tr}_E[T[\rho \otimes \omega_E]] \quad (3)$$

is, in general, not covariant. In particular, a useful catalyst must be asymmetric; otherwise, the induced dynamics on S were covariant and would not require the catalyst.

In this context, asymmetric states are also called *quantum reference frames*, since they break the underlying symmetry and thereby allow to (partially) distinguish symmetry operations: If ρ is asymmetric, then at least for some group elements $g \in G$ the state ρ is (partially) distinguishable from $W_S(g)\rho W_S(g)^\dagger$. An example of a reference frame for rotations is a collection of spins in strongly polarized states. A superposition (in contrast to an incoherent mixture) of energy eigenstates provides a reference frame for the group of time translations (a phase-reference in the case of equally spaced energy levels). A *perfect* quantum reference corresponds to a quantum state ρ that is perfectly distinguishable from $W_S(g)\rho W_S(g)^\dagger$, i.e. the trace-distance

$$D(W_S(g)\rho W_S(g)^\dagger, \rho) := \frac{1}{2} \left\| W_S(g)\rho W_S(g)^\dagger - \rho \right\|_1 \tag{4}$$

is equal to 1 unless $g = 1$. Perfect reference frames cannot exist for continuous groups: If W_S is a strongly continuous (possibly projective) unitary representation on a (possibly infinite-dimensional) Hilbert-space, then $g \mapsto D(W_S(g)\rho W_S(g)^\dagger, \rho)$ is a continuous function on G and hence cannot be equal to the discontinuous map $g \mapsto 1 - \delta_{1g}$ [1].

This paper aims to show two results about catalysts in the context of conservation laws. First, we show that a useful catalyst must build up correlations to either the system of interest S or the environmental degrees of freedom E (or both) that dilate the covariant channel on SC to a covariant unitary channel. In particular, if the process on SC is already unitary, then C must become correlated to S. Since a quantum system in a pure state is always uncorrelated to any other system, catalysts must be mixed to be useful. This is in strong contrast to the setting of LOCC. Our result generalizes previous results showing that pure catalysts are useless in the case where only energy is preserved [15] or when the states of interest are pure [41]. We discuss the role of correlations in catalysis further below.

Our second result concerns the case where a quantum reference frame C is used to locally implement approximately unitary dynamics on S that may violate the conservation laws. Such a situation is quite common in a semi-classical limit, where, for example, a laser (described by a classical field) is used to induce a state transition of an atom from the ground state to the superposition of the ground state and an excited state, violating energy conservation. It is already known that a quantum reference frame that allows to locally implement approximately unitary dynamics must be much larger than S and strongly asymmetric [9, 23, 39–41, 44–46, 49, 56–59, 74]. In this sense

it must be close to macroscopic. For example, a laser described by a (quantum) coherent state with high expected photon number is of the right form. It is also known that a quantum reference frame generically degrades in quality when it is used due to back-action [1, 7, 8, 51]. We show that a quantum reference frame that allows implementation of local unitary dynamics to high precision can always be used so that it is approximately catalytic: its quantum state is only slightly perturbed. Therefore, such a quantum reference frame is also macroscopic in the sense of receiving little back-action. In particular, if it is initially in a pure state, it only becomes weakly correlated to S. The second result holds for covariant operations with respect to projective unitary representation of any group, not just connected Lie groups.

The paper is structured as follows: In section 2 we provide the necessary background on the framework that we work in, namely the resource theory of asymmetry. We also introduce the closely related resource theory of athermality as our first main result immediately transfers to that setting as well. In section 3 we then discuss the role of correlations for catalysis and present our first main result and its proof, which builds on Wiegmann's theorem in matrix theory. In section 4 we discuss how quantum reference frames can be used to locally circumvent conservation laws and present our second main result together with its proof, which relies on the information-disturbance tradeoff in quantum mechanics.

2 The resource theories of asymmetry and athermality

We now introduce the resource theory of asymmetry and then also the resource theory of athermality, which provides a model for thermodynamics. For more background on the resource theory of asymmetry, see Refs. [9, 22, 41, 44] and references therein and for more background on the resource theory of athermality, see Refs. [11, 12, 17, 21, 24, 27, 29, 73].

The resource theory of asymmetry formalizes the consequences of symmetries and conservation laws for the manipulation of quantum systems.

Symmetries are implemented in quantum physics by projective unitary representations of a symmetry group G on Hilbert-space [2]. It is then required that the Hamiltonian generating the time-evolution commutes with this representation. If the symmetry group G is a Lie group, then its Lie algebra $\mathfrak{g}$ is the set of infinitesimal generators of the group. We can view an infinitesimal generator X as an abstract conserved quantity because its representative X_S on a system

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[1]This argument even works in operator algebraic settings: If the system S is described by a von Neumann algebra $\mathcal{M}$ with $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ a (normal) action of G, then $\|\alpha_g(\rho) - \rho\|_{\mathcal{M}_*}$ is continuous for all normal states $\rho \in \mathcal{M}_*$.

[2]In fact, to be fully general, one should also consider anti-unitary operators, but here we restrict to (projective) unitary representations.

S is a Hermitian operator commuting with the system Hamiltonian H_S. Notationally, we distinguish an element X of the Lie algebra and its Hermitian representative X_S on a system S only by the label, so that $W_S(e^X) = e^{-iX_S}$.

In the resource theory of asymmetry with respect to a group G, it is assumed that every system S carries a projective unitary representation $g \mapsto W_S(g)$ of G commuting with its Hamiltonian H_S, with the consistency condition that any two separate systems, S_1 and S_2, carry jointly the tensor-product representation $g \mapsto W_{S_1}(g) \otimes W_{S_2}(g)$ when considered as a single system $S = S_1 S_2$. Then any unitary U realizing a physical operation on a (possibly multi-partite) system S must commute with the representation:

$$[U, W_S(g)] = 0 \quad \forall g \in G. \tag{5}$$

In the case of a connected Lie group, this is equivalent to

$$[U, X_S] = 0 \quad \forall X \in \mathfrak{g}. \tag{6}$$

Therefore, all X_S, with $X \in \mathfrak{g}$, are indeed conserved quantities. In the case of finite-dimensional Hilbert-spaces and for a connected Lie group G, we can, without loss of generality, assume that all systems are equipped with proper unitary representations [6]. For this, we pass to the universal covering group of G, i.e. the unique simply-connected group generated by the Lie algebra $\mathfrak{g}$. For a compound system $S = S_1 S_2$ we have

$$X_S = X_{S_1} \otimes \mathbb{1}_{S_2} + \mathbb{1}_{S_1} \otimes X_{S_2} \quad X \in \mathfrak{g}. \tag{7}$$

