

Constant-sized correlations are sufficient to self-test maximally entangled states with unbounded dimension

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Let p be an odd prime and let r be the smallest generator of the multiplicative group \mathbb{Z}_p^* . We show that there exists a correlation of size $\Theta(r^2)$ that self-tests a maximally entangled state of local dimension $p - 1$. The construction of the correlation uses the embedding procedure proposed by Slofstra (*Forum of Mathematics, Pi.* (2019)). Since there are infinitely many prime numbers whose smallest multiplicative generator is in the set $\{2, 3, 5\}$ (D.R. Heath-Brown *The Quarterly Journal of Mathematics* (1986) and M. Murty *The Mathematical Intelligencer* (1988)), our result implies that constant-sized correlations are sufficient for self-testing of maximally entangled states with unbounded local dimension.

The certification of a quantum device is an important building block for many quantum information processing tasks, especially when the devices are provided by some untrusted vendor. We would like such certification to be done based solely on the observed measurement statistics and with the only assumption that any local device cannot communicate with the other local devices. The measurement statistics are referred to as correlations. It has been shown that certain quantum correlations require the distant parties to share a particular entangled state up to some local isometry. This phenomenon is referred to as self-testing.

The fact that the verifier only interacts classically with the unknown device makes self-testing a powerful tool for applications in quantum cryptography and computational complexity theory. It allows a classical party to delegate quantum computations to some untrusted service provider and verify that the computations are performed

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honestly and correctly [9, 25]. Self-testing also becomes a critical component of the security proofs of device-independent quantum cryptographic protocols [12, 18]. Since the setting of multi-prover interactive proof systems (MIP) is similar to a self-test, self-tests also help to bound the computational power of entanglement-assisted MIP protocols [10, 14, 24].

The case of self-testing of the EPR pair is fully understood. The techniques for this case were first introduced in [19], then improved in [3]. Self-testing of tensor products of maximally entangled qubits were proved in [22, 23], with the latter being the one with the smallest question and answer sets. Self-testing of a general bipartite entangled states with local dimension d was proved in [8], which uses four questions but each question had d answers. The number of questions was later reduced to two in [27], but the number of answers was still d .

Our work aims at minimizing the sizes of the question and answer sets of a correlation that can self-test a maximally entangled state with large local dimension. For comparison, all the correlations used in the results listed above have either the number of questions or answers dependent on the local dimension of the entangled state. Our progress is summarized in the following theorem.

Theorem 1. *Let $r \in \{2, 3, 5\}$ and let D denote the infinite set of all odd primes p whose smallest primitive root is r . Then for any $p \in D$, the following maximally entangled state of local dimension $p - 1$,*

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{p-1}} \sum_{j=1}^{p-1} |j\rangle|p-j\rangle,$$

can be self-tested with constant-sized question and answer sets.

Proof overview

To prove Theorem 1, we construct a bipartite quantum correlation $P_{p,r}$. We denote the two parties by Alice and Bob. Alice and Bob's question sets are of size $\Theta(r)$ and their answer sets are constant-sized for each odd prime number p with smallest primitive root r . We say the size of the question set is of order $\Theta(r)$ to mean that there exist constants c and d such that the size of the question set is $cr + d$, and we say that r is a primitive root of p if r is a multiplicative generator of the group \mathbb{Z}_p^* . Note that when $r \in \{2, 3, 5\}$ the size of neither the question set nor the answer set of $P_{p,r}$ depends on p . We can view $P_{p,r}$ in a matrix where each entry in the matrix is labelled by Alice's and Bob's question-answer pair, so the size of $P_{p,r}$ is the size of this matrix, which equals the product of the number of Alice's question-answer pairs and the number of Bob's question answer pairs. Therefore, the size of $P_{p,r}$ is of order $\Theta(r^2)$. We prove $P_{p,r}$ self-tests a maximally entangled state of dimension $(p - 1)$.

The correlation $P_{p,r}$ contains a perfect correlation associated with a binary linear system. To induce such a correlation, Alice is given a random equation from a linear system and she should give an assignment to each variable of the equation. Bob is asked to assign a value to a random variable of the chosen equation. The correlation is perfect if Alice's assignment satisfies the equation and Bob's assignment to the variable matches Alice's assignment. A widely-used and thoroughly-studied example is the perfect correlation of the following linear system

$$\begin{aligned} x_1 + x_2 + x_3 = 0 & & x_4 + x_5 + x_6 = 0 \\ x_7 + x_8 + x_9 = 0 & & x_1 + x_4 + x_7 = 0 \\ x_2 + x_5 + x_8 = 1 & & x_3 + x_6 + x_9 = 0. \end{aligned}$$

The game version of this linear system is known as the Magic square game [2, 20]. Using two copies of $|EPR\rangle$, the perfect correlation can be induced. It has been shown that if a strategy can induce this correlation, the shared state must be $|EPR\rangle^{\otimes 2}$ up to some local isometry [29]. The key observation that leads to the self-testing proof is that an inducing strategy must contain binary observables X and Z that satisfy the anti-commutation relation

$$ZXZ = -X.$$

Intuitively, we can think of this perfect correlation as a way to enforce the anti-commutation relation [7].

We design a linear system $\hat{A}x = 0$ such that the perfect correlation $P_{\hat{A}}$ associated with the linear system can enforce the relation

$$UOU^\dagger = O^r, \quad (1)$$

for unitaries U and O . The inspiration comes from Slofstra's seminal work [28], where he outlined a new way to design binary linear system such that the associated perfect correlation can enforce conjugacy relations of the form $XYX^\dagger = Z$ for unitaries X, Y and Z . Following Slofstra's design, the numbers of equations and variables of $\hat{A}x = 0$ are of order $\Theta(r)$.

The reason that we choose Equation (1) to be the relation enforced by $P_{\hat{A}}$ is the following. Inducing $P_{\hat{A}}$ guarantees that the strategy contains unitary operators U and O satisfying Equation (1). Moreover, if we can certify that the unitary O has the eigenvalue $\omega_p := e^{i2\pi/p}$, Equation (1) automatically guarantees that the spectrum of O contains $\{\omega_p^j | 1 \leq j \leq p - 1\}$. It also guarantees that the eigenspace of each distinct eigenvalue has the same dimension.

In order to certify an eigenvalue of O , we use the optimal correlation P_μ that achieves the maximal violation of the $\cot(\mu)$ -weighted CHSH inequality proposed in [1, 16] as a variation of the CHSH inequality [5]

$$\begin{aligned} I_{\cot(\mu)} &= \cot(\mu) \langle M_0 N_0 + M_0 N_1 \rangle + \langle M_1 N_0 - M_1 N_1 \rangle \\ &\leq 2|\cot(\mu)|, \end{aligned} \quad (2)$$

for binary observables M_x, N_y with $x, y \in \{0, 1\}$. It has been proved that this correlation can self-test $|EPR\rangle$ [15]. We further prove that in an inducing strategy, the product of Bob's observables has eigenvalues $e^{i2\mu}$ and $e^{-i2\mu}$.

