Relative Facts of Relational Quantum Mechanics are Incompatible with Quantum Mechanics

Jay Lawrence[1], Marcin Markiewicz[2], and Marek Żukowski[2]

[1]Department of Physics and Astronomy, Dartmouth College, Hanover, NH 03755, USA
[2]International Centre for Theory of Quantum Technologies (ICTQT), University of Gdansk, 80-308 Gdansk, Poland

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Relational Quantum Mechanics (RQM) claims to be an interpretation of quantum theory [see Rovelli (2022), which appears in the *Oxford Handbook of the History of Interpretation of Quantum Physics*]. However, there are significant departures from quantum theory: (i) in RQM measurement outcomes arise from interactions which entangle a system S and an observer A without decoherence, and (ii) such an outcome is a "fact" relative to the observer A, but it is not a fact relative to another observer B who has not interacted with S or A during the foregoing measurement process. For B the system $S \otimes A$ remains entangled. We derive a GHZ-like contradiction showing that relative facts described by these statements are incompatible with quantum theory. Hence Relational Quantum Mechanics should not be considered an interpretation of quantum theory, according to a criterion for interpretations that we have introduced. The criterion states that whenever an interpretation introduces a notion of outcomes, these outcomes, whatever they are, must follow the probability distribution specified by the Born rule.

1 Introduction

Nearly a century after the birth of quantum theory, we have more different interpretations of it than ever, as well as a growing number of proposed alternative theories. Many of these so-called interpretations are discussed and grouped into categories by Cabello Cabello (2017). There are two broad categories (with subcategories that do not concern us), namely (i) "intrinsic realism" - those which hold that "the probabilities of measurement outcomes are determined by intrinsic properties of the observed system," and (ii) "participatory realism" - which hold that "quantum theory does not deal with intrinsic properties of the observed system, but with the experiences an observer or agent has of the observed system." One might also characterize these two categories with the adjectives "system-centric" and "observer-centric," respectively. Most of the current developments have occurred in the second category. These include QBism, Rovelli's Relational Quantum Mechanics and related ideas of Brukner, as well as the Copenhagen interpretation. More radical developments since the publication of Cabello's paper incorporate ideas of metaphysics, philosophy and psychology, together with interpretations of quantum theory Nurgalieva and Renner (2020); Adlam (2021); Cavalcanti (2021); Brukner (2022). Discussions of these "interdisciplinary" developments would take us beyond the scope of this work.

In this paper we will focus on Relational Quantum Mechanics (RQM), which was introduced by Rovelli Rovelli (1996). RQM is a body of ideas which has been extended with the help of a number of coworkers over the years. It has helped to spawn variations by others, with subtle differences, but these works together represent an emerging theme within the observer-centric category. Their ideas are illustrated by scenarios involving two or more observers. In the simplest case of two observers, one observer (A) performs a measurement on a (typically simple) physical system S (say a qubit), and then the second observer B measures the joint system consisting of S and A. We shall refer to these and more complicated versions as "layered-observer" scenarios. One can view them as inspired by the well-known Wigner's Friend scenario Wigner (1995), and an extension by Deutsch Deutsch (1985). These two earlier works had different motivations and drew conclusions unlike the current ones; they do not

belong to the current theme. But, since they provide inspiration for current developments, we include brief descriptions in Appendix A for interested readers.

The current theme was initiated by Rovelli in his seminal work Rovelli (1996). The ideas are illustrated by a scenario that introduces this work: An observer A interacts with a system S (say a single qubit), and realizes an outcome. The outcome is one of the possible values (an eigenvalue) of the measured observable. This value is a *relative fact* for A. It is *not* a relative fact for a second observer B who has not interacted with S or A during the process. RQM assumes that B faces the situation in which S and A are in an entangled state representing a superposition of all possible outcomes. Thus, according to RQM, B has a different, but equally valid account of the events as compared with A's account. This is special to RQM. Common features of the above layered-observer theories involve similarly unconventional ideas about measurement; for example, the rejection of the collapse postulate,[1] and measurement without decoherence - the embrace of relative facts. RQM in particular holds that a measurement is simply an entangling interaction between *any* two systems; it does not require a macroscopic measuring device. We emphasize that when referring to RQM's description we use the term "observer" in this very broad sense: it can refer to any physical system interacting with another system. Such significant departures from quantum measurement theory create tension between these new theories and quantum mechanics. We will show how this tension manifests itself as a contradiction when it comes to predicted measurement outcomes.

General interest in layered-observer scenarios has grown since the appearance of the articles Frauchiger and Renner (2018) and Brukner (2018). These draw inspiration from the extended Wigner's Friend scenario of Deutsch Deutsch (1985), (Appendix A), which provided some intriguing background ideas. Hence we will refer interchangeably to "extended Wigner-Friend" and "layered-observer" scenarios. Two of us proved a no-go theorem for extended Wigner-Friend scenarios involving relative outcomes for the Friend Żukowski and Markiewicz (2021), to counter the argumentation of Frauchiger and Renner (2018) and Brukner (2018), which use the no-collapse approach for some of the observers who play the role of Friends.

As the above examples show, there is a trend in the literature not to accept quantum measurement theory, and instead claim that one can have "a unitary quantum mechanics" without collapse, and importantly also without the decoherence theory of measurement. In such theories, including RQM, measurement is an establishment of a system-device correlation (entanglement), and the introduction of 'relative facts' as the measurement results with respect to an observer controlling the device.