We may further enrich the set of possible operations $\mathcal{O}_{\text{symm}}$ by assuming that we can prepare "for free" arbitrary systems E in an arbitrary *symmetric density matrix* ω_E, which fulfills $[\omega_E, W_E(g)] = 0$ for all $g \in G$. Symmetric states may be interpreted as those which can be prepared without access to a reference frame for the group [9]. By further being able to discard subsystems (formally tracing them out), it is then possible to implement any quantum channel T between two systems S and S' that is *covariant*:

$$T[W_S(g)\rho_S W_S(g)^\dagger] = W_{S'}(g)T[\rho_S]W_{S'}(g)^\dagger \quad \forall g \in G. \tag{8}$$

Indeed, the *covariant Stinespring theorem* states that any covariant quantum channel T can be written as

$$T[\rho] = \text{Tr}_E[U\rho \otimes \omega_E U^\dagger], \tag{9}$$

where ω_E is a symmetric state that, in fact, may be assumed to be pure [31]. The preparation of a symmetric state on S can be seen as a covariant quantum channel from a trivial system to S.

Summarizing, the resource theory of asymmetry consists of quantum channels that are covariant with respect to the (projective) unitary representations of

a symmetry group G. As already mentioned in the introduction, any quantum state that is not symmetric can be understood as a reference frame for the group G and may be utilized to implement non-covariant quantum channels via a covariant unitary U: If $[\omega_E, W_E(G)] \neq 0$, then in general the quantum channel

$$\rho \mapsto \text{Tr}_2[U\rho \otimes \omega_E U^\dagger] \tag{10}$$

is not covariant, even though U commutes with $W_S(g) \otimes W_E(g)$ for all $g \in G$.

Let us now discuss the resource theory of athermality. It is closely related to the resource theory of asymmetry for the group $G = \mathbb{R}$ of time translations, which enforces strict energy conservation in the sense that any unitary operation has to commute with the full Hamiltonian. In this case, the group representation is simply given by unitary time-evolution with the system's Hamiltonian, $t \mapsto W_S(t) = \exp(-itH_S)$. In contrast to the resource theory of asymmetry where arbitrary symmetric states can be prepared "for free", the resource theory of athermality only allows for the Gibbs states at some fixed inverse temperature $\beta > 0$, i.e. states of the form $\omega_S = \exp(-\beta H_S)/Z$ with $Z = \text{Tr}[\exp(-\beta H_S)]$, instead of arbitrary symmetric states. The underlying idea is that such systems represent heat baths at some fixed environment temperature. Therefore, we also only allow for those covariant quantum channels T that can be realized by appending a Gibbs state of an arbitrary Hamiltonian H_E, applying an energy-conserving unitary and tracing out a subsystem:

$$T[\rho] = \text{Tr}_E[U\rho \otimes \omega_E U^\dagger], \tag{11}$$

where $\omega_E = \exp(-\beta H_E)/Z$ and $[U, H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E] = 0$. We refer to the resulting quantum channels as "thermal operations" [27]. Any thermal operation has the Gibbs state at the fixed inverse temperature β as a fixed-point: $T[\exp(-\beta H_S)] = \exp(-\beta H_S)$. This can be seen as enforcing the second law of thermodynamics: without spending resources, we cannot produce a non-equilibrium state from thermal equilibrium. Any unitary quantum channel that is covariant with respect to time translations automatically has the Gibbs state as a fixed point and is a valid quantum channel in this framework.

3 Correlations and catalysis

We now discuss the role of correlations for catalysis and discuss our first main result. Traditionally, catalysis in the context of quantum mechanics, in particular LOCC, always referred to a situation where $\rho_S \to \rho_S'$ is impossible under the given set of operations $\mathcal{O}$, but $\rho_S \otimes \sigma_C \to \rho_S' \otimes \sigma_C$ is possible. In other words, no correlations between S and C were

allowed. This assumption can be justified by considering several catalytic transformations. For example, assume S interacts with C_1 in such a way that S and C_1 became correlated, but C_1 did not change its state. Afterwards, S interacts with C_2 in a similar fashion. Then it will typically be the case that C_1 and C_2 become correlated. Therefore, while each is catalytic on their own, the joint-system $C_1 C_2$ is not catalytic. Similarly, if a second system S' interacts with C_1 in the same way as S did, then afterwards, the reduced density matrix of S' will be precisely the same as that of S. However, the two will typically be correlated [10, 61]. In other words, once we allow for the build-up of correlations, these correlations will typically spread in an uncontrolled way (the only way to completely prevent this is to demand that the states of catalysts are pure).

However, assume now that the interaction of S and C_1 is not unitary so that auxiliary degrees of freedom E are involved, as is typically the case in a resource theory. Then even though these degrees of freedom may be considered "for free", they can similarly result in uncontrolled spreading of correlations if C_1 (or S for that matter) becomes correlated to them. Hence, if one does not restrict correlations with E, why restrict those between S and C? Moreover, E is typically treated implicitly so that the correlations to E do not appear in the mathematical treatment.

On the other hand, it has recently been shown in a number of works that it can be quite beneficial to allow for correlations between catalyst and system as they can significantly enhance the set of reachable states from a given initial state [10, 32, 35, 48, 52, 54, 67, 68], even if no auxiliary degrees of freedom E are allowed for [10, 67, 68]. In fact, oftentimes such *correlated catalysis* expands the set of reachable states for a single system to the asymptotic set of reachable states in the thermodynamic limit: Suppose for any $\epsilon > 0$ there exists a number n such that the state transition $\rho^{\otimes n} \to \rho'_{n,\epsilon}$ is possible without a catalyst, where $\rho'_{n,\epsilon}$ approximates $\rho'^{\otimes n}$ up to error ϵ in trace-distance. It is quite common that such a situation occurs even though the transition $\rho \to \rho'$ is impossible with good accuracy. The asymptotic limit of many uncorrelated copies therefore strongly regularizes possible state-transitions. The same regularization can be achieved with correlated catalysis: In the setting described above, for any $\epsilon > 0$ a state-transition $\rho \to \rho'_\epsilon$ is possible with correlated catalysis, where ρ'_ϵ is ϵ-close to ρ' in trace-distance. Moreover, often these beneficial effects are possible while building up arbitrary little correlations between S and C as measured by the trace distance to the closes product state. At the same time, it is typically unknown how strongly C becomes correlated to E.