Therefore, the full correlation $P_{p,r}$ is induced by an ideal strategy that can induce $P_{\hat{A}}$ and $P_{-\pi/p}$ for some odd prime p whose primitive root is r . We can use $P_{-\pi/p}$ to certify that the unitary O has eigenvalues ω_p and ω_p^{-1} . Combining this observation with the fact that O satisfies the relation Equation (1), we can prove that the correlation $P_{p,r}$ self-tests the state $|\tilde{\psi}\rangle$ defined by

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{p-1}} \sum_{j=1}^{p-1} |j\rangle |p-j\rangle.$$

More precisely, we prove that if some quantum strategy using a shared state $|\psi\rangle$ induces $P_{p,r}$, then there exist a local isometry $\Phi_A \otimes \Phi_B$ and a state $|junk\rangle$ such that

$$\Phi_A \otimes \Phi_B(|\psi\rangle) = |junk\rangle \otimes |\tilde{\psi}\rangle. \quad (3)$$

Note that the isometry $\Phi_A \otimes \Phi_B$ only involves local operations on Alice's and Bob's sides. The isometry $\Phi_A \otimes \Phi_B$ captures the essence of the generalized swap-isometry proposed in [30], but it is constructed in a different way.

The last step of proving Theorem 1 involves a number theory result. It has been shown that there exists $r \in \{2, 3, 5\}$ such that there are infinitely many primes whose smallest primitive is r [13, 21]. The set D in the statement of Theorem 1 is the set of all such primes and we can prove Theorem 1 by applying the self-testing result of $P_{p,r}$ to all $p \in D$.

Structure of the paper

We start with notations and background information in Section 1. We present a correlation that can enforce the relation $UOU^\dagger = O^r$ in Section 2 and present an extension of P_μ in Section 3. Then we define the correlation $P_{p,r}$ and prove its self-testing property in Section 4. Conclusions and open problems are discussed in Section 5.

1 Preliminaries

In this section, we introduce our notations and bipartite quantum correlations. The EPR pair is denoted by

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (4)$$

The d -th root of unity is denoted by $\omega_d := e^{2\pi i/d}$. The set $\{0, 1, \dots, n-1\}$ is denoted by $[n]$.

When we write operator relation with respect to a state, we use the following style. Let \mathcal{H}_A and \mathcal{H}_B be two Hilbert spaces. When it is clear from the context that unitaries U_A and U_B act on \mathcal{H}_A and \mathcal{H}_B respectively, we write $U_A \otimes U_B |\psi\rangle$ as $U_A U_B |\psi\rangle$, $U_A \otimes \mathbb{I} |\psi\rangle$ as $U_A |\psi\rangle$, and $\mathbb{I} \otimes U_B |\psi\rangle$ as $U_B |\psi\rangle$ for some $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

To introduce *bipartite quantum correlations*, we consider a *nonlocal scenario* with two players, Alice and Bob. Each of them is requested to

give an answer for some question. Alice's question is chosen from the set $[n_A]$ and her answer should be from the set $[m_A]$. Bob's question is chosen from the set $[n_B]$ and his answer should be from the set $[m_B]$. Therefore, a nonlocal scenario is described by the tuple $([n_A], [n_B], [m_A], [m_B])$. A *bipartite correlation* of a nonlocal scenario $([n_A], [n_B], [m_A], [m_B])$ is a function $P : [n_A] \times [n_B] \times [m_A] \times [m_B] \rightarrow \mathbb{R}_{\geq 0} : (i, j, k, l) \mapsto P(k, l|i, j)$ where $P(k, l|i, j)$ is the probability for Alice to answer k and Bob to answer l when the question to Alice is i and to Bob is j . The size of the correlation P is $n_A n_B m_A m_B$.

Recall that a set of self-adjoint operators on \mathcal{H} , $\{P_j \mid j \in [n]\}$, is a *projective measurement* if $P_i^2 = P_i$ for all $i \in [n]$, $P_i P_j = 0$ for all $i \neq j$, and $\sum_{j \in [n]} P_j = \mathbb{I}_{\mathcal{H}}$. Each P_i is called a projector. A *binary observable* on a Hilbert space \mathcal{H} is a self-adjoint operator M such that $M^2 = \mathbb{I}_{\mathcal{H}}$.

A *quantum projective measurement strategy* for a nonlocal scenario $([n_A], [n_B], [m_A], [m_B])$ is a tuple

$$(|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{\{P_i^{(k)} \mid k \in [m_A]\} \mid i \in [n_A]\}, \{\{Q_j^{(l)} \mid l \in [m_B]\} \mid j \in [n_B]\}),$$

where \mathcal{H}_A and \mathcal{H}_B are Hilbert spaces of arbitrary dimension, $\{\{P_i^{(k)} \mid k \in [m_A]\} \mid i \in [n_A]\}$ and $\{\{Q_j^{(l)} \mid l \in [m_B]\} \mid j \in [n_B]\}$ are two sets of projective measurements on \mathcal{H}_A and \mathcal{H}_B respectively. Note that the tensor product structure indicates that the two parties cannot communicate with each other, which is the reason why we say such scenario is nonlocal. This quantum strategy induces the bipartite quantum correlation

$$P(k, l|i, j) = \langle \psi | P_i^{(k)} \otimes Q_j^{(l)} | \psi \rangle, \quad (5)$$

for each $(i, j, k, l) \in [n_A] \times [n_B] \times [m_A] \times [m_B]$.

Next we introduce a type of correlations associated with a binary linear system. Let $Ax = 0$ be an $m \times n$ binary linear system, where each row of A has κ nonzero entries and each entry of A is from \mathbb{Z}_2 . For each $i \in [m]$, we define

$$I_i = \{j \in [n] \mid A(i, j) = 1\}$$

$$S_i = \{x \in \mathbb{Z}_2^{I_i} \cong \mathbb{Z}_2^\kappa \mid \sum_{j \in I_i} x(j) \equiv 0 \pmod{2}\},$$

where the isomorphism between $\mathbb{Z}_2^{I_i}$ and \mathbb{Z}_2^κ is implicit.

A correlation $P : [m] \times [n] \times \mathbb{Z}_2^\kappa \times \mathbb{Z}_2 \rightarrow \mathbb{R}$ is a *perfect correlation* of $Ax = 0$ if

$$\sum_{x,y:x \in S_i, x(j)=y} P(x,y|i,j) = 1,$$

for all $i \in [m]$ and $j \in I_i$. The implication of a perfect correlation of $Ax = 0$ is summarized in the following lemma.

Lemma 1.1. *For an m -by- n binary linear system $Ax = 0$ where each row of A has κ nonzero entries, if a quantum strategy*

$$\begin{aligned} &(|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \\ &\{\{P_i^{(x)}|x \in \mathbb{Z}_2^\kappa\}|i \in [m]\}, \\ &\{\{Q_j^{(y)}|y \in \mathbb{Z}_2\}|j \in [n]\}) \end{aligned}$$

induces a perfect correlation of $Ax = 0$, then there exist a set of binary observables $\{M_j|j \in [n]\}$ on \mathcal{H}_A and a set of binary observables $\{N_j|j \in [n]\}$ on \mathcal{H}_B such that

$$M_j N_j |\psi\rangle = |\psi\rangle$$

for all $j \in [n]$,

$$\prod_{j \in I_i} M_j |\psi\rangle = \prod_{j \in I_i} N_j |\psi\rangle = |\psi\rangle$$

for all $i \in [m]$, and

$$(M_j M_k) |\psi\rangle = (M_k M_j) |\psi\rangle$$

for all $i \in [m]$ and $j, k \in I_i$.