The rejection of collapse ignores an essential aspect of the measurement postulates of quantum mechanics. These postulates entail the Born probability rule for measurement outcomes, with collapse as the post-measurement state-update rule. The latter guarantees that a repeated measurement will produce the same result as the first. In the initial formulations of quantum mechanics, it was presumed that collapse reflects the classical nature of macroscopic measuring devices, and the associated irreversible amplification process. It is now generally understood that collapse is described (and justified) by decoherence theory, in which the measuring apparatus is treated quantum mechanically, and assuming that its interactions with the system and the environment are unitary. The interactions with the environment cause a rapid and irreversible decay of the off-diagonal elements of the density matrix (the coherences), leaving the diagonal elements as the Born rule probabilities of measurement outcomes, see e.g. Zurek (1982), Schlosshauer (2005), Schlosshauer (2019), Zurek (2022). Thus, collapse manifests as a Bayesian conditional update. A measurement cannot occur without the decoherence and irreversibility that inevitably accompany the action of a measuring apparatus, producing a classical (probabilistic) description, see e.g. Haake and Żukowski (1993).

In this paper, we address specifically the Relational Interpretation, as anticipated in the concluding paragraph of Żukowski and Markiewicz (2021), and we focus on the central element of relative facts. In common with Żukowski

[1] An observer-independent concept as understood in quantum mechanics (defined in the first paragraph of Sec. 2), as compared with the observer-dependent concept in RQM.

and Markiewicz (2021), our proof is based on a three-qubit Greenberger-Horne-Zeilinger (GHZ) state: we exploit the perfect correlations predicted by quantum mechanics for measurement outcomes. These correlations impose constraints on the products of relative facts for two different observers (A and B). We will show that relative facts cannot satisfy these constraints, and cannot therefore be incorporated within quantum mechanics. In this way, Relational Quantum Mechanics is incompatible with quantum theory. Therefore, according to the following criterion, RQM should not be considered an interpretation of quantum theory:

- *Criterion for an interpretation:* If an interpretation of quantum theory introduces some conceptualization of outcomes of a measurement, then probabilities of these outcomes must follow the quantum predictions as given by the Born rule.

This includes any constraints, probabilistic or deterministic, implied by the Born rule. Outcomes may include perceptions of observers.[2] Our proof will show that RQM does not satisfy this criterion.

We should emphasize that our proof in this work is based on the definition of relative facts within the RQM framework, and that it does not apply to relative facts as might be defined within the framework of QBism Pienaar (2021a). The primary difference is that a relative fact in QBism refers to the statement of beliefs of an agent, and it is natural to expect differences between agents, which need not imply inconsistencies. In contrast, relative facts in Relational Quantum Mechanics refer to properties of a physical system S with which A interacts.

Outline of the paper. In Section 2, we review the postulates of RQM and comment on the tension with quantum mechanics. In Sec. 3, we lay out the ground rules and assumptions to be followed in the proof of the GHZ contradiction. The proof itself is presented in Sec. 4. It is self-contained; familiarity with the earlier GHZ-type reasonings is not assumed. In Section 5 (Further Remarks) we demonstrate the compatibility of mathematical steps in our proof with the rules of Relational Quantum Mechanics, and we argue that relative facts are a form of hidden variables. In Appendix A, we review the historical origins of extended Wigner-Friend scenarios, and in B, we discuss a recent amendment to RQM.

[2]Wigner and Deutsch refer to such outcomes in the scenarios described in Appendix A.

2 Tension between RQM and quantum mechanics

Here we present the essential postulates of RQM of which we shall make use in our arguments. These will be accompanied by commentaries on the tension with quantum mechanics. We refer here to the quantum theory which is used to predict quantum phenomena, which has not been falsified for the last 98 years. It is a theory which may be defined by the postulates as listed in standard textbooks.

2.1 Principles of RQM (with commentary)

Our description of RQM draws upon three articles, most prominently (i) Rovelli's entry in the Oxford Handbook of Quantum Interpretations Rovelli (2022), which sets out the main postulates of RQM. The other two articles propose amendments to these main postulates. The earlier of these is (ii) a reply of Di Biagio and Rovelli Di Biagio and Rovelli (2022) to a recent critique by Pienaar Pienaar (2021b), and the more recent is (iii) an article by Adlam and Rovelli Adlam and Rovelli (2022).

We shall write the principles of RQM (marked by bullet points) with direct quotes of Rovelli where possible, and otherwise we shall paraphrase.

- "RQM interprets quantum mechanics as a theory about physical *events*, or *facts*".
- "A fact is quantitatively described by the value of a variable or a set of variables."
- "Facts are ... realised *only* at the *interactions* between (any) two physical systems."

Commentary: According to the second and third bullet points, the realization of a fact in RQM corresponds to a measurement outcome, and this occurs without decoherence or the collapse of the state vector. It conforms to the RQM idea that "interaction is measurement." We will henceforth refer to this concept as an "RQM measurement". In quantum mechanics, the interaction

between two systems, the measured system and the "pointer" variable, comprises what is called a "premeasurement," which is a decoherence-free correlation between two systems without an outcome. There is no known experiment in which outcomes are obtained without the help of a macroscopic apparatus and without decoherence.

