An Improved Sample Complexity Lower Bound for (Fidelity) Quantum State Tomography

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We show that $\Omega(rd/\epsilon)$ copies of an unknown rank-r, dimension-d quantum mixed state are necessary in order to learn a classical description with $1-\epsilon$ fidelity. This improves upon the tomography lower bounds obtained by Haah et al. and Wright (when closeness is measured with respect to the fidelity function).

1 Background

We consider quantum state tomography: given n copies of a mixed state ρ, output a classical description of a state σ that is close to ρ. In this note we measure closeness between ρ and σ via their *fidelity* $F(\rho,\sigma)$, defined as the supremum of $|\langle\varphi \mid \psi\rangle|^2$ over all purifications $|\psi\rangle, |\varphi\rangle$ of ρ,σ respectively[1]. In *fidelity tomography*, the goal is for the output state σ to satisfy $F(\rho,\sigma) \geq 1-\epsilon$. Haah et al. [2] showed that $n = O(rd\log(d/\epsilon)/\epsilon)$ is sufficient for fidelity tomography where r is the rank of the density matrix ρ. They also proved a $n = \Omega\left(\frac{rd}{\delta^2\log(d/r\delta)}\right)$ lower bound for *trace distance tomography*, where the goal is to output a state σ whose trace distance $\|\rho-\sigma\|_1 = \frac{1}{2}\mathrm{Tr}(|\rho-\sigma|)$ with ρ is at most δ. The Fuchs-van de Graff inequalities then imply a $n = \Omega\left(\frac{rd}{\epsilon\log(d/r\epsilon)}\right)$ lower bound for fidelity tomography; thus their upper and lower bounds are tight up to logarithmic factors. O'Donnell and Wright [4] proved that $n = O(rd/\delta^2)$ input samples are sufficient for trace distance tomography. In his PhD thesis, Wright [7] showed that $n = \Omega(rd)$ samples are necessary for both fidelity and trace distance tomography; this is optimal when the desired trace distance error δ or infidelity ϵ is fixed to a constant, but otherwise is loose when the error is treated as a quantity going to zero.

In this note we improve the lower bound of Haah et al. [2] and Wright [7] in the fidelity tomography case and show that $n = \Omega(rd/\epsilon)$ input samples are needed. It is natural to conjecture that the optimal bound for fidelity tomography is $n = \Theta(rd/\epsilon)$; however we leave obtaining a matching upper bound for future work.

2 The argument

We prove our lower bound via reduction to the *pure state* tomography scenario, in which the input samples ρ are guaranteed to be pure states (in other words, the rank of the density matrix is 1). It was proved by Bruß and Macchiavello [1] that $n = \Theta(d/\epsilon)$ samples are necessary and sufficient to achieve fidelity $1-\epsilon$; this was based on a tight connection between optimal pure state tomography and optimal pure state cloning [3, 5].

Suppose there is an algorithm $\mathcal{A}$ that, on input n copies of a rank-r, dimension-d mixed state ρ, outputs with high probability a classical description of a state σ that has fidelity $1-\epsilon$ with ρ. Then we use this algorithm to construct another algorithm $\mathcal{B}$ that performs tomography on *pure*, dimension-rd

[1] We note that there are two versions of fidelity considered in the literature; this is the *squared* one.

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states using $O\Big(n + \frac{r^2}{\epsilon}\Big)$ input samples and achieves $1 - O(\epsilon)$ fidelity. The performance of algorithm $\mathcal{B}$ is subject to the bounds of Bruß and Macchiavello [1] – in other words, $\mathcal{B}$ must use at least $\Omega(rd/\epsilon)$ input samples. Thus it must be that

$$n = \Omega(rd/\epsilon) - O(r^2/\epsilon) = \Omega(rd/\epsilon) \ ,$$

as desired.

The algorithm $\mathcal{B}$ works as follows. Let $|\psi\rangle_{XY}$ denote the rd-dimensional pure input sample where X denotes an r-dimensional register and Y denotes a d-dimensional register.

1. The algorithm $\mathcal{B}$ takes n input samples $|\psi\rangle_{XY}$ and traces out the X register in each copy to obtain n copies of a mixed state $\rho \in \mathbb{C}^{d\times d}$.

2. Run algorithm $\mathcal{A}$ on the n copies of ρ to obtain (with high probability) a classical description of a rank-r, dimension-d state σ that has fidelity $1 - \epsilon$ with ρ.

3. Compute a classical description of the rank-r projector Π onto the support of σ.

4. Take $O(r^2/\epsilon)$ additional copies of the input state $|\psi\rangle_{XY}$ and measure the Y register of each copy using the projective measurement $\{\Pi, I - \Pi\}$, and keep the post-measurement states $|\widetilde{\psi}\rangle$ of the copies where the Π outcome was obtained.

5. Use the tomography procedure of [1] for dimension-r^2 pure states on the copies of $|\widetilde{\psi}\rangle$ where we treat the states as residing in the dimension-r^2 subspace

$$\mathbb{C}^r \otimes \mathrm{supp}(\Pi) \subseteq \mathbb{C}^r \otimes \mathbb{C}^d \ .$$

   Let $|\varphi\rangle \in \mathbb{C}^r \otimes \mathrm{supp}(\Pi)$ denote the result of this pure state tomography procedure. The algorithm $\mathcal{B}$ then outputs the classical description of $|\varphi\rangle$ as its estimation of $|\psi\rangle$.