The proof of the lemma can be found in the proof of Theorem 4 of [6] and in Section 3 of [28] so we omit it here.

Another correlation that we are interested in is the correlation that gives the maximal violation of the $\cot(\mu)$ -weighted CHSH inequality defined in Equation (2).

We first give the inducing strategy of this correlation. Let $\mu \in [-\pi, \pi)$. Define

$$\begin{aligned} \tilde{P}_0^{(0)} &= |0\rangle\langle 0|, & \tilde{P}_0^{(1)} &= |1\rangle\langle 1|, \\ \tilde{P}_1^{(0)} &= \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|), & \tilde{P}_1^{(1)} &= \mathbb{I} - \tilde{P}_1^{(0)}, \end{aligned}$$

and

$$\begin{aligned} \tilde{Q}_0^{(0)} &= (\cos(\frac{\mu}{2})|0\rangle + \sin(\frac{\mu}{2})|1\rangle) \\ &\quad \times (\cos(\frac{\mu}{2})\langle 0| + \sin(\frac{\mu}{2})\langle 1|), \\ \tilde{Q}_0^{(1)} &= \mathbb{I} - \tilde{Q}_0^{(0)}, \\ \tilde{Q}_1^{(0)} &= (\cos(\frac{\mu}{2})|0\rangle - \sin(\frac{\mu}{2})|1\rangle) \\ &\quad \times (\cos(\frac{\mu}{2})\langle 0| - \sin(\frac{\mu}{2})\langle 1|), \\ \tilde{Q}_1^{(1)} &= \mathbb{I} - \tilde{Q}_1^{(0)}. \end{aligned}$$

Definition 1.2. *The correlation $P_\mu : [2] \times [2] \times [2] \times [2] \rightarrow \mathbb{R}$ is induced by the strategy*

$$\begin{aligned} &(|EPR\rangle, \{\{\tilde{P}_x^{(a)}|a \in [2]\}|x \in [2]\}, \\ &\quad \{\{\tilde{Q}_y^{(b)}|b \in [2]\}|y \in [2]\}), \end{aligned}$$

such that $P_\mu(a,b|x,y) = \langle EPR|P_x^{(a)}Q_y^{(b)}|EPR\rangle$.

The correlation P_μ is known as the optimal correlation of the $\cot(\mu)$ -weighted CHSH inequality. The self-testing property of P_μ is summarized in the following Lemma, which was first proved in [15, Proposition A.3].

Lemma 1.3. *For $\mu \in [-\pi, \pi)$, if a quantum strategy $(|\psi\rangle, \{\{P_x^{(a)}|a \in [2]\}|x \in [2]\}, \{\{Q_y^{(b)}|b \in [2]\}|y \in [2]\})$ induces P_μ , then there exist a local isometry $\Phi = \Phi_A \otimes \Phi_B$ and an auxiliary state $|junk\rangle$ such that*

$$\Phi(P_x^{(a)} \otimes Q_y^{(b)} |\psi\rangle) = |junk\rangle \otimes (\tilde{P}_x^{(a)} \otimes \tilde{Q}_y^{(b)}) |EPR\rangle$$

for $x, y \in \{-1, 0, 1\}$ where the subscript -1 refers to the identity operator and $\tilde{P}_x^{(a)}, \tilde{Q}_y^{(b)}$ for $x, y, a, b \in \{0, 1\}$ are defined above.

Instead of giving the full proof, we list some key relations of this proof that will be reused later. From the projective measurements of the strategy, we get binary observables $P_x := P_x^{(0)} - P_x^{(1)}$ and $Q_y := Q_y^{(0)} - Q_y^{(1)}$ for $x, y \in [2]$. Define

$$\begin{aligned} Z_A &:= P_0, & X_A &:= P_1, \\ Z_B &:= \frac{Q_0 + Q_1}{2 \cos \mu}, & X_B &:= \frac{Q_0 - Q_1}{2 \sin \mu}, \end{aligned}$$

then

$$Z_A |\psi\rangle = Z_B |\psi\rangle, \tag{6}$$

$$X_A |\psi\rangle = X_B |\psi\rangle, \tag{7}$$

$$X_A (\mathbb{I} + Z_B) |\psi\rangle = X_B (\mathbb{I} - Z_A) |\psi\rangle, \tag{8}$$

$$Z_A (\mathbb{I} + X_B) |\psi\rangle = Z_B (\mathbb{I} - X_A) |\psi\rangle, \tag{9}$$

$$Z_A X_A |\psi\rangle = -X_A Z_A |\psi\rangle, \tag{10}$$

$$X_A Z_A |\psi\rangle = -X_B Z_B |\psi\rangle. \tag{11}$$

2 Enforcing $UOU^{-1} = O^r$

In this section, we show how to construct a binary linear system such that the perfect correlation associated with it can enforce the relation $UOU^{-1} = O^r$. Since the construction relies heavily on group presentations, we give the group theory background first. For more contexts, please refer to [26].

2.1 Group theory background

Let S be a set of letters. We denote by $\mathcal{F}(S)$ the free group generated by S , which consists of all finite words made from $\{s, s^{-1} | s \in S\}$ such that no ss^{-1} or $s^{-1}s$ appears as a substring for any s , where s^{-1} denotes the inverse of s . The group law is given by concatenation and cancellation.

Definition 2.1 (Group presentation). *Given a set S , let $\mathcal{F}(S)$ be the free group generated by S and let R be a subset of $\mathcal{F}(S)$. Then $\langle S : R \rangle$ denotes the quotient of $\mathcal{F}(S)$ by the normal subgroup generated by R . If a group G is isomorphic to $\langle S : R \rangle$, then we say G has a presentation $\langle S : R \rangle$.*

If both sets S and R are finite and a group G is defined by $\langle S : R \rangle$, then we say the group $G = \langle S : R \rangle$ is *finitely-presented*. The elements of S are the *generators* and the elements of R are the *relations*. A relation $r \in R$ is written as $r = e$ to convey its significance in the quotient group G . There are two types of relations that we will work with. Considering a subset of generators $\{s_j | 1 \leq j \leq n\} \subseteq S$, the relations of the form $\prod_{j=1}^n s_j = e$, where e is the identity element, are called *linear relations*; and the relations of the form $s_i s_j s_i^{-1} = s_k$ for some $i \neq j$ are called *conjugacy relations*.

There are two special types of groups that we will work with in this section. We give the definitions below.

Definition 2.2 (Conjugacy group). *Let $C \subseteq [n] \times [n] \times [n]$, and*

$$G = \langle \{s_i | i \in [n]\} : \{s_i^2 = e | i \in [n]\} \cup \{s_i s_j s_i = s_k | (i, j, k) \in C\} \rangle.$$

We say a group is a conjugacy group if it has a presentation of this form.

A conjugacy group is a special case of what Slofstra defined as a linear-plus-conjugacy group (Definition 26 of [28]), since a conjugacy group

does not have linear relations. The other type of group is directly related to a linear system.