These considerations motivate our first main result. It shows that a catalyst in the resource theories of asymmetry and athermality can only be useful if it either becomes correlated to S or E (or both).

Theorem 1 (Correlations are neccessary for catalysis). *Let G be a connected Lie group and let $g \mapsto W_S(g)$, $g \mapsto W'_S(g)$ and $g \mapsto W_C(g)$ be finite-dimensional unitary representations on systems S and C, respectively. Let ρ_S, ρ'_S be density matrices on S and σ_C a density matrix on C. Finally, suppose a unitary U on SC fulfills*

$$U \rho_S \otimes \sigma_C U^\dagger = \rho'_S \otimes \sigma_C \tag{12}$$

and for all $g \in G$

$$U W_S(g) \otimes W_C(g) = W'_S(g) \otimes W_C(g) U. \tag{13}$$

Then there exists a unitary V on S such that $V \rho_S V^\dagger = \rho'_S$ and $V W_S(g) = W'_S(g) V$ for all $g \in G$.

The non-trivial statement of the theorem is that the unitary V can also be chosen to intertwine the two group representations W_S and W'_S. The theorem can be interpreted most clearly when we substitute the system S in the statement of the theorem by a bipartite system SE, where E contains the degrees of freedom required to dilate a covariant quantum channel T on SC to a unitary channel:

$$T[\rho_S \otimes \sigma_C] = \text{Tr}_E[U \rho_S \otimes \omega_E \otimes \sigma_C U^\dagger]. \tag{14}$$

Here, ω_E is a general symmetric state in the resource theory of asymmetry and a Gibbs state in the resource theory of athermality. Theorem 1 then states that if a system C takes part in a process but neither experiences back-action (does not change its density matrix σ_C) nor becomes correlated to SE, then the system C was not necessary to realize the state transition on S (and E). This is because the covariant quantum channel T' defined by

$$T'[\rho_S] = \text{Tr}_E[V \rho_S \otimes \omega_E V^\dagger] \tag{15}$$

has the same effect on S. Moreover, T' is also a thermal operation if T was. In other words: For a catalyst to be useful in realizing otherwise impossible state transitions, it must become correlated to either S, E, or both, and this statement holds both in the resource theory of asymmetry for connected Lie groups and in the resource theory of athermality. Note that the usual setting of uncorrelated catalysis allows for correlations between S and E as well as between C and E but not between C and S.

Therefore, a catalyst in a pure state is useless in contrast to the setting of LOCC. While this statement is known in the cases of either a single conserved quantity (i.e. $G = \mathbb{R}$) [15] or the case where ρ_S is pure [41], our result shows that it holds for any finite number of conserved quantities and any state ρ_S. More importantly, it explains why this is true: A catalyst in a pure state cannot be correlated to another system. If we interpret C as a reference frame, this means that a

reference frame that is actually used (even only once) must either degrade or become correlated to other systems.

One may wonder whether there actually are situations where it is helpful to have a system C that only becomes correlated to S but does not experience back-action. This is indeed the case: There are state transitions $\rho_S \to \rho'_S$ that can be realized in this way, but that cannot be realized using only covariant operations [15]. However, it is impossible to transform a symmetric state to an asymmetric state in this way, as shown by so-called no-broadcasting theorems [38, 43]. One can also construct examples in the resource theory of athermality where an asymmetric (coherent) catalyst that becomes correlated to S allows for state transitions that are impossible to implement without a catalyst [36].

It is important to emphasize that our results hold for one particular choice of density matrices: We do *not* require that σ_C remains unchanged and uncorrelated for *all* possible density matrices on S, but only for the specific density matrix ρ_S. We have mentioned above that a system C in a quantum state that is not symmetric with respect to a group representation can be viewed as an (imperfect) reference frame for the group: By interacting with it using only covariant quantum channels, one can implement some non-covariant quantum channels on a system S. It is already known that in this case, for at least some state on S (including the maximally mixed state), the final state on C cannot be brought back exactly to its initial state by a covariant quantum channel [42]. This means that the reference frame "degrades" for such inputs. Conversely, if the reference frame does not degrade for any input on S, then the dynamics on S is already covariant. The degradation of quantum reference frames has been studied in several concrete examples, see Refs. [1, 7, 8, 51].

Theorem 1 then can also be interpreted as showing that whenever one uses a quantum reference beneficially, it must either experience some back-action (change its density matrix) or become correlated to the system of interest S. In particular, back-action is unavoidable if the reference is initially in a pure state.

In section 4 we will see a counter-point to this result: if a quantum reference frame is of such high quality that it allows implementing approximately unitary dynamics on S, then it suffers only mild back-action.

The amount of correlations that the catalyst C establishes with the system and environment SE can be measured in terms of the *mutual information* $I(C : SE)$. From the unitary invariance of the von Neumann entropy H it then follows that

$$I(C : SE)_{U\rho_{SE} \otimes \sigma_C U^\dagger} = H(\rho'_{SE}) - H(\rho_{SE}). \quad (16)$$

Thus, in the context of the resource theory of asymmetry, a catalyst is only useful if the entropy on SE strictly increases. Similarly, one can check that the

rank of ρ'_{SE} cannot be smaller than that of ρ_{SE}. Recently, it was shown in Refs. [10, 67, 68] that if ρ_S and ρ'_S are two finite-dimensional density matrices such that $H(\rho_S) < H(\rho'_S)$ and $\text{rank}(\rho_S) \leq \text{rank}(\rho'_S)$, then there exists a finite-dimensional density matrix σ_C and a unitary U such that

$$\text{Tr}_2[U\rho_S \otimes \sigma_C U^\dagger] = \rho'_S, \quad \text{Tr}_1[U\rho_S \otimes \sigma_C U^\dagger] = \sigma_C. \quad (17)$$

However, as far as we are aware, it is currently an open problem to decide when U and σ_C exist under the additional requirement that C carries a unitary representation of a given group G, and U intertwines the respective representations as in Theorem 1 (see Refs. [3, 25, 41] for progress on the general state-convertibility problem in the resource theory of asymmetry).