- "Facts are *relative* to systems that interact. That is, they are *"labelled"* by the interacting systems."

This point is developed in the next two bullet points below, which we call *RQM postulates for Wigner-Friend scenarios*:

- "...the Friend interacts with a system and a fact is realised *with respect to the Friend*." But this fact is not realised with respect to Wigner, who was not involved in the interaction..."

- "With respect to Wigner, it only corresponds to the establishment of an 'entanglement' ... between the Friend and the System."

Commentary: These two bullet points conveniently summarize the "active" RQM principles that we shall use in our proof in Sec. 4. They allow two different (but "equally valid", according to RQM) accounts of the events associated with the interaction between the Friend and the System. In quantum mechanics, however, the Friend can be aware of an outcome only by reading the output of the apparatus.

- "Any system is an 'observer:' a system with respect to which facts happen. Decoherence characterizes observers with respect to which stable facts happen."

Commentary: Quantum mechanics has a more restrictive concept of an observer: An observer is any object (animate or inanimate), which is capable of recording the (macroscopic and irreversible) result of a quantum measurement. A microscopic object, such as an atom belonging to a Bell pair, does not qualify as an observer. There is no meaning to the statement that an atom "realizes" the value of its partner's spin component σ_z.

- The probability amplitude, $W(b,a)$, determines the probability for a fact a given that a fact b occurred. $W(b,a)$ determine probability amplitudes *only* if facts a and b are relative *to the same systems*.

- Assume that events a, $\{b_i\}$ and c happen in a sequence, and that the intermediate events within the set $\{b_i\}$ are mutually exclusive. If c and b_i have different "labels", that is, pertain to different systems, the transition probability between a and c reads:

$$P_{\text{unitary}}(c|a) = \left| \sum_i W(c,b_i) W(b_i,a) \right|^2 . \quad (1)$$

Commentary: In our proof, all mathematical steps will be consistent with the above RQM statements about probability amplitudes, as we shall confirm in Sec. 5.

2.2 Amendments

A first amendment to the principles quoted above was presented in Di Biagio and Rovelli (2022). It is known as *the no-comparison rule*,

- "It is meaningless to compare events relative to different systems, unless this is done relative to a (possibly third) system. [...] comparisons can only be made by a (quantum–mechanical) interaction. [...] Wigner [...] might compare the result of his measurement on S with that of F only by physically interacting with F in an appropriate manner. There is no meaning in comparing facts relative to Wigner with facts relative to Friend, [...] apart from this direct physical interaction."

A second amendment by Adlam and Rovelli (2022) is intended to replace the "first amendment" above.

- "*cross-perspective links:*" In a scenario where some observer Alice measures a variable V of a system S, then provided that Alice does not undergo any interactions which destroy the information about V stored in Alice's physical variables, if Bob subsequently measures the physical variable representing Alice's information about the variable V, then Bob's measurement result will match Alice's measurement result.

Commentary: The motivation for this amendment (see p. 3, paragraph 2 of Adlam and Rovelli (2022)) is to ensure that a repeated measurement by another observer produces the same outcome (the state-update rule of quantum mechanics) or more specifically, that a measurement of Alice's physical variable by Bob will yield a result consistent with her measurement. Further comments about cross-perspective links are given in Appendix B.

We should emphasize that our no-go theorem for relative facts of RQM in quantum theory does not require the use of the cross-perspectives links postulate. However, if one assumes this postulate, then our criterion for an interpretation (Section 1) together with its concrete implementation in the context of relative facts (Section 3) does not have to be invoked to show incompatibility between RQM and quantum mechanics.

3 Strategy of Proof

There are many points of tension between RQM and quantum mechanics. This tension exists at the level of postulates or concepts which define the theory. To prove that two theories, so defined, are actually different, or incompatible, we must work at the level of predicted measurement outcomes (as in our *criterion for an interpretation* in Sec. 1). We shall begin by considering a scenario built on the Wigner-Friend postulates, which in turn are based on the properties of relative facts. Eventually we will arrive at a contradiction between the predictions of RQM and quantum mechanics. Specifically, we will show that relative facts predicted by RQM cannot take values that satisfy the constraints predicted by quantum mechanics for measurement outcomes. The tension between RQM and quantum mechanics at the postulate level becomes a clear incompatibility at the level of predicted outcomes.

The scenario that we will construct is as follows: The Friend (A) will make RQM measurements on a system S consisting of three qubits in a GHZ state, thus acquiring three relative facts. Following this, Wigner (B), who has interacted with neither S nor A during this measurement, will make RQM measurements on the compound system of S and A, thus acquiring three relative facts of his own. The perfect correlations present in the initial state of S propagate through the evolution of the scenario and are manifested as constraints on the final outcomes, the relative facts of A and of B. The final equations expressing these constraints are analogous to those of Mermin in his proof Mermin (1990) that noncontextual hidden variables are incompatible with quantum mechanics. This may be of interest to readers familiar with Mermin's work, but our proof is self-contained, and familiarity with previous proofs is not assumed.