Definition 2.3 (Solution group). *Let $Ax = 0$ be an $m \times n$ binary linear system. The solution group of this system is the group*

$$\begin{aligned} \Gamma(A) := & \langle \{x_i | i \in [n]\} : \{x_j^2 = e | j \in [n]\} \\ & \cup \{ \prod_{j \in I_i} x_j = e | i \in [m] \} \\ & \cup \{x_l x_k = x_k x_l | k, l \in I_i \text{ for some } i \in [m]\} \rangle. \end{aligned}$$

A representation of a group G on a finite-dimensional Hilbert space \mathcal{H} is a group homomorphism from G to the group of unitary operators on \mathcal{H} , denoted by $\mathcal{U}(\mathcal{H})$. The first reason that we study solution groups is that a representation of a solution group gives us a quantum strategy that induces a perfect correlation of the linear system [28, Theorem 18]. The second reason is that, given a solution group, we can construct a binary linear system by converting a relation of the form $x_i x_j x_k = e$ into a linear equation $x_i + x_j + x_k = 0$. In the next subsection, we are going to construct a solution group first and then extract the binary linear system from it.

Lastly, for groups G and K , an *embedding* of G into K is an injective group homomorphism $\phi : G \rightarrow K$. The relation between conjugacy groups and solution groups is that any conjugacy group can be embedded in a solution group, as proved in [28, Proposition 27].

2.2 Constructing the linear system

This subsection is devoted to proving the following proposition.

Proposition 2.4. *Let $r \geq 2$ be a positive integer. There exists a binary linear system $\hat{A}x = 0$ such that the following holds. If a quantum strategy $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{\{P_x^{(a)}\}\}, \{\{Q_y^{(b)}\}\})$ induces a perfect correlation of $\hat{A}x = 0$, then there exist binary observables $\{M_{u_1}, M_{u_2}, M_{o_1}, M_{o_2}\}$ on \mathcal{H}_A and $\{N_{u_1}, N_{u_2}, N_{o_1}, N_{o_2}\}$ on \mathcal{H}_B such that*

$$\begin{aligned} M_{u_1} M_{u_2} (M_{o_1} M_{o_2}) M_{u_2} M_{u_1} |\psi\rangle &= (M_{o_1} M_{o_2})^r |\psi\rangle \\ N_{u_1} N_{u_2} (N_{o_1} N_{o_2}) N_{u_2} N_{u_1} |\psi\rangle &= (N_{o_1} N_{o_2})^r |\psi\rangle. \end{aligned}$$

Note that in the statement of Proposition 2.4 r can be any positive integer greater than 1, and it doesn't have to be from the set $\{2, 3, 5\}$. To prove Proposition 2.4, we need the following lemma to

combine operator relations with respect to the same shared state.

Lemma 2.5 (Lemma 7 of [4]). *Let $|\psi\rangle \in \mathcal{H}$ be a quantum state. Suppose there exist unitaries $\{V\} \cup \{V_i \mid i \in [k]\} \cup \{M_i \mid i \in [n]\}$ on \mathcal{H} commuting with $\{N_i \mid i \in [n]\}$ on \mathcal{H} such that*

$$M_i|\psi\rangle = N_i|\psi\rangle$$

for each $i \in [n]$, and

$$V|\psi\rangle = \prod_{i \in [k]} V_i|\psi\rangle.$$

Then,

$$V \prod_{i \in [n]} M_i|\psi\rangle = \left(\prod_{i \in [k]} V_i \right) \left(\prod_{i \in [n]} M_i \right) |\psi\rangle.$$

Proof of Proposition 2.4. The linear system $\hat{A}x = 0$ is constructed from a solution group, wherein the following group is embedded. For $r \geq 2$, define

$$G := \langle u, o : uou^{-1} = o^r \rangle. \quad (12)$$

By [28, Proposition 33], G can be embedded into a conjugacy group $G_c = \langle S_c : R_c \rangle$ where S_c contains $\{u_1, u_2, o_1, o_2\}$ and $u_1^2 = u_2^2 = o_1^2 = o_2^2 = e$. We also know that the embedding $\phi : G \rightarrow G_c$ maps u to u_1u_2 and o to o_1o_2 . In other words, in G_c

$$u_1u_2(o_1o_2)u_2u_1 = (o_1o_2)^r. \quad (13)$$

By [28, Proposition 27], G_c can be embedded into a solution group $\Gamma(\hat{A}) := \langle S_\Gamma, R_\Gamma \rangle$. Moreover, $\{u_1, u_2, o_1, o_2\} \subseteq S_\Gamma$ and the embedding $\phi' : G_c \rightarrow \Gamma(\hat{A})$ maps s to s for each $s \in \{u_1, u_2, o_1, o_2\}$. Therefore, G is embedded in $\Gamma(\hat{A})$ and we get the binary linear system $\hat{A}x = 0$.

Since G_c is embedded in $\Gamma(\hat{A})$, we know that the relation in Equation (13) can be reconstructed by substituting in relations in R_Γ . Then, the statement of the proposition follows from Lemmas 1.1 and 2.5. \square

Following [28, Remark 30], we can get that $\hat{A}x = 0$ has $n(r) := 16s + 75$ variables and $m(r) := 14s + 62$ equations, where each equation has 3 nonzero variables. We assume that in this system x_0 corresponds to o_1 and x_1 corresponds to o_2 .

Next we show that there exists a quantum strategy that can induce a perfect correlation of

$\hat{A}x = 0$. The correlation is denoted by $P_{\hat{A}}$ and the strategy is denoted by $S_{\hat{A}}$, which is based on a representation of $\Gamma(\hat{A})$.

We first give a representation of G_c . Let p be an odd prime number such that r is a primitive root of p . Note that here we don't require that r is the smallest primitive root of p . Let $\mathbb{C}^{p-1} = \text{span}(\{|j\rangle \mid 1 \leq j \leq p-1\})$. A second basis of \mathbb{C}^{p-1} is $\{|x_j\rangle \mid 1 \leq j \leq p-1\}$, where

$$|x_j\rangle = -\frac{1}{\sqrt{2}}(|j\rangle + i|p-j\rangle), \quad (14)$$

$$|x_{p-j}\rangle = \frac{-\omega_{2p}^j}{\sqrt{2}}(|j\rangle - i|p-j\rangle) \quad (15)$$

for $1 \leq j \leq \frac{p-1}{2}$. Note that another form of this basis is $\{|x_{r^j}\rangle \mid j \in [p-1]\}$, where the subscript r^j is taken modulo p implicitly. Based on the second basis, we define a third basis of \mathbb{C}^{p-1} , $\{|u_k\rangle \mid k \in [p-1]\}$ defined by

$$|u_k\rangle = \frac{1}{\sqrt{p-1}} \sum_{j=0}^{p-2} \omega_{p-1}^{jk} |x_{r^j}\rangle.$$

On \mathbb{C}^{p-1} , we define

$$O_1 = \sum_{j=1}^{(p-1)/2} \omega_p^j |x_j\rangle \langle x_{p-j}| + \omega_p^{-j} |x_{p-j}\rangle \langle x_j|, \quad (16)$$

$$O_2 = \sum_{j=1}^{(p-1)/2} |x_j\rangle \langle x_{p-j}| + |x_{p-j}\rangle \langle x_j|, \quad (17)$$

$$U_1 = |u_0\rangle \langle u_0| - |u_{(p-1)/2}\rangle \langle u_{(p-1)/2}| \\ + \sum_{k=1}^{(p-3)/2} \left(\omega_{p-1}^k |u_k\rangle \langle u_{p-1-k}| + \omega_{p-1}^{-k} |u_{p-1-k}\rangle \langle u_k| \right), \quad (18)$$

$$U_2 = |u_0\rangle \langle u_0| + |u_{(p-1)/2}\rangle \langle u_{(p-1)/2}| \\ + \sum_{k=1}^{(p-3)/2} (|u_{p-1-k}\rangle \langle u_k| + |u_k\rangle \langle u_{p-1-k}|). \quad (19)$$