The assumption of a *connected* Lie group in the theorem is necessary. The theorem fails for any non-connected Lie group G: Let G be a Lie group with finitely many connected components, and let H be the quotient group G/G_0 w.r.t. the connected component of the neutral element (which is always a normal subgroup). Note that H is a finite group and consider the Hilbert space $\mathcal{H}_C = \ell^2(H)$. We may view the left regular representation of H, given by $W_C(x)|y\rangle = |xy\rangle$, $x, y \in H$, as a unitary representation of G (with kernel equal to G_0). Given states, ρ and σ on any system S which carries a representation W_S of G with $G_0 \subset \ker W_S$, i.e., a system where W_S is essentially a representation of H, one can always find a covariant quantum channel $\mathcal{E}$ on SC which maps $\rho \otimes |x\rangle\langle x|$ to $\sigma \otimes |x\rangle\langle x|$ for some arbitrary but fixed $x \in H$. For example, this channel could be given by [41]

$$\mathcal{E}[X] = \sum_{y \in H} \text{Tr}[(\mathbb{1} \otimes |y\rangle\langle y|)X] \times \ldots$$
$$\times W_C(yx^{-1})\sigma W_C(x^{-1}y) \otimes |y\rangle\langle y|. \quad (18)$$

The reason why such a construction is possible is that the states $|x\rangle$ for $x \in H$ are perfect reference frames for H: $\langle x|y\rangle = \delta_{xy}$ for all $y \in H$. For general finite groups G, any quantum channel on S can be realized by a covariant quantum channel on SC if C carries the left regular representation of G. Concretely, let $T[\cdot] = \text{Tr}_E[V\sigma_E \otimes (\cdot)V^\dagger]$ be the channel to be implemented. Define $V_y := (\mathbb{1}_E \otimes W_S(y))V(\mathbb{1}_E \otimes W_S(y)^\dagger)$. Then the covariant channel

$$\mathcal{E}(X) = \sum_y \text{Tr}_E[(V_y \otimes \langle y|)\sigma_E \otimes X(V_y^\dagger \otimes |y\rangle)] \otimes |y\rangle\langle y|$$

fulfills

$$\mathcal{E}[\rho \otimes |1\rangle\langle 1|] = T[\rho] \otimes |1\rangle\langle 1|. \quad (19)$$

3.1 The proof

We will now prove Theorem 1. The important observation is that since we consider connected Lie groups

G, the infinitely many equations (13) reduce to a finite set of equations. Namely, let $\{X^{(i)}\}_{i=1}^{\dim(G)}$ be a (Hermitian) basis for the Lie algebra of G (corresponding to conserved quantities). The $X^{(i)}$ are represented on the systems S and C by the given representations as $X_S^{(i)}$, $Y_S^{(i)}$ and $X_C^{(i)}$, respectively. (That is, $Y_S^{(i)}$ represents $X^{(i)}$ on system S as induced by the representation W_S'.) We also introduce the strictly positive matrices

$$\omega_S^{(i)} \otimes \omega_C^{(i)} := \exp(-X_S^{(i)}) \otimes \exp(-X_C^{(i)}),$$
$$\hat{\omega}_S^{(i)} \otimes \omega_C^{(i)} := \exp(-Y_S^{(i)}) \otimes \exp(-X_C^{(i)}).$$

Up to normalization, we may interpret the operators $\omega_{S/C}^{(i)}$ as Gibbs states of the associated conserved quantities at temperature $T = 1$. The eqs. (12) and (13) are equivalent to the $\dim(G) + 1$ equations

$$U\rho \otimes \sigma U^\dagger = \rho' \otimes \sigma \qquad (20)$$
$$U\omega_S^{(i)} \otimes \omega_C^{(i)} U^\dagger = \hat{\omega}_S^{(i)} \otimes \omega_C^{(i)}. \qquad (21)$$

To arrive at this result, simply observe that $[U, X] = 0$ if and only if $[U, \exp(X)] = 0$ for a unitary U and a Hermitian matrix X. Thus, we have a system of $m + 1 = \dim(G) + 1$ equations of the form

$$UA^{(i)} \otimes C^{(i)} U^\dagger = B^{(i)} \otimes C^{(i)}, \qquad (22)$$

where $A^{(i)}$, $B^{(i)}$ and $C^{(i)}$ are positive operators.

Theorem 1 is now an immediate consequence of the following proposition:

Proposition 1. *Let $\{A_i = A^{(i)} \otimes C^{(i)}\}_{i=0}^m$ and $\{B_i = B^{(i)} \otimes C^{(i)}\}_{i=0}^m$ be two sets of positive semi-definite matrices such that all matrices with $i \geq 1$ have full rank. Then if $UA_iU^\dagger = B_i$ for a unitary U there exists a unitary V such that $VA^{(i)}V^\dagger = B^{(i)}$.*

The main technical tool we will make use of is Wiegmann's theorem [65], which is a generalization of Specht's theorem [55] (see the survey by Shapiro [53] for more results on unitary invariance). To state it, we define a *word* $w(x_0, \ldots, x_m)$ as a monomial in the $m + 1$ non-commuting variables x_i. For example, one possible word for $m = 2$ is given by $x_0^0 x_2^2 x_1^1 x_0^4$ (we do not assume that $x^0 = 1$).

Theorem 2 (Wiegmann [65])*. Let $(A_0, \ldots, A_m)$ and $(B_0, \ldots, B_m)$ be two tuples of complex $d \times d$-matrices. Then there exists a unitary matrix U such that $UA_iU^\dagger = B_i$ for all $i = 0, \ldots, m$ if and only if*

$$\mathrm{Tr}[w(A_0, A_0^\dagger, A_1, A_1^\dagger, \ldots, A_m, A_m^\dagger)]$$
$$= \mathrm{Tr}[w(B_0, B_0^\dagger, B_1, B_1^\dagger, \ldots, B_m, B_m^\dagger)] \qquad (23)$$

for every word w in $2(m + 1)$ variables.

Since only positive operators appear in our case, we restrict ourselves to words in $m+1$ variables. Now we fix an arbitrary word $w(x_0 \ldots, x_m)$ of $(m+1)$ variables and show that the conditions of the proposition imply

$$\mathrm{Tr}[w(A^{(0)}, \ldots, A^{(m)})] = \mathrm{Tr}[w(B^{(0)}, \ldots, B^{(m)})]. \quad (24)$$

Then Wiegmann's theorem implies that the required unitary V exists.