We analyze the algorithm $\mathcal{B}$. Let

$$|\psi\rangle_{XY} = \sum_{i=1}^{r} \lambda_i |u_i\rangle \otimes |v_i\rangle$$

denote the Schmidt decomposition of $|\psi\rangle$ where $\{|u_1\rangle, \ldots, |u_r\rangle\}$ is an orthonormal basis for $\mathbb{C}^r$ and $\{|v_1\rangle, \ldots, |v_r\rangle\}$ is an orthogonal set of vectors in $\mathbb{C}^d$. We can then write ρ as

$$\rho = \mathrm{Tr}_X(|\psi\rangle\langle\psi|) = \sum_{i=1}^{r} \lambda_i^2 |v_i\rangle\langle v_i| \ .$$

Note that ρ is a rank-r density matrix.

Let $\sigma = \sum_{i=1}^{r} \mu_i |w_i\rangle\langle w_i|$ denote the estimate produced by Step 2. By the guarantees of algorithm $\mathcal{A}$, we have that (with high probability) $F(\rho, \sigma) \geq 1 - \epsilon$. Let $\Pi = \sum_{i=1}^{r} |w_i\rangle\langle w_i|$ denote the projector onto the support of σ. By the definition of fidelity, there exists a purification $|\varphi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^d$ of σ such that

$$F(\rho, \sigma) = |\langle\psi \mid \varphi\rangle|^2 \geq 1 - \epsilon \ .$$

On the other hand,

$$|\langle\psi \mid \varphi\rangle|^2 = |\langle\psi \mid (I \otimes \Pi) |\varphi\rangle|^2 \leq \langle\psi|I \otimes \Pi|\psi\rangle \cdot \langle\varphi \mid \varphi\rangle = \langle\psi|I \otimes \Pi|\psi\rangle$$

where the inequality uses Cauchy-Schwarz. Let $|\widetilde{\psi}\rangle = (I \otimes \Pi) |\psi\rangle / \| I \otimes \Pi |\psi\rangle \|$, and observe that

$$\begin{aligned} |\langle\widetilde{\psi} \mid \psi\rangle|^2 &= \frac{1}{\| I \otimes \Pi |\psi\rangle \|^2} \, | \langle\psi| I \otimes \Pi |\psi\rangle |^2 \\ &= \frac{1}{| \langle\psi| I \otimes \Pi |\psi\rangle |} \, | \langle\psi| I \otimes \Pi |\psi\rangle |^2 \\ &= | \langle\psi| I \otimes \Pi |\psi\rangle | \geq 1 - \epsilon \ . \end{aligned}$$

The number of copies of $|\widetilde{\psi}\rangle$ available in Step 5 is, with high probability, at least $\Omega(r^2/\epsilon)$. Thus the estimate $|\varphi\rangle$ computed by Step 5 will satisfy

$$|\langle\varphi \mid \widetilde{\psi}\rangle|^2 \geq 1-\epsilon$$

with high probability.

We now turn to a simple geometric proposition: if $|\langle\widetilde{\psi} \mid \psi\rangle| \geq 1-\eta$ and $|\langle\varphi \mid \widetilde{\psi}\rangle| \geq 1-\eta$, then $|\langle\varphi \mid \psi\rangle| \geq 1-4\eta$. This is because $|\langle\widetilde{\psi} \mid \psi\rangle| \geq 1-\eta$ implies

$$\| \, |\widetilde{\psi}\rangle - e^{i\alpha} \, |\psi\rangle \, \|^2 = 2 - 2|\langle\widetilde{\psi} \mid \psi\rangle| \leq 2\eta$$

where $e^{i\alpha}$ is a complex phase satisfying $|\langle\widetilde{\psi} \mid \psi\rangle| = e^{i\alpha}\langle\widetilde{\psi} \mid \psi\rangle$. Similarly $\| \, |\widetilde{\psi}\rangle - e^{i\beta} \, |\varphi\rangle \, \|^2 \leq 2\eta$ for some complex phase $e^{i\beta}$. Then by triangle inequality we have

$$\begin{aligned} 2 - 2|\langle\varphi \mid \psi\rangle| &\leq \|e^{i\beta} \, |\varphi\rangle - e^{i\alpha} \, |\psi\rangle \, \|^2 \\ &\leq 2\| \, |\widetilde{\psi}\rangle - e^{i\alpha} \, |\psi\rangle \, \|^2 + 2\| \, |\widetilde{\psi}\rangle - e^{i\beta} \, |\varphi\rangle \, \|^2 \leq 8\eta \, . \end{aligned}$$

The proposition follows via rearrangement.

Therefore with high probability, the estimate $|\varphi\rangle$ produced by algorithm $\mathcal{B}$ satisfies

$$|\langle\varphi \mid \psi\rangle|^2 \geq (1-8\epsilon)^2 \geq 1-16\epsilon \, .$$

3 Conclusion

We proved a $\Omega(rd/\epsilon)$ sample complexity lower bound for fidelity tomography for rank-r, dimension-d mixed states where $1-\epsilon$ is the fidelity of the resulting estimate. This is proved via reduction to the $\Omega(d/\epsilon)$ lower bound for pure state tomography established by [1]. In contrast, the lower bounds of [2] and [7] are based on communication complexity arguments. Natural questions include: (a) whether the upper bound for fidelity tomography can be improved to $O(rd/\epsilon)$, and (b) whether a $\Omega(rd/\delta^2)$ lower bound can be established for trace distance tomography. One obstacle to extending our argument to the trace distance setting is that we do not know whether applying the projection Π to the state $|\psi\rangle$ (if Π is the projector onto the support of a state σ that is δ-close to ρ in trace distance) yields a state that is $O(\delta)$-close to $|\psi\rangle$ in trace distance. The Gentle Measurement Lemma [6] implies that the post-measurement state is $O(\sqrt{\delta})$-close to $|\psi\rangle$; this ultimately yields a $\Omega(rd/\delta)$ lower bound on trace distance tomography, which we believe is not optimal.

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