It can be checked that

$$O_1 O_2 = \sum_{j \in [p-1]} \omega_p^{r^j} |x_{r^j}\rangle \langle x_{r^j}|,$$

$$U_1 U_2 = \sum_{j \in [p-1]} |x_{r^{j-1}}\rangle \langle x_{r^j}|,$$

$$U_1 U_2 (O_1 O_2) U_2 U_1 = (O_1 O_2)^r.$$

Hence, we can follow the proof of [28, Proposition 33] to extend $\rho : G_c \rightarrow \mathcal{U}(\mathbb{C}^{p-1}) : u_1 \mapsto U_1, u_2 \mapsto$

$U_2, o_1 \mapsto O_1, o_2 \mapsto O_2$ to a representation of G_c , still denoted by ρ . Then, following the proofs of [28, Proposition 27 and Lemma 29], ρ can be extended to a representation of $\Gamma(\hat{A})$, $\rho' : \Gamma(\hat{A}) \rightarrow \mathcal{U}(\mathbb{C}^{p-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. In particular, for any $s \in \{u_1, u_2, o_1, o_2\}$,

$$\rho'(s) = \rho(s) \otimes \mathbb{I}_{\mathbb{C}^2} \otimes \mathbb{I}_{\mathbb{C}^2}.$$

Let

$$|\tilde{\psi}\rangle := \frac{1}{\sqrt{p-1}} \sum_{j=1}^{p-1} |x_j\rangle |x_{p-j}\rangle, \quad (20)$$

and $\Pi_s^{(0)}, \Pi_s^{(1)}$ be the projectors onto the $+1$ and -1 -eigenspaces of $\rho'(s)$ for each $s \in S_\Gamma$. Then we can construct a strategy

$$\begin{aligned} S_{\hat{A}} = & (|\tilde{\psi}\rangle \otimes |EPR\rangle^{\otimes 2}, \\ & \{\{P_i^{(x)} | x \in \mathbb{Z}_2^3\} | i \in [m(r)]\}, \\ & \{\{\Pi_s^{(y)} | y \in \mathbb{Z}_2\} | s \in S_\Gamma\}), \end{aligned}$$

where $P_i^{(x)}$ can be constructed from $\{\Pi_s^{(0)}, \Pi_s^{(1)} | s \in S_\Gamma\}$. Note that since the variables of \hat{A} are in one-to-one correspondence with the generators in S_Γ , we label Bob's projectors by the corresponding generators.

Definition 2.6. *The correlation $P_{\hat{A}} : [m(r)] \times S_\Gamma \times \mathbb{Z}_2^3 \times \mathbb{Z}_2 \rightarrow R$ is defined by*

$$\begin{aligned} P_{\hat{A}}(x, y | i, s) \\ = & [\langle \tilde{\psi} | \otimes \langle EPR |^{\otimes 2} (P_i^{(x)} \Pi_s^{(y)}) (|\tilde{\psi}\rangle \otimes |EPR\rangle^{\otimes 2}). \end{aligned}$$

It can be checked that $P_{\hat{A}}$ is a perfect correlation of $\hat{A}x = 0$.

3 Extending the correlation P_μ

3.1 Revisiting the self-testing property of P_μ

In this section, we show how to use P_μ to argue the eigenvalue of an unknown unitary.

Proposition 3.1. *For $\mu \in [-\pi, \pi)$, if a quantum strategy $(|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{\{P_x^{(a)} | a \in [2]\} | x \in [2]\}, \{\{Q_y^{(b)} | b \in [2]\} | y \in [2]\})$ induces P_μ , then there exist quantum states $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that*

$$\begin{aligned} Q_0 Q_1 |\psi_1\rangle &= e^{-i2\mu} |\psi_1\rangle, \\ Q_0 Q_1 |\psi_2\rangle &= e^{i2\mu} |\psi_2\rangle, \end{aligned}$$

where $Q_y := Q_y^{(0)} - Q_y^{(1)}$ for $y \in [2]$.

Proof. Let $P_x = P_x^{(0)} - P_x^{(1)}$ for $x \in \{0, 1\}$. The states in Proposition 3.1 are

$$|\psi_1\rangle = (P_0^{(0)} + iP_1 P_0^{(1)}) |\psi\rangle, \quad (21)$$

$$|\psi_2\rangle = (P_0^{(0)} - iP_1 P_0^{(1)}) |\psi\rangle. \quad (22)$$

We first show that $\| |\psi_1\rangle \| = 1$.

$$\begin{aligned} \| |\psi_1\rangle \|^2 \\ = & \langle \psi | (P_0^{(0)} + P_0^{(1)} - iP_0^{(1)} P_1 P_0^{(0)} + iP_0^{(0)} P_1 P_0^{(1)}) | \psi \rangle \\ = & 1 - i \langle \psi | P_0^{(1)} P_1 P_0^{(0)} - P_0^{(0)} P_1 P_0^{(1)} | \psi \rangle. \end{aligned}$$

Recall that $Z_A = P_0$, $X_A = P_1$, $Z_B = (Q_0 + Q_1)/2 \cos(\mu)$ and $X_B = (Q_0 - Q_1)/2 \sin(\mu)$. Using eqs. (7) and (11), we know

$$\begin{aligned} P_1 P_0^{(0)} |\psi\rangle &= \frac{X_A(\mathbb{I} + Z_A)}{2} |\psi\rangle \\ &= \frac{X_B(\mathbb{I} - Z_B)}{2} |\psi\rangle \\ &= \frac{(\mathbb{I} + Z_B) X_B}{2} |\psi\rangle \\ P_0^{(1)} |\psi\rangle &= \frac{\mathbb{I} - Z_A}{2} |\psi\rangle \\ &= \frac{\mathbb{I} - Z_B}{2} |\psi\rangle, \end{aligned}$$

so $\langle \psi | P_0^{(1)} P_1 P_0^{(0)} | \psi \rangle = 0$. With similar reasoning, we get $\langle \psi | P_0^{(0)} P_1 P_0^{(1)} | \psi \rangle = 0$. Therefore, $\| |\psi_1\rangle \| = 1$. The derivation of $\| |\psi_2\rangle \| = 1$ is very similar, so we omit it here.