We now define the length L of the word w as follows: We write out the word in the form $\ell_1^{n_1} \cdots \ell_L^{n_L}$, where each letter ℓ_j is taken from $(x_0, \ldots, x_m)$ and no two adjacent letters are the same. To give an example, if $m = 2$ and our word is $w(x_0, x_1, x_2) = x_0^2 x_1^3 x_0^4$ we have

$$x_0^2 x_1^3 x_0^4 = x_0^{n_0} x_1^{n_1} x_0^{n_3}, \qquad (25)$$

so that the word has $L = 3$ letters.

Definition 3. *Given a word w of length L and a tuple of m positive semi-definite matrices $(A_0, \ldots, A_m)$ we define a function $f_w(s|A_0, \ldots, A_m) : \mathbb{R}^L \to \mathbb{C}$ by replacing each exponent n_j with a real variable s_j in the word $w(A_0, \ldots, A_m)$ and taking the trace.*

In the definition, we need to specify how we take real powers of positive semi-definite matrices, which is done by functional calculus (with $0^0 = 0$). Note that well-definedness of $s \mapsto X^s$ for a normal matrix X requires positivity of all eigenvalues, i.e., positive semi-definiteness of X.

If we consider our previous example $w(x_0, x_1, x_2) = x_0^0 x_2^2 x_1^1 x_0^4$, then the function we now consider is given by

$$\mathbb{R}^4 \ni s \mapsto f_w(s|A_0, A_1, A_2) = \mathrm{Tr}[A_0^{s_1} A_2^{s_2} A_1^{s_3} A_0^{s_4}]. \quad (26)$$

Note that a word of k variables may only involve a smaller number of variables. For example $w(x_0, x_1, x_2) = x_1$. We call the variables that actually appear in a word the *participating* variables.

Lemma 4. *Let w be a word of length L and of $2(m+1)$ variables. Then the function $f_w(s|A_0, \ldots, A_m)$ has the following properties:*

1. *Factorization: $f_w(s|A_0 \otimes C_0, \ldots, A_m \otimes C_m) = f_w(s|A_0, \ldots, A_m)f_w(s|C_0, \ldots, C_m)$*

2. *Analyticity: $s \mapsto f_w(s|A_0, \ldots, A_m)$ is analytic.*

3. *$f_w(0, \ldots, 0|A_0, \ldots, A_m) \geq \mathrm{rank}(A_0)$ if A_i has full rank for $i \geq 1$.*

Proof. The first property (factorization) follows from $\mathrm{Tr}(X \otimes Y) = \mathrm{Tr}X \, \mathrm{Tr}Y$ and from $(X \otimes Y)^s = X^s \otimes Y^s$, which only makes sense for positive X and Y. Analyticity follows from the fact that $s \mapsto A^s$ is an analytic function (in contrast to the function $A \mapsto A^s$). Writing out $f_w(s|A_0, \ldots, A_m)$ in terms of the spectral decompositions of the involved matrices shows that it is a finite sum of finite products of analytic

functions and hence analytic. For the last property, there are two cases: If A_0 participates in the word $w(A_0, \ldots, A_m)$, then $f_w(0, \ldots, 0|A_0, \ldots, A_m) = \mathrm{rank}(A_0)$, since $A_i^0 = \mathbb{1}$ for $i \geq 1$ and $A_0^0 = P_{\mathrm{supp}(A_0)}$. Otherwise, $f_w(0, \ldots, 0|A_0, \ldots, A_m)$ coincides with the dimension d of the matrices since all participating matrices have full rank. $\qquad\square$

Now fix an arbitrary word w of length L. By taking appropriate (real) powers of the eqs. $UA^{(i)} \otimes C^{(i)}U^\dagger = B^{(i)} \otimes C^{(i)}$, multiplying them appropriately and taking the trace, we find

$$
\begin{aligned}
&f_w(s|A^{(0)} \otimes C^{(0)}, \ldots, A^{(m)} \otimes C^{(m)}) \\
&\quad = f_w(s|B^{(0)} \otimes C^{(0)}, \ldots, B^{(m)} \otimes C^{(m)}). \quad (27)
\end{aligned}
$$

By the factorization property in Lemma 4, we find

$$
\begin{aligned}
\Big[f_w(s|A^{(0)}, \ldots, A^{(m)}) &- f_w(s|B^{(0)}, \ldots, B^{(m)}) \Big] \\
&\times f_w(s|C^{(0)}, \ldots, C^{(m)}) = 0. \quad (28)
\end{aligned}
$$

Since all involved functions are analytic by Lemma 4, we have a product of two analytic functions that vanishes identically. This implies that at least one of the factors needs to vanish identically. But by property 3 in Lemma 4 and the requirement that $C^{(i)}$ has full rank for $i \geq 1$, we know that $f_w(s|C^{(0)}, \ldots, C^{(m)})$ cannot vanish identically. Therefore

$$
f_w(s|A^{(0)}, \ldots, A^{(m)}) = f_w(s|B^{(0)}, \ldots, B^{(m)}) \quad (29)
$$

for all $s \in \mathbb{R}^L$. Hence, in particular, we have

$$
\mathrm{Tr}[w(A^{(0)}, \ldots, A^{(m)})] = \mathrm{Tr}[w(B^{(0)}, \ldots, B^{(m)})]. \quad (30)
$$

which we wanted to show.

We close this section with a remark about Theorem 1. One may wonder whether the condition that the $C^{(i)}$ have full rank for $i \geq 1$ is necessary. Indeed at least some requirement on the support of the matrices is necessary. For example, let $A^{(0)} = \sigma_x, B^{(0)} = \sigma_y, A^{(1)} = \sigma_x, B^{(1)} = \sigma_z$. Then for each $i = 1, 2$ there exist *distinct* unitaries U_i such that $U_i A^{(i)} U_i^\dagger = B^{(i)}$ but not a single unitary that works for both $i = 1$ and $i = 2$. However, by choosing $C^{(i)} = |i\rangle\langle i|$ and $U = \sum_i U_i \otimes |i\rangle\langle i|$, all conditions of the proposition apart from the condition on the rank would be fulfilled. It is also not sufficient that all the $C^{(i)}$ have pairwise overlapping supports: One can come up with examples of three pairs of Hermitian matrices $(A^{(i)}, B^{(i)})$ such that every *two* pairs of the matrices are jointly unitarily equivalent, but not all three of them at once, and at the same time there exists three positive matrices with pairwise overlapping supports $C^{(i)}$ and a unitary U such that $UA^{(i)} \otimes C^{(i)}U^\dagger = B^{(i)} \otimes C^{(i)}$ for $i = 1, 2, 3$. An example is given in the Appendix.