In the above scenario, we shall employ RQM measurements obeying the RQM rules of Sec. 2, most importantly the Wigner-Friend points. We recall, with emphasis, that the term "RQM measurement" refers to the RQM concept (Sec. 2) of a measurement as an entangling interaction between two systems through which relative facts are realized. We use this term to avoid confusion with the concept of measurement in standard quantum measurement theory Zurek (1982), Schlosshauer (2005), Schlosshauer (2019), Zurek (2022), in which entangling interactions correspond only to pre-measurements (where decoherence does not occur), but the completed measurements, with outcomes, *require* decoherence. RQM calls the latter results "stable facts".

We shall also apply the rules of quantum mechanics, avoiding steps that are explicitly forbidden by RQM in its original version without the last amendment of *cross-perspective links*, for the reasons discussed at the end of Section 2. For example, we shall nowhere invoke the collapse postulate, and we shall nowhere compare the *values* of relative facts of different observers. However, we will implement our "criterion for an interpretation" by applying quantum mechanical constraints (the Born rule) to *products* of relative facts of different observers, by insisting that:

- Relative facts for different observers, who make different measurements, must take values that obey the constraints imposed by quantum mechanics.

The above is motivated by the following reasoning concerning the logical structure of quantum theory. Relative facts are a supplement to quantum mechanics. RQM considers them to be outcomes (realizations) of RQM measurements. If they do not satify the constraints arising from the Born rule probabilities, then RQM and quantum theory are incompatible with one another.

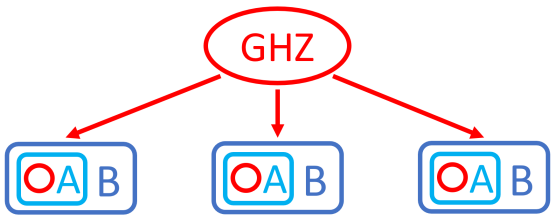

Figure 1: Three qubits are prepared in a GHZ state. Observer A then performs separate measurements on each qubit S_m, $m = 1, 2, 3$. Finally observer B performs separate measurements on each pair $S_m \otimes A_m$, which together comprise the compound object $S \otimes A$. *The arrows and separations in the figure do not depict spatial separation of subsystems:* they merely emphasize that measurements are performed on the individual objects (labeled by m).

And, according to our criterion, RQM is *not* an interpretation of quantum theory.

4 Proof of GHZ Contradiction

In the next three subsections we shall first define the prepared state of the system S; next, the RQM measurements on S by the observer A; and lastly the RQM measurements on the compound system $S \otimes A$ by the observer B. In a fourth subsection we shall identify the perfect GHZ correlations, and in subsection five we complete the proof.

4.1 The System S

Our system S consists of three qubits ($S = S_1 \otimes S_2 \otimes S_3$) in a GHZ state, as pictured in Fig. 1. We shall write this state below, but first we introduce our notation for the three complementary (one-qubit) bases. They will be referred to later in descriptions of the anticipated measurements by A and B, as well as the initial state itself.

The basis states are denoted by $\left|\pm 1^{(n)}\right\rangle_{S_m}$, where ± 1 are the state indices (in any basis), $n = 1, 2, 3$, denote the bases, and m specifies the qubit. The bases $n = 2, 3$ expressed in terms of the standard basis $n = 1$ are

$$\left|\pm 1^{(2)}\right\rangle_{S_m} = \frac{1}{\sqrt{2}}\left(\left|+1^{(1)}\right\rangle_{S_m} \pm \left|-1^{(1)}\right\rangle_{S_m}\right), \tag{2}$$

and

$$\left|\pm 1^{(3)}\right\rangle_{S_m} = \frac{1}{\sqrt{2}}\left(\left|+1^{(1)}\right\rangle_{S_m} \pm i\left|-1^{(1)}\right\rangle_{S_m}\right). \tag{3}$$

These bases $n = 1, 2, 3$, form the standard trio of mutually complementary bases, typically called Z, X, Y, respectively. Following convention for qubit states, $l = \pm 1$ are the eigenvalues of the associated Pauli operator (σ_n) in the states $\left|l^{(n)}\right\rangle_{S_m}$.

We choose a GHZ state as the initial state of S,

$$|GHZ\rangle_{S_1S_2S_3} = \frac{1}{\sqrt{2}}\left(\otimes_{m=1}^{3}\left|+1^{(1)}\right\rangle_{S_m} + \otimes_{m=1}^{3}\left|-1^{(1)}\right\rangle_{S_m}\right), \tag{4}$$

involving standard basis states for each qubit.

4.2 RQM measurements by A

Observer A now makes RQM measurements separately on each qubit S_m, interacting so as to perform each measurement in the basis $n = 3$. According to RQM, each measurement then consists of a unitary entangling evolution, $\hat{U}_m^{SA}$, defined by:

$$\hat{U}_m^{SA}\left(\left|l^{(3)}\right\rangle_{S_m}|\text{initial}\rangle_{A_m}\right) = \left|l^{(3)}\right\rangle_{S_m}\left|l^{(3)}\right\rangle_{A_m} \equiv \left|l^{(3)}\right\rangle_{SA_m}, \tag{5}$$

where we have introduced shorthand notation SA_m for $S_m \otimes A_m$. The three interactions together comprise an RQM measurement on the system S in the product basis $\otimes_{m=1}^{3}\left|l_m^{(3)}\right\rangle_{S_m}$.