Next, we show $Q_0 Q_1 |\psi_1\rangle = e^{-i2\mu} |\psi_1\rangle$ and $Q_0 Q_1 |\psi_2\rangle = e^{i2\mu} |\psi_2\rangle$. From eq. (6), we get that

$$\begin{aligned} Z_B P_0^{(0)} |\psi\rangle &= \frac{Z_B(\mathbb{I} + Z_A)}{2} |\psi\rangle \\ &= \frac{Z_B + \mathbb{I}}{2} |\psi\rangle \\ &= \frac{\mathbb{I} + Z_A}{2} |\psi\rangle \\ &= P_0^{(0)} |\psi\rangle. \end{aligned}$$

With similar reasoning, we get

$$Z_B P_0^{(1)} |\psi\rangle = -P_0^{(1)} |\psi\rangle.$$

Substituting the expression of Z_B , we see that

$$\begin{aligned} (Q_0 + Q_1) P_0^{(0)} |\psi\rangle &= 2 \cos(\mu) P_0^{(0)} |\psi\rangle, \\ (Q_0 + Q_1) P_0^{(1)} |\psi\rangle &= -2 \cos(\mu) P_0^{(1)} |\psi\rangle. \end{aligned}$$

From eqs. (6) to (8) and (10), we get that

$$\begin{aligned} X_B P_0^{(0)} |\psi\rangle &= X_A P_0^{(1)} |\psi\rangle, \\ X_B P_0^{(1)} |\psi\rangle &= X_A P_0^{(0)} |\psi\rangle. \end{aligned}$$

Substituting in the expression of X_B , we get that

$$\begin{aligned} (Q_0 - Q_1) P_0^{(0)} |\psi\rangle &= 2 \sin(\mu) P_1 P_0^{(1)} |\psi\rangle, \\ (Q_0 - Q_1) P_0^{(1)} |\psi\rangle &= 2 \sin(\mu) P_1 P_0^{(0)} |\psi\rangle. \end{aligned}$$

Simple calculation gives us that

$$\begin{aligned} Q_0 P_0^{(0)} |\psi\rangle &= [\cos(\mu) P_0^{(0)} + \sin(\mu) P_1 P_0^{(1)}] |\psi\rangle, \\ Q_1 P_0^{(0)} |\psi\rangle &= [\cos(\mu) P_0^{(0)} - \sin(\mu) P_1 P_0^{(1)}] |\psi\rangle, \\ Q_0 P_0^{(1)} |\psi\rangle &= [-\cos(\mu) P_0^{(1)} + \sin(\mu) P_1 P_0^{(0)}] |\psi\rangle, \\ Q_1 P_0^{(1)} |\psi\rangle &= [-\cos(\mu) P_0^{(1)} - \sin(\mu) P_1 P_0^{(0)}] |\psi\rangle. \end{aligned}$$

Then,

$$\begin{aligned} Q_0 Q_1 P_0^{(0)} |\psi\rangle &= [\cos(2\mu) P_0^{(0)} + \sin(2\mu) P_1 P_0^{(1)}] |\psi\rangle, \\ Q_0 Q_1 P_1 P_0^{(1)} |\psi\rangle &= [\cos(2\mu) P_1 P_0^{(1)} - \sin(2\mu) P_0^{(0)}] |\psi\rangle. \end{aligned}$$

Finally, the fact that $|\psi_1\rangle$ and $|\psi_2\rangle$ are $e^{-2i\mu}$ and $e^{2i\mu}$ eigenvectors of $Q_0 Q_1$ follows Equations (22) and (23). \square

3.2 The correlation $\hat{P}_{-\pi/p}$

In the rest of the work, we fix $\mu = -\pi/p$ for some odd prime p . We will introduce a correlation that is extended from $P_{-\pi/p}$ and the correlation is denoted by $\hat{P}_{-\pi/p}$. We define $\hat{P}_{-\pi/p} : [5] \times [5] \times [3] \times [3] \rightarrow \mathbb{R}$ by defining its inducing quantum strategy.

In \mathbb{C}^{p-1} , we define a subspace $V = \text{span}(\{|1\rangle, |p-1\rangle\})$ and we denote the projector onto V by Π_V . For question $x = 0$, let

$$\bar{P}_0^{(a)} = \bar{Q}_0^{(a)} = \begin{cases} \Pi_V & \text{if } a = 0 \\ \mathbb{I} - \Pi_V & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For questions $x = 1, 2$, let $\Lambda_{O_x}^{(0)}, \Lambda_{O_x}^{(1)}$ be the projectors onto the $+1$ - and -1 -eigenspaces of O_x , where O_1 and O_2 are defined in Equations (16) and (17), and let

$$\bar{P}_x^{(a)} = \bar{Q}_x^{(a)} = \begin{cases} \Lambda_{O_x}^{(0)} & \text{if } a = 0 \\ \Lambda_{O_x}^{(1)} & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For question $x = 3$, let

$$\bar{P}_3^{(a)} = \bar{Q}_3^{(a)} = \begin{cases} |1\rangle\langle 1| & \text{if } a = 0 \\ |p-1\rangle\langle p-1| & \text{if } a = 1 \\ \mathbb{I} - \Pi_V & \text{otherwise.} \end{cases}$$

For question $x = 4$, let

$$\bar{P}_4^{(a)} = \bar{Q}_4^{(a)} = \begin{cases} \frac{(|1\rangle + |p-1\rangle)(\langle 1| + \langle p-1|)}{2} & \text{if } a = 0 \\ \frac{(|1\rangle - |p-1\rangle)(\langle 1| - \langle p-1|)}{2} & \text{if } a = 1 \\ \mathbb{I} - \Pi_V & \text{otherwise.} \end{cases}$$

Substituting Equations (14) and (15) into Equation (20), we get

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{p-1}} \sum_{j=1}^{(p-1)/2} \omega_{2p}^j (|j\rangle|j\rangle + |p-j\rangle|p-j\rangle).$$

The inducing strategy is

$$\begin{aligned} S_{-\pi/p} &= (|\tilde{\psi}\rangle, \{\{\bar{P}_x^{(a)} | a \in [3]\} | x \in [5]\}, \\ &\quad \{\{\bar{Q}_y^{(b)} | b \in [3]\} | y \in [5]\}). \end{aligned}$$

Definition 3.2. *The correlation $\hat{P}_{-\pi/p} : [5] \times [5] \times [3] \times [3] \rightarrow \mathbb{R}$ is induced by $S_{-\pi/p}$:*

$$\hat{P}_{-\pi/p}(a, b | x, y) = \langle \tilde{\psi} | \bar{P}_x^{(a)} \bar{Q}_y^{(b)} | \tilde{\psi} \rangle.$$

As an analogue of Proposition 3.1, the implication of $\hat{P}_{-\pi/p}$ is summarized in the next proposition.

Proposition 3.3. *Let $(|\psi\rangle) \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\{\{P_x^{(a)} | a \in [3]\} | x \in [5]\}$, $\{\{Q_y^{(b)} | b \in [3]\} | y \in [5]\}$ be a strategy that induces $\hat{P}_{-\pi/p}$, and let*

$$|\psi_1\rangle = \frac{1}{2} (P_3^{(0)} + iP_4 P_3^{(1)} - iP_4 P_3^{(0)} + P_3^{(1)}) |\psi\rangle, \quad (23)$$

where $P_4 = P_4^{(0)} - P_4^{(1)}$. Then, $\| |\psi_1\rangle \|^2 = 1/(p-1)$ and

$$P_1 P_2 |\psi_1\rangle = \omega_p^{-1} |\psi_1\rangle \quad (24)$$

$$Q_1 Q_2 |\psi_1\rangle = \omega_p |\psi_1\rangle, \quad (25)$$

where $P_x := P_x^{(0)} - P_x^{(1)}$ and $Q_y := Q_y^{(0)} - Q_y^{(1)}$ for $x, y \in \{1, 2\}$.