4 Good quantum reference frames degrade little

Theorem 1 implies that a quantum reference frame in a pure state that is used beneficially necessarily degrades by building up correlations. In this section, we show a counter-statement: if the quantum reference frame can be used to implement approximately unitary dynamics on S, then it can be used in such a way that it hardly degrades (of course, one can always use it in a non-optimal way as well). Before coming to the Theorem, let us explain heuristically why we can expect this to be true. Label the reference frame by C and suppose that the joint dynamics on SC is unitary. Let the initial state of C by $|\phi\rangle$ and consider two orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ on S. Then if the induced dynamics on S is approximately given by the unitary V, we must have:

$$
U|\psi_1\rangle|\phi\rangle \approx (V|\psi_1\rangle)|\phi_1'\rangle \quad (31)
$$
$$
U|\psi_2\rangle|\phi\rangle \approx (V|\psi_2\rangle)|\phi_2'\rangle. \quad (32)
$$

But by linearity, we must also have

$$
\begin{aligned}
U(\alpha|\psi_1\rangle + \beta|\psi_2\rangle)|\phi\rangle &\approx \alpha(V|\psi_1\rangle)|\phi_1'\rangle + \beta(V|\psi_2\rangle)|\phi_2'\rangle \\
&\approx (\alpha V|\psi_1\rangle + \beta V|\psi_2\rangle)|\phi_3'\rangle.
\end{aligned} \quad (33)
$$

This is only possible if $|\phi_1'\rangle \approx |\phi_2'\rangle \approx |\phi_3'\rangle$, i.e. the final state of C only depends little on the initial state on S (up to global phases). We see here a particular example of the *information-disturbance trade-off* in quantum mechanics [19, 20, 33]: If only little information can be gained about the initial state on S by measuring C, then the disturbance must be small. Conversely, if there is little disturbance on S, then only little information about the initial state of S can be gained by measuring the final state of C.

However, what the discussion so far hasn't shown is why the final state on C is close to its initial state. C could be rotated unitarily as well. One could hope this unitary to be covariant so that it could be undone by a (unitary) covariant quantum channel. But the whole point of using the reference frame is to be able to implement on S a unitary that is *not* covariant, so why should the effective unitary on C be covariant? Nevertheless, we will show that C can be brought back close to its initial state using a covariant quantum channel. The intuitive explanation for why this is the case is the following: Since the unitary on S is not covariant, it must fail to commute with at least some conserved quantity, meaning that the quantity is, in fact, not conserved under the dynamics induced by U. Implementing the unitary therefore requires a coherent source or sink with which the quantity can be exchanged. However, our considerations above show that the final state on S is essentially independent of how much of the conserved quantity is exchanged with S. This is possible only if

the amount of the conserved quantity in C is highly uncertain, to begin with (which in turn requires C to be much larger than S in the sense that it can store much larger amounts of the conserved quantity). But in this case, any unitary rotation induced by S on C must leave the state approximately invariant. This explanation leaves many open questions, in particular, if there are several non-commuting conserved quantities, but it provides a rough idea. Let us now come to our theorem. To state it, we use the *diamond-norm* for quantum channels, which is defined as:

$$\|T_1 - T_2\|_\diamond := \sup_{n \in \mathbb{N}} \sup_\rho \|(T_1 \otimes \mathrm{Id}_n)[\rho] - (T_2 \otimes \mathrm{Id}_n)[\rho]\|_1 \,,$$

where Id_n denotes the identity quantum channel on an n-dimensional quantum system and ρ denote arbitrary quantum states on the joint system. The diamond-norm quantifies the distinguishability of two channels on a given system taking into account possible correlations to other systems.

Theorem 5. *Consider quantum systems S and C with projective unitary representations of some (arbitrary) group G, a covariant quantum channel T on SC and a state σ_C on C such that the induced channel on S*

$$T_S[\rho_S] = \mathrm{Tr}_C[T[\rho_S \otimes \sigma_C]] \tag{34}$$

is close to unitary: $\left\|T_S - V(\cdot)V^\dagger\right\|_\diamond \leq \epsilon$. Then there exists a covariant quantum channel T' on SC with identical induced dynamics on S, $T'_S = T_S$, but such that

$$D(\mathrm{Tr}_S[T'[\rho_S \otimes \sigma_C]], \sigma_C) \leq 2\sqrt{2\epsilon}, \tag{35}$$

for all states ρ_S on S.

In contrast to Theorem 1, this result does not transfer to the resource theory of athermality since even if T is a thermal operation, we currently cannot guarantee that T' also is. We leave this as an open problem.

The fact that the system C can be used in such a way that it changes little is reminiscent of the phenomenon of *embezzlement* in the theory of entanglement [63]. There, a suitable resource state (the embezzler) may be used to implement, via local unitary operations, arbitrary state-transitions between pure states on a finite dimensional bipartite quantum system while experiencing arbitrary little back-action. Embezzlers also exist in other resource theories, including asymmetry and athermality, see for example [37] for a comprehensive discussion. From this point of view, Theorem 5 shows that if a quantum reference frame is sufficiently good to implement arbitrary unitaries to high precision, then it must be an embezzler. However, we emphasize that embezzlement is usually concerned with the ability to enable arbitrary state transitions with little back-action, whereas Theorem 5 is concerned with implementing arbitrary *unitary quantum channels* with little back-action.

A particularly interesting obstruction arising from conservation laws is the impossibility of measuring exactly (in the sense of von Neumann) an observable that does not commute with a conserved quantity. This statement is known as Wigner-Araki-Yanase (WAY) theorem [4, 66, 70], and has recently been re-interpreted and generalized in several ways (see, for example, [2, 13, 34, 40, 46, 47, 49, 50]). From the point of view of the resource theory of asymmetry, the theorem may be understood as saying that the measurement apparatus implementing an approximate measurement of an observable that does not commute with a conserved quantity must include a subsystem that serves as a quantum reference frame. Our theorem then implies that a measurement apparatus that can be used to implement such a measurement to very high accuracy can also be used so that its internal quantum reference frame hardly degrades. In particular, the measurement apparatus may be used again once its pointer is reset.