To understand the resulting entangled state of $S \otimes A$ in a transparent manner, we first expand the initial GHZ state (of just the S_m) in the $n = 3$ basis:

$$|GHZ\rangle_{S_1S_2S_3} = \sum_{p,q,r=\pm 1} c_{pqr}^{333}\left|p^{(3)}\right\rangle_{S_1}\left|q^{(3)}\right\rangle_{S_2}\left|r^{(3)}\right\rangle_{S_3}. \tag{6}$$

The expansion coefficients are clearly given by

$$c_{pqr}^{333} = \frac{1}{\sqrt{2}}\sum_{l=0}^{1}\langle p^{(3)}|l^{(1)}\rangle\langle q^{(3)}|l^{(1)}\rangle\langle r^{(3)}|l^{(1)}\rangle, \tag{7}$$

where subscripts in c^{333}_{pqr} are basis state indices, and superscripts are the respective bases they refer to. Combining the three entangling operations (5) with (6), we have

$$|GHZ\rangle_{SA} = \otimes_{m=1}^{3}\hat{U}_m^{SA}\left(|GHZ\rangle_S \otimes_{j=1}^{3} |\text{initial}\rangle_{A_j}\right) = \sum_{pqr=\pm 1} c^{333}_{pqr}\left|p^{(3)}\right\rangle_{SA_1}\left|q^{(3)}\right\rangle_{SA_2}\left|r^{(3)}\right\rangle_{SA_3}. \quad (8)$$

According to RQM, observer A realizes a trio of relative facts – specific values (± 1) for each of p, q, and r, all referring to basis $n = 3$. We shall refer to these as $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, respectively, for later reference. According to RQM, as mentioned earlier, these are *not* relative facts for B, who has not interacted with S or A during the above process. We shall discuss B's role in the next subsection, but let us first make a couple of observations.

The equality of coefficients in Eqs. (6) and (8) shows the exact isomorphism between GHZ states of the S and $S \otimes A$ systems. From this we may conclude that, although the system $S \otimes A$ has a larger Hilbert space, it accesses an *effective* Hilbert space of the same dimension (eight) as that of the three-qubit system S. This means that the number of possible independent measurement outcomes with nonzero probability, as inferred from either equation, is the same.

We should also note here that there are no correlations between measurement outcomes in the $n = 3$ basis. One can see this by showing that all possible measurement outcomes will occur with equal probability; that is, all of the coefficients in the superposition (7) have $|c^{333}_{pqr}|^2 = \frac{1}{8}$. Correlations will appear with measurements taken in the $n = 2$ basis.

4.3 RQM measurement by B on the system $S \otimes A$

Now B interacts with each of the subsystems $S_m A_m$ so as to perform RQM measurements in the basis $n = 2$. Note that this basis (and the others) are defined in the effective (eight-dimensional) Hilbert space occupied by the state $|GHZ\rangle_{SA}$ (8), as described two paragraphs above. RQM assigns this state to the system $S \otimes A$ now faced by B. To explore the consequence of B's interactions, we re-express $|GHZ\rangle_{SA}$ in terms of the $n = 2$ basis for $S_m \otimes A_m$, whose states are related to those of the $n = 3$ basis, $\left|\pm 1^{(3)}\right\rangle_{SA_m}$, by:

$$\left|\pm 1^{(2)}\right\rangle_{SA_m} \equiv \frac{1}{\sqrt{2}}\left(\left|+1^{(3)}\right\rangle_{SA_m} \pm i\left|-1^{(3)}\right\rangle_{SA_m}\right) \quad (9)$$

modulo an irrelevant overall phase factor. B's interactions with SA may now be expressed by unitary transformations $\hat{U}_m^{SAB}$ which have the familiar entangling features:

$$\hat{U}_m^{SAB}\left(\left|l^{(2)}\right\rangle_{SA_m}|\text{initial}\rangle_{B_m}\right) = \left|l^{(2)}\right\rangle_{SA_m}\left|l^{(2)}\right\rangle_{B_m} \equiv \left|l^{(2)}\right\rangle_{SAB_m}, \quad (10)$$

where B_m denotes the subsystem of B which interacts with SA_m, and SAB_m stands for $S_m \otimes A_m \otimes B_m$. The state of the system SAB after these interactions reads:

$$|GHZ\rangle_{SAB} = \sum_{pqr} c^{222}_{pqr}\left|p^{(2)}\right\rangle_{SAB_1}\left|q^{(2)}\right\rangle_{SAB_2}\left|r^{(2)}\right\rangle_{SAB_3}. \quad (11)$$

According to RQM, the entangling interactions comprise a measurement, and B realizes relative facts - specific values (± 1) for each of p, q, and r, but now referred to basis $n = 2$. We call these relative facts $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ for later reference. At this stage, the observers A and B have completed all RQM operations required for this gedanken experiment. We can now consider correlations between different possible outcomes.

4.4 GHZ correlations

For the sake of transparency, before considering measurement outcomes per se, we discuss correlations inherent in the state $|GHZ\rangle_{SAB}$ (11) itself, as manifested in the expansion coefficients c_{pqr}. As a first example, we note that $|GHZ\rangle_{SAB}$ is isomorphic with $|GHZ\rangle_S$ when expressed in the basis $n = 2$ for all qubits. Hence one can easily show that the amplitudes $c^{222}_{pqr} \neq 0$ only if:

$$p^{(2)}q^{(2)}r^{(2)} = 1. \quad (12)$$

This represents one of the four perfect GHZ correlations, which impose constraints on products of the individual eigenvalues. The analog of Eq. (7) shows that the three individual measurement

outcomes are random, but the product of all three is deterministic (and in this case, equal to unity).