To help with the proof of this proposition, we first give some values of $\hat{P}_{-\pi/p}$.

$$\hat{P}_{-\pi/p}(a, b|0, 0) = \begin{cases} 2/(p-1) & \text{if } a = b = 0 \\ (p-3)/(p-1) & \text{if } a = b = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{P}_{-\pi/p}(a, b|3, 3) = \begin{cases} 1/(p-1) & \text{if } a = b = 0 \\ 1/(p-1) & \text{if } a = b = 1 \\ (p-3)/(p-1) & \text{if } a = b = 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{P}_{-\pi/p}(a, b|0, 3) = \begin{cases} 1/(p-1) & \text{if } a = 0, b \in [2] \\ (p-3)/(p-1) & \text{if } a = 1, b = 2 \\ 0 & \text{otherwise.} \end{cases}$$

		$x = 3$		$x = 4$	
		$a = 0$	$a = 1$	$a = 0$	$a = 1$
$y = 1$	$b = 0$	$\frac{\cos^2(\pi/2p)}{p-1}$	$\frac{\sin^2(\pi/2p)}{p-1}$	$\frac{1-\sin(\pi/p)}{2(p-1)}$	$\frac{1+\sin(\pi/p)}{2(p-1)}$
	$b = 1$	$\frac{\sin^2(\pi/2p)}{p-1}$	$\frac{\cos^2(\pi/2p)}{p-1}$	$\frac{1+\sin(\pi/p)}{2(p-1)}$	$\frac{1-\sin(\pi/p)}{2(p-1)}$
$y = 2$	$b = 0$	$\frac{\cos^2(\pi/2p)}{p-1}$	$\frac{\sin^2(\pi/2p)}{p-1}$	$\frac{1+\sin(\pi/p)}{2(p-1)}$	$\frac{1-\sin(\pi/p)}{2(p-1)}$
	$b = 1$	$\frac{\sin^2(\pi/2p)}{p-1}$	$\frac{\cos^2(\pi/2p)}{p-1}$	$\frac{1-\sin(\pi/p)}{2(p-1)}$	$\frac{1+\sin(\pi/p)}{2(p-1)}$

Table 1: The correlation for $x \in \{3, 4\}$, $y \in \{1, 2\}$ and $a, b \in [2]$.

Proof. From the definition of $\hat{P}_{-\pi/p}$, it is easy to see that

$$P_x^{(2)}|\psi\rangle = Q_x^{(2)}|\psi\rangle = 0$$

for $x, y \in [3]$. Then

$$\begin{aligned} P_x^2|\psi\rangle &= [P_x^{(0)} + P_x^{(1)}]|\psi\rangle + P_x^{(2)}|\psi\rangle \\ &= |\psi\rangle \end{aligned}$$

for $x \in \{1, 2\}$. Similarly, we see that $Q_y^2|\psi\rangle = |\psi\rangle$ for $y \in \{1, 2\}$.

Next, we will show

$$\begin{aligned} P_0^{(0)}|\psi\rangle &= (P_3^{(0)} + P_3^{(1)})|\psi\rangle \\ &= (P_4^{(0)} + P_4^{(1)})|\psi\rangle \\ &= Q_0^{(0)}|\psi\rangle \\ &= (Q_3^{(0)} + Q_3^{(1)})|\psi\rangle \\ &= (Q_4^{(0)} + Q_4^{(1)})|\psi\rangle \\ P_0^{(1)}|\psi\rangle &= P_3^{(2)}|\psi\rangle = P_4^{(2)}|\psi\rangle = \\ Q_0^{(1)}|\psi\rangle &= Q_3^{(2)}|\psi\rangle = Q_4^{(2)}|\psi\rangle, \end{aligned}$$

and

$$\begin{aligned} P_3^{(0)}|\psi\rangle &= Q_3^{(0)}|\psi\rangle & P_3^{(1)}|\psi\rangle &= Q_3^{(1)}|\psi\rangle \\ P_4^{(0)}|\psi\rangle &= Q_4^{(0)}|\psi\rangle & P_4^{(1)}|\psi\rangle &= Q_4^{(1)}|\psi\rangle. \end{aligned}$$

Instead of giving the full proof, we show $P_0^{(0)}|\psi\rangle = Q_0^{(0)}|\psi\rangle$ to demonstrate the ideas. From the correlation, we know $\langle\psi|P_0^{(0)}Q_0^{(0)}|\psi\rangle = \langle\psi|P_0^{(0)}|\psi\rangle = \langle\psi|Q_0^{(0)}|\psi\rangle = 2/(p-1)$. Then,

$$\begin{aligned} &\|P_0^{(0)}|\psi\rangle - Q_0^{(0)}|\psi\rangle\|^2 \\ &= \langle\psi|P_0^{(0)}|\psi\rangle + \langle\psi|Q_0^{(0)}|\psi\rangle - 2\langle\psi|P_0^{(0)}Q_0^{(0)}|\psi\rangle \\ &= 0, \end{aligned}$$

which implies that $P_0^{(0)}|\psi\rangle = Q_0^{(0)}|\psi\rangle$.

Then, we can show that $\hat{P}_{-\pi/p}$ can be "reduced" to $P_{-\pi/p}$ by proving that

$$S = \left(\frac{P_0^{(0)}|\psi\rangle}{\|P_0^{(0)}|\psi\rangle\|}, \{ \{ P_x^{(0)}, P_x^{(1)} \} | x \in \{3, 4\} \}, \{ \{ Q_y^{(0)}, Q_y^{(1)} \} | y \in \{1, 2\} \} \right)$$

can induce $P_{-\pi/p}$, and that

$$S' = \left(\frac{P_0^{(0)}|\psi\rangle}{\|P_0^{(0)}|\psi\rangle\|}, \{ \{ P_x^{(0)}, P_x^{(1)} \} | x \in \{1, 2\} \}, \{ \{ Q_y^{(0)}, Q_y^{(1)} \} | y \in \{3, 4\} \} \right)$$

can induce $P_{-\pi/p}$ with Alice and Bob's roles flipped. To prove S can induce $P_{-\pi/p}$, we need to examine the terms of the form $\langle\psi|P_0^{(0)}P_x^{(a)}Q_y^{(b)}P_0^{(0)}|\psi\rangle$ for $x = 3, 4$, $y = 1, 2$ and $a, b = 0, 1$. We find that these terms relate to $\langle\psi|P_x^{(a)}Q_y^{(b)}|\psi\rangle$ by

$$\begin{aligned} &\langle\psi|P_x^{(a)}Q_y^{(b)}|\psi\rangle \\ &= \langle\psi|(P_0^{(0)} + P_0^{(1)})P_x^{(a)}Q_y^{(b)}(P_0^{(0)} + P_0^{(1)})|\psi\rangle \\ &= \langle\psi|P_0^{(0)}P_x^{(a)}Q_y^{(b)}P_0^{(0)}|\psi\rangle, \end{aligned}$$

where we use the facts that $P_x^{(a)}P_0^{(1)}|\psi\rangle = P_x^{(a)}P_x^{(2)}|\psi\rangle = 0$ for the relevant values of (x, y, a, b) . Therefore,

$$\frac{\langle\psi|P_0^{(0)}P_x^{(a)}Q_y^{(b)}P_0^{(0)}|\psi\rangle}{\|P_0^{(0)}|\psi\rangle\|^2} = \frac{\langle\psi|P_x^{(a)}Q_y^{(b)}|\psi\rangle}{\|P_0^{(0)}|\psi\rangle\|^2},$$

for the relevant values of (x, y, a, b) , and it is easy to verify that S induces $P_{-\pi/p}$. The proof of S'

induces $P_{-\pi/p}$ with Alice and Bob's roles flipped is similar, so we omit it here.