4.1 The proof of theorem 5

The proof consists of several steps: First, we restrict to the case where T is a unitary quantum channel, and σ_C is pure. Second, we use the quantitative version of the information-disturbance tradeoff from Ref. [33] to argue that the final state on C is almost independent of the input ρ_S on S and still approximately pure. Third, since the final state on C barely depends on ρ_S, we consider the situation in which $\rho_S = \mathbb{1}_S/d_S$ is maximally mixed. In this case, the dynamics on C is covariant as well and doubly stochastic. We then show that when a doubly stochastic, covariant channel maps a pure state close to a pure state, there exists a covariant recovery map R that maps the final state on C close to the initial state σ_C. The channel T' can now be chosen as $(I_S \otimes R) \circ T$ (with I_S denoting the identity channel on S). Finally, we use the covariant Stinespring theorem to generalize the result to the case where T is not unitary, and σ_C is mixed.

We now go through the steps in detail. So in the following, we assume that $T = U(\cdot)U^\dagger$ is a unitary, covariant quantum channel and $\sigma_C = |\phi\rangle\langle\phi|$. Refs. [33, 64] then shows that if $\left\|T_S - V(\cdot)V^\dagger\right\|_\diamond \leq \epsilon$ for some unitary V, then there exists a unitary W such that

$$\sup_{\||\psi\rangle_S\|=1} \left\| U|\psi\rangle_S|\phi\rangle_C - V|\psi\rangle_S W|\phi\rangle_C \right\|^2 \leq 2\epsilon \tag{36}$$

We now make use of the fidelity of quantum states, defined as $F(\rho, \sigma) = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1 = \mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}]$. It is non-decreasing under partial traces, so that $F(\rho_{SC}, \sigma_{SC}) \leq F(\rho_C, \sigma_C)$ for any two density matrices ρ_{SC} and σ_{SC}. Moreover, for pure states, we have

$$F(\Psi, \Phi) \geq 1 - \frac{1}{2}\||\Psi\rangle - |\Phi\rangle\|^2 \tag{37}$$

where we use the short-hand notation $\Psi = |\Psi\rangle\langle\Psi|$. For a given state ρ on S let

$$\hat{T}_\rho[\sigma_C] = \text{Tr}_S[U\rho \otimes \sigma_C U^\dagger] \qquad (38)$$

be the final state on C. Then for a pure state $\rho = |\psi\rangle\langle\psi|$ the information-disturbance tradeoff (36) implies

$$F(\hat{T}_\psi[\sigma_C], W\sigma_C W^\dagger) \geq 1 - \epsilon. \qquad (39)$$

However, if $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, then $\hat{T}_\rho[\sigma_C] = \sum_i p_i\hat{T}_{\psi_i}[\sigma_C]$. Since the fidelity is concave in its arguments, we, therefore, find

$$F(\hat{T}_\rho[\sigma_C], W\sigma_C W^\dagger) \geq 1 - \epsilon \qquad (40)$$

for all density matrices ρ on S. We now make use of a simple observation regarding the fidelity and so-called doubly stochastic quantum channels, i.e. quantum channels T such that $T[\mathbb{1}] = \mathbb{1}$. For a doubly stochastic quantum channel, its Hilbert-Schmidt dual T^*, defined by $\text{Tr}[AT[B]] = \text{Tr}[T^*[A]B]$ for all A, B, is also a quantum channel. Moreover, if T is covariant, then so is T^*. Now, if $\Psi = |\Psi\rangle\langle\Psi|$ and σ is an arbitrary density matrix, then $F(\Psi,\sigma)^2 = \langle\Psi|\sigma|\Psi\rangle = \text{Tr}[\Psi\sigma]$. Using the definition of the Hilbert-Schmidt dual, we thus get

$$F(T^*[\Psi],\Phi)^2 = \text{Tr}[T^*[\Psi]\Phi] = \text{Tr}[\Psi T[\Phi]] = F(\Psi, T[\Phi])^2$$

for any two pure states $\Psi = |\Psi\rangle\langle\Psi|$ and $\Phi = |\Phi\rangle\langle\Phi|$. The channel $\hat{T}_{\mathbb{1}_S/d_S}$ is doubly stochastic because it has a dilation in terms of a maximally mixed state. Therefore its dual $R := (\hat{T}_{\mathbb{1}_S/d_S})^*$ is a covariant quantum channel and fulfills

$$F(\sigma_C, R[W\sigma_C W^\dagger]) = F(\hat{T}_{\mathbb{1}_S/d_S}[\sigma_C], W\sigma_C W^\dagger)$$
$$\geq 1 - \epsilon. \qquad (41)$$

In the following, we make use of the Fuchs-van de Graaf inequality [18]

$$D(\rho,\sigma) \leq \sqrt{1 - F(\rho,\sigma)^2}. \qquad (42)$$

Together with (40) and (41), it implies

$$D(\hat{T}_\rho[\sigma_C], W\sigma_C W^\dagger) \leq \sqrt{2\epsilon} \qquad (43)$$

as well as

$$D(\sigma_C, R[W\sigma_C W^\dagger]) \leq \sqrt{2\epsilon}. \qquad (44)$$

Then using a triangle-inequality and the data-processing inequality for the trace-distance yields

$$D(R \circ \hat{T}_\rho[\sigma_C], \sigma_C) \leq D(R \circ \hat{T}_\rho[\sigma_C], R[W\sigma_C W^\dagger])$$
$$+ D(R[W\sigma_C W^\dagger], \sigma_C) \qquad (45)$$
$$\leq D(\hat{T}_\rho[\sigma_C], W\sigma_C W^\dagger)$$
$$+ D(\sigma_C, R[W\sigma_C W^\dagger]) \qquad (46)$$
$$\leq 2\sqrt{2\epsilon}. \qquad (47)$$

This finishes the proof in the case of unitary T and pure σ_C by setting $T' = (\text{Id}_S \otimes R) \circ T$. We now use this result to prove the general case, where T need not be unitary and σ_C need not be pure. By the covariant Stinespring theorem, there exists a symmetric pure state $\omega_E = |\chi\rangle\langle\chi|_E$ and a covariant *unitary* channel $\tilde{T}$ on SCE such that

$$T[\rho_{SC}] = \text{Tr}_E[\tilde{T}[\rho_{SC} \otimes \omega_E]]. \qquad (48)$$