We shall identify three more perfect correlations, which will impose similar constraints on other combinations of measurement outcomes already obtained by A and B, without requiring any further measurements. We can derive these additional constraints as follows. Unitary transformations like (10) can be put in a form explicitly acting on all three parts SAB_m. One can introduce $\hat{\mathbf{U}}^{SAB}_{(m)} = \hat{U}^{SAB}_m \otimes \hat{I}_{nk}$, where $\hat{I}_{nk}$ is a unit operator acting on systems $S_nAB \otimes S_kAB$, where $m, n, k = 1, 2, 3$ and cyclic permutations of these numbers. Note that the three transformations $\hat{\mathbf{U}}^{SAB}_{(m)}$ $(m = 1, 2, 3)$ mutually commute, and thus their order of application is immaterial.

We can now find the additional constraints on eigenvalues (p, q, r) by applying each of the three operations to the state $|GHZ\rangle_{SA}$. Starting with $\hat{\mathbf{U}}^{SAB}_{(1)}$, one generates the following Hilbert space vector,

$$\begin{aligned}
&|GHZ\rangle_{(SAB_1)(SA_2)(SA_3)} \\
&= \hat{\mathbf{U}}^{SAB}_{(1)} \left(|GHZ\rangle_{SA} \otimes_{j=1}^{3} |\text{initial}\rangle_{B_j} \right) \\
&= \left(\sum_{pqr} c^{233}_{pqr} \left|p^{(2)}\right\rangle_{SAB_1} \left|q^{(3)}\right\rangle_{SA_2} \left|r^{(3)}\right\rangle_{SA_3} \right) \\
&\otimes_{j=2}^{3} |\text{initial}\rangle_{B_j} \,. \qquad (13)
\end{aligned}$$

Exploiting the isomorphism between this state and $|GHZ\rangle_{S_1S_2S_3}$ when expressed in bases $n = 2, 3, 3$, respectively, one can show that $c^{233}_{pqr} \neq 0$ *only if*:

$$p^{(2)}q^{(3)}r^{(3)} = -1. \qquad (14)$$

This is the second perfect GHZ correlation, written as a state-imposed constraint on combinations of individual eigenvalues. The third perfect correlation arises from the application of $\hat{\mathbf{U}}^{SAB}_{(2)}$ to $|GHZ\rangle_{SA}$. The resulting Hilbert space vector is $|GHZ\rangle_{(SA_1)(SAB_2)(SA_3)}$. This has non-zero amplitudes for

$$p^{(3)}q^{(2)}r^{(3)} = -1. \qquad (15)$$

Finally, $\hat{\mathbf{U}}^{SAB}_{(3)}$ generates the Hilbert space vector $|GHZ\rangle_{(SA_1)(SA_2)(SAB_3)}$, with non-zero amplitudes for

$$p^{(3)}q^{(3)}r^{(2)} = -1. \qquad (16)$$

Thus we have four constraints, Eqs (12), (14), (15), and (16), on products of individual eigenvalues that characterize the GHZ state (11), and others related by unitary transformations.

4.5 GHZ contradiction with relative facts

The GHZ states are superpositions which allow all combinations of individual eigenvalues whose products satisfy the constraints (12), (14), (15), and (16). The output of each set of RQM measurements is just one of these possibilities, forming a trio of relative facts: First, A's measurement on $|GHZ\rangle_{SA}$ (Eq. 8), results in $\{p^{(2)}, q^{(2)}, r^{(2)}\} \to \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$, a specific set of facts relative to observer A (but not to B). Second, B's measurement on $|GHZ\rangle_{SAB}$ (Eq. 11), results in $\{p^{(3)}, q^{(3)}, r^{(3)}\} \to \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$, a specific set of facts relative to observer B (but not to A). It is known to both observers that two sets of measurements were performed resulting in six relative facts,[3] even though only three of these are known to Alice (according to RQM), and only the other three are known to Bob.

With this, we can implement our technical criterion, as stated in the bullet point in Sec. 3: the relative facts, $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$, whatever these may be, must follow the predictions of quantum theory, irrespective of whether they are realized with respect to the same or different observers, who make measurements in the same or different bases.

Thus, rewriting the GHZ constraints of Eqs. (12), (14), (15), and (16) in terms of relative facts, we have

$$\begin{aligned}
\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3 &= +1, \\
\mathcal{B}_1\mathcal{A}_2\mathcal{A}_3 &= -1, \\
\mathcal{A}_1\mathcal{B}_2\mathcal{A}_3 &= -1, \\
\mathcal{A}_1\mathcal{A}_2\mathcal{B}_3 &= -1. \qquad (17)
\end{aligned}$$

The combination of all four equations leads to

$$\mathcal{A}_1^2\mathcal{A}_2^2\mathcal{A}_3^2\mathcal{B}_1^2\mathcal{B}_2^2\mathcal{B}_3^2 = -1. \qquad (18)$$

This means that there is no set of real solutions $\{\mathcal{A}_m\}$ for which a set of real solutions $\{\mathcal{B}_m\}$ exists, and its converse. The logical possibilities are:

- If we allow that the relative facts $\{\mathcal{A}_m\}$ are realized by the observer A, then Bob's measurements cannot produce the relative facts $\{\mathcal{B}_m\}$ satisfying the constraints [17] imposed by quantum mechanics.

[3] This conforms to the example of the Wigner-Friend scenario described by Rovelli in (Rovelli, 2021)

- If we assume, on the other hand, that B's measurements produce relative facts $\{\mathcal{B}_m\}$, then the relative facts $\{\mathcal{A}_m\}$ do not exist, and in fact never existed. For if they had, then B could not have realized the required $\{\mathcal{B}_m\}$.

In either case, relative facts predicted by RQM lead to contradictions with the predictions of quantum mechanics based on GHZ correlations. Therefore, the concept of relative facts cannot be accommodated within quantum mechanics without rendering it internally inconsistent; Relational Quantum Mechanics is incompatible with quantum mechanics. Since the GHZ correlations follow from the Born rule, Relational Quantum Mechanics, according to our *criterion*, cannot be considered an interpretation of quantum mechanics.

If one additionally invokes the cross-perspective links postulate, then results $\mathcal{A}_i$ are effectively assumed to exist, and are merely unknown to Bob (see Appendix B), and thus we have a contradiction without invoking our technical criterion from Sec. 3, which is not a part of RQM. Therefore in such a case the contradiction found by us is an internal contradiction within RQM (augmented with cross-perspective links postulate).

5 Further Remarks

5.1 Consistency with the RQM assumptions concerning probability amplitudes

Note that the constraint (14) is in full concurrence with the RQM rule (1). In the case of quantum mechanics it stems from:

$$P\left(\left(p^{(2)},q^{(3)},r^{(3)}\right)\Big|GHZ\right) = \left|\left\langle p^{(2)},q^{(3)},r^{(3)}\Big|GHZ\right\rangle_S\right|^2, \tag{19}$$

which holds due to the isomorphism between the final state (13) and the ordinary GHZ state expressed in respective local bases. In RQM the transition probability $P\left(\left(p^{(2)},q^{(3)},r^{(3)}\right)|GHZ\right)$ applies on the "initial" side to the three qubits, and on the "outcomes" (relative values) side to B_1, A_2 and A_3 subsystems respectively. The relative facts $\left(p^{(2)},q^{(3)},r^{(3)}\right)$ have different RQM labels (which means that they are realised with respect to different systems). The formula for the quantum amplitude appearing in the right-hand-side of (19) can be re-expressed in terms of the RQM amplitudes in accordance with the rule (1):

$$\left\langle p^{(2)},q^{(3)},r^{(3)}\Big|GHZ\right\rangle_S = c^{233}_{pqr} = \sum_{p^{(3)}=\pm1} w\left(p^{(2)},p^{(3)}\right) w\left(\left(p^{(3)},q^{(3)},r^{(3)}\right),GHZ\right). \tag{20}$$

On the one hand each of the RQM amplitudes applies to different systems in the *in* and *out* sides, e.g. in $w\left(p^{(2)},p^{(3)}\right)$, $p^{(2)}$ is a relative fact about system $S\otimes A$ realised with respect to the observer B, whereas relative fact $p^{(3)}$ is about the initial system S realised with respect to the observing system A. On the other hand the values of the RQM amplitudes, put here generally as $w(y,x)$, are due to isomorphisms between the final states after entangling interactions and the initial GHZ state, formulas (4-13), given by the respective quantum amplitudes $\langle x|y\rangle_S$ calculated for the qubits $S = S_1\otimes S_2\otimes S_3$, that is $w(x,y) = \langle x|y\rangle_S$.

5.2 Relation with no-go theorem for non-contextual hidden variables

The four equations (17) are identical to those written by Mermin [Equations (6) in Mermin (1990)] in his refutation of hidden variables theories of either local or non-contextual character. His proof is based on a GHZ state of three spins [our Eq. (4)] and generalizes the original theorem of Greenberger, Horne, and Zeilinger (GHZ) Greenberger *et al.* (1989).

We should stress that the analogy with Mermin's proof is only mathematical, and not physical. Apart from the three-qubit GHZ states, the physical situations and scenarios are different. In Mermin's no-go for non-contextuality proof there is one observer, and the reasoning involves four separate experiments. In our case, two observers perform a single experiment. In Mermin's case, the GHZ contradiction arises only when one explicitly assigns non-contextual character to the variables in the equations.[4] In our case, we follow RQM, in which no hidden variables exist initially.

[4] In the version of Mermin's proof involving local hidden variables, there are three spatially separated observers, each choosing between two local measurements.

However, they come into existence when Alice interacts with S and realizes relative facts. These relative facts are, effectively, hidden variables.

Expanding on the above, Alice's relative facts are hidden variables according to the following definition: A hidden variable is any additional notion, or feature of a physical variable, parameter or state, which is present in a given interpretation and does not appear in the quantum mechanical formalism. Such variables are more subtle in the Relational Interpretation: *they are produced in interactions between systems, not during preparation procedure of the initial quantum state.*

5.3 Final note

If relative facts were to have any meaning *in* quantum theory, they would have to satisfy the constraints that could emerge from the quantum mechanical description of any experimental situation. We have shown that this is impossible.

As a concluding remark we provide the following quotation from Wheeler Wheeler and Zurek (1983):

- For a proper way of speaking we recall once more that it makes no sense to talk of the phenomenon until it has been brought to a close by an irreversible act of amplification: "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon."