Now consider a purification $|\Phi\rangle_{CC'}$ of σ_C on a system C'. Then the channel $\tilde{T} \otimes \text{Id}_{C'}$ is a covariant unitary channel on $SCEC'$ implementing $V(\cdot)V^\dagger$ on S using a pure reference frame on CEC' in the state $|\Phi\rangle\langle\Phi|_{CC'} \otimes |\chi\rangle\langle\chi|_E$. Since $\tilde{T} \otimes \text{Id}_{C'}$ does not act on C' and since this property is preserved when going to the Hilbert-Schmidt dual, by the previous reasoning, there is a covariant channel $\tilde{R} = \tilde{R}_{CE} \otimes \text{Id}_{C'}$ on CEC' such that the covariant quantum channel

$$\tilde{T}' = (I_S \otimes \tilde{R}) \circ (\tilde{T} \otimes \text{Id}_{C'}) = [(\text{Id}_S \otimes \tilde{R}_{CE}) \circ \tilde{T}] \otimes \text{Id}_{C'}$$
$$= \tilde{T}'_{SCE} \otimes \text{Id}_{C'} \qquad (49)$$

leaves CEC' approximately invariant and does not act on C' while implementing $V(\cdot)V^\dagger$ with the same precision. Because trace-distance is non-increasing under partial traces, the state on C is also approximately invariant with at most the same error. We can therefore define T' by

$$T'[\rho_{SC}] = \text{Tr}_E[\tilde{T}'_{SCE}[\rho_{SC} \otimes \omega_E]], \qquad (50)$$

which is covariant because ω_E is symmetric and $\tilde{T}'_{SCE}$ is covariant. This finishes the proof in the general case.

5 Discussion and outlook

Our first result shows that a catalyst in the context of the resource theories of asymmetry and athermality must become correlated to some other degrees of freedom to be of any use. The amount of correlations measured in terms of mutual information is precisely the change of entropy of system and environment combined, but our results do not say anything about how these correlations can be distributed. It is known in the resource theory of athermality that it can be quite beneficial to allow the catalyst to become (even only slightly) correlated to S, see Refs. [21, 48, 52, 54, 69]. But do the same results hold if one bounds also the correlations with the thermal environment E, or is there a tradeoff, so that correlations to E have to increase as those to S decrease?

Our second result shows that a good quantum reference frame can be used in such a way that its state hardly changes. In particular, the result shows that a quantum reference frame that can implement any unitary transformation to high accuracy on a given system must automatically be an embezzler in the

sense that it is almost catalytic [62]. We have left the corresponding statement for the resource theory of athermality as an open problem, see also [5] for related considerations.

Here, we have restricted ourselves to systems described by finite-dimensional Hilbert spaces. It would be interesting to know whether similar results can be derived for infinite dimensional systems. This would allow for the treatment of unitary representations of non-compact and non-abelian groups, such as the Galilean or Poincaré group, which are relevant for treating reference frames for space-time localization and orientation.

Note added. Results closely related to Theorem 5, also using the quantitative information-disturbance tradeoff as a proof-technique, have been previously derived in the context of so-called "no-programming theorems" for quantum computers [26, 71] and for estimating the fundamental energy costs of implementing unitary gates on a quantum computer [14, 72] (where the energy-source corresponds to the quantum reference frame in our formulation). We thank Yuxiang Yang for making us aware of the close resemblance between these results and ours after the first version of this manuscript appeared as a preprint.

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A The rank-condition in Proposition 1

We here give the promised example of three pairs of Hermitian matrices $(A^{(i)}, B^{(i)})$ such that every *two* pairs of the matrices are jointly unitarily equivalent, but not all three of them at once, and at the same time there exist three positive matrices with pairwise overlapping supports $C^{(i)}$ and a unitary U such that $U A^{(i)} \otimes C^{(i)} U^\dagger = B^{(i)} \otimes C^{(i)}$ for $i = 1, 2, 3$.

We first describe a general way to find such examples and then give an explicit example.

We will choose $C^{(i)}$ as

$$C^{(i)} = |i\rangle\langle i| + |i+1\rangle\langle i+1|, \tag{51}$$

where addition is modulo 3. Any product of the $C^{(i)}$ involving all the different i is zero. Therefore, all words of the variables $A^{(i)} \otimes C^{(i)}$ where all the i are participating are traceless. Hence by Wiegmann's theorem, as long as every two pairs $(A^{(i)}, B^{(i)})$ are unitarily equivalent, then the $A^{(i)} \otimes C^{(i)}$ are jointly unitarily equivalent to the $B^{(i)} \otimes C^{(i)}$.

So we are looking for matrices such that there exists unitaries U, V and W with

$$U A^{(i)} U^\dagger = B^{(i)}, \quad i = 1, 2 \tag{52}$$

$$V A^{(i)} V^\dagger = B^{(i)}, \quad i = 2, 3 \tag{53}$$

$$W A^{(i)} W^\dagger = B^{(i)}, \quad i = 3, 1, \tag{54}$$

but we want that the Hermitian matrices $A^{(i)}$ and $B^{(i)}$ are not jointly unitarily equivalent for all $i = 1, 2, 3$. For simplicity we choose $U = \mathbb{1}$ (this simply amounts to re-defining the $B^{(i)}$ so it does not limit the generality of what follows). The above equations imply that

$$A^{(1)} = W A^{(1)} W^\dagger \tag{55}$$

$$A^{(2)} = V A^{(2)} V \tag{56}$$

$$A^{(3)} = V^\dagger W A^{(3)} W^\dagger V. \tag{57}$$

Let us define $\tilde{U} = V^\dagger W$, so that the equations read

$$[A^{(1)}, W] = 0, \ [A^{(2)}, V] = 0, \ [A^{(3)}, \tilde{U}] = 0 \tag{58}$$

and

$$B^{(1)} = A^{(1)}, \ B^{(2)} = A^{(2)}, \ B^{(3)} = V A^{(3)} V^\dagger. \tag{59}$$

By Wiegmann's theorem, all we need now is two unitaries V, W and three Hermitian matrices $A^{(i)}$ fulfilling the above conditions such that

$$\begin{aligned} \mathrm{Tr}[B^{(1)} B^{(2)} B^{(3)}] &= \mathrm{Tr}[A^{(1)} A^{(2)} V A^{(3)} V^\dagger] \\ &\neq \mathrm{Tr}[A^{(1)} A^{(2)} A^{(3)}]. \end{aligned} \tag{60}$$

An explicit such choice is as follows:

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{(3)} = \mathrm{i} \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 \end{pmatrix}, \tag{61}$$

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{62}$$

which yields

$$\left| \mathrm{Tr}[B^{(1)} B^{(2)} B^{(3)}] - \mathrm{Tr}[A^{(1)} A^{(2)} A^{(3)}] \right| = 2\sqrt{3}. \tag{63}$$