See also Miller and Wheeler (1984), and a review Ma *et al.* (2016).

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A The First Wigner's Friend Scenarios

Wigner (1963)

E. P. Wigner eloquently expressed the view that the state vector of a quantum system is a statement of an observer's knowledge about the system, based on the available information. At the same time he followed quantum theory, including the collapse postulate. With the goal of locating the site of collapse in the von Neumann measurement chain von Neumann (1955), he introduced his Friend in 1963 Wigner (1995), and placed her in the chain between the measurement apparatus and Wigner himself. The Friend informed Wigner that she had observed a definite outcome, creating a tension with Wigner's expectation, based on unitary evolution, that one should instead find a superposition. This is the so-called "Wigner's Friend Paradox," although for Wigner it was not a paradox: For Wigner, it formed the argument that the "collapse" occurred in the consciousness of his Friend. Nevertheless, the "paradox" is a prominent theme in the current literature.

Deutsch (1985)

Deutsch introduced a more elaborate scenario in 1985 Deutsch (1985) that has inspired some of the current variations, to different purposes. Deutsch's motivation was to demonstrate the possibility, in principle, of distinguishing between the Copenhagen and Many-Worlds interpretations. To this end, he believed it to be necessary for one observer (say Wigner) to perform measurements on a compound system containing another observer (say Friend), and the spin of an atom which Friend has measured. Admittedly, the thought experiment is beyond experimental capabilities at present, if not forever. Nevertheless, here is a very brief outline:

The Friend measures the spin of the atom and reports to Wigner that she has observed a definite outcome, but does not reveal which outcome. In a radical departure from the previous scenario, Wigner now performs an experiment on the compound system of the spin and the Friend, $S \otimes A$, including the consciousness of the Friend. The experiment involves the reversal of the Hamiltonian dynamics of the Friend's consciousness, the atom's trajectories, and the spin detection apparatus - culminating finally in a remeasurement of the atom's spin. A pure state indicates Many-

Worlds evolution, whereas a mixed state indicates Copenhagen-like collapse.

It is important to stress a couple of points in assessing Deutsch's influence on later work. First, the Friend's spin measurement is described only vaguely. There is a reference to "sense organs," which represent some unspecified combination of the atom's detection apparatus and the Friend's consciousness. Nonetheless, it is clear that the measurement process is purely unitary. Second, Wigner's expectation of a superposition state of $S \otimes A$ is also based on his (momentary) assumption of unitary evoution. Both assumptions are consistent with the Many-Worlds framework assumed by Deutsch. However, these assumptions (of unitarity) are carried over into later scenarios in treatments that do not admit collapse into their descriptions, while retaining a one-world perspective. This is the source of a number of inconsistencies in later works, see e.g. Frauchiger and Renner (2018) and Brukner (2018), and the critique in Żukowski and Markiewicz (2021).

B Cross-perspective links

The *cross-perspective links* postulate clearly defines the prescription for Bob to get information about Alice's result. The method is deterministic. Alice's relative fact determines Bob's outcome in this prescription, despite the other postulate of RQM which says that he faces the situation in which S and A are entangled. Thus, Alice's relative fact is effectively a hidden variable for Bob.

This postulate is aimed at establishing in RQM the repeatability of outcomes of measurements by any observer who subsequently measures in the same basis. It creates an obvious tension with the postulate that after Alice's interaction/RQM-measurement Bob faces an entangled state of the system with Alice, with the Schmidt basis for $S \otimes A$ being the measurement basis of A. The introduction of *cross-perspective links* clarifies the status of Alice's relative facts. Namely, if we assume that for every two observers A and B, there exists a precisely defined operational way for B to *deterministically* recover outcomes of a measurement done by A, this means that A's outcomes have definite values, which are in principle accessible to the other observers, but without a direct *specific* interaction are just unknown to them. However, since the process of revealing them is fully deterministic, as effectively assumed by *cross-perspective links*, this lack of knowledge is fully "classical", and has nothing to do with quantum randomness, superpositions etc. Hence, it is now clear that axioms of RQM extended with *cross-perspective links* imply that every observer has access to his/her outcomes, and a deterministic procedure to get to know outcomes of any other observers, which means that in fact all outcomes exist for all observers, but some of them are just unknown. This conclusion is supported by the statement in Adlam and Rovelli (2022), p. 8, first paragraph, which reads as follows: "*...with the addition of the postulate of cross-perspective links it no longer seems possible to insist that everything is relational – or at least, it is no longer necessary to do so – because this postulate implies that the information stored in Alice's physical variables about the variable V of the system S is accessible in principle to any observer who measures her in the right basis, so at least at an emergent level this information about V is an observer-independent fact. This suggests that the set of 'quantum events' should be regarded as absolute, observer-independent features of reality in RQM, although quantum states remain purely relational. Thus we continue to endorse the sparse-flash ontology for RQM as advocated in refs [15,16]: however we now regard the pointlike quantum events or 'flashes' as absolute, observer-independent facts about reality, rather than relativizing them to an observer.*"

In summary, the meaning of the new postulate is that RQM, when addressing the results of A in relation to observer B, gives to the results the status of existence. They are merely "unknown" to B, being accessible by measurement as dictated in *cross-perspective links*.