The state $|\psi_1\rangle$ can also be written as

$$|\psi_1\rangle = \frac{1}{2}(\mathbb{I} - iP_4)(P_3^{(0)} + iP_4P_3^{(1)})|\psi\rangle. \quad (26)$$

The derivation of $\|\psi_1\|$ is very similar to the corresponding part in the proof of Proposition 3.1, so we omit it here. Since S can induce $P_{-\pi/p}$, by Proposition 3.1, we know that

$$\begin{aligned} & Q_1Q_2(P_3^{(0)} + iP_4P_3^{(1)})P_0^{(0)}|\psi\rangle \\ &= \omega_p(P_3^{(0)} + iP_4P_3^{(1)})P_0^{(0)}|\psi\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} & (P_3^{(0)} + iP_4P_3^{(1)})P_0^{(0)}|\psi\rangle \\ &= (P_3^{(0)} + iP_4P_3^{(1)})(P_3^{(0)} + P_3^{(1)})|\psi\rangle \\ &= (P_3^{(0)} + iP_4P_3^{(1)})|\psi\rangle. \end{aligned}$$

Hence, using the fact that Q_1Q_2 commutes with $(\mathbb{I} - iP_4)$, we know

$$Q_1Q_2|\psi_1\rangle = \omega_p|\psi_1\rangle.$$

What remains to be proved is $P_1P_2|\psi_1\rangle = \omega_p^{-1}|\psi_1\rangle$. In order to prove it, we need another form of $|\psi_1\rangle$, which is

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{2}(Q_3^{(0)} - iQ_4Q_3^{(1)} + iQ_4Q_3^{(0)} + Q_3^{(1)})|\psi\rangle. \\ &= (\mathbb{I} + iQ_4)(Q_3^{(0)} - iQ_4Q_3^{(1)})|\psi\rangle, \end{aligned} \quad (27)$$

where $Q_4 := Q_4^{(0)} - Q_4^{(1)}$. Comparing the two forms of $|\psi_1\rangle$, it suffices to show

$$P_4(P_3^{(1)} - P_3^{(0)})|\psi\rangle = Q_4(Q_3^{(0)} - Q_3^{(1)})|\psi\rangle.$$

This equation can be derived in the following way

$$\begin{aligned} & P_4(P_3^{(1)} - P_3^{(0)})|\psi\rangle \\ &= P_4(P_3^{(1)} - P_3^{(0)})P_0^{(0)}|\psi\rangle \\ &= (Q_3^{(1)} - Q_3^{(0)})Q_4Q_0^{(0)}|\psi\rangle \\ &= Q_4(Q_3^{(0)} - Q_3^{(1)})Q_0^{(0)}|\psi\rangle \\ &= Q_4(Q_3^{(0)} - Q_3^{(1)})|\psi\rangle, \end{aligned}$$

where we use the fact that Equation (10) is satisfied in the inducing strategies S and S' . In the end, we apply Proposition 3.1 to S' to see that

$$\begin{aligned} & P_1P_2|\psi_1\rangle \\ &= (\mathbb{I} + iQ_4)P_1P_2(Q_3^{(0)} - iQ_4Q_3^{(1)})|\psi\rangle \\ &= \omega_p^{-1}|\psi_1\rangle, \end{aligned}$$

which completes the proof. \square

4 The self-testing correlation

In this section, we introduce the correlation $P_{p,r} : [n(r) + m(r) + 3] \times [n(r) + m(r) + 3] \times [8] \times [8] \rightarrow \mathbb{R}$, which can be thought of as the combination of $P_{\hat{A}}$ and $\hat{P}_{-\pi/p}$.

We define $P_{p,r}$ by giving the inducing strategy, which is based on $S_{\hat{A}}$ of $P_{\hat{A}}$ and $S_{-\pi/p}$ of $\hat{P}_{-\pi/p}$. Let

$$\tilde{P}_x^{(a)} = \begin{cases} \Pi_x^{(a)} & \text{if } x \in [n(r)] \text{ and } a \in [2], \\ P_{x-n(r)}^{(a)} & \text{if } n(r) \leq x \leq n(r) + m(r) - 1, \\ \bar{P}_0^{(a)} \otimes \mathbb{I}_{\mathbb{C}^2}^{\otimes 2} & \text{if } x = n(r) + m(r) \text{ and } a \in [2], \\ \bar{P}_{x+2-n(r)-m(r)}^{(a)} \otimes \mathbb{I}_{\mathbb{C}^2}^{\otimes 2} & \text{if } x = n(r) + m(r) + 1, n(r) + m(r) + 2 \text{ and } a \in [3], \\ 0 & \text{otherwise;} \end{cases}$$

$$\tilde{Q}_y^{(b)} = \begin{cases} \Pi_y^{(b)} & \text{if } y \in [n(r)] \text{ and } b \in [2], \\ P_{y-n(r)}^{(b)} & \text{if } n(r) \leq y \leq n(r) + m(r) - 1, \\ \bar{Q}_0^{(b)} \otimes \mathbb{I}_{\mathbb{C}^2}^{\otimes 2} & \text{if } y = n(r) + m(r) \text{ and } b \in [2], \\ \bar{Q}_{y+2-n(r)-m(r)}^{(b)} \otimes \mathbb{I}_{\mathbb{C}^2}^{\otimes 2} & \text{if } y = n(r) + m(r) + 1, n(r) + m(r) + 2 \text{ and } b \in [3], \\ 0 & \text{otherwise.} \end{cases}$$

The inducing strategy is

$$\tilde{S} = (|\tilde{\psi}\rangle \otimes |EPR\rangle^{\otimes 2}, \{\{\tilde{P}_x^{(a)}|a \in [8]\}|x \in [n(r) + m(r) + 3]\}, \{\{\tilde{Q}_y^{(b)}|b \in [8]\}|y \in [n(r) + m(r) + 3]\}).$$

Note that in the definition of \tilde{S} , a bijection between $[n(r)]$ and S_Γ is implicit.

Definition 4.1. *The correlation $P_{p,r} : [n(r) + m(r) + 3] \times [n(r) + m(r) + 3] \times [8] \times [8] \rightarrow \mathbb{R}$ is induced by \tilde{S} as*

$$P_{p,r}(a, b|x, y) = \langle \tilde{\psi} | \otimes \langle EPR |^{\otimes 2} \tilde{P}_x^{(a)} \tilde{Q}_y^{(b)} [|\tilde{\psi}\rangle \otimes |EPR\rangle^{\otimes 2}].$$

It is not hard to see that

- When $x \in [n(r)]$, $n(r) \leq y \leq n(r) + m(r) - 1$ and $a \in [2]$,

$$P_{p,r}(a, b|x, y) = P_{\hat{A}}(b, a|y - n(r), x).$$

- When $n(r) \leq x \leq n(r) + m(r) - 1$, $y \in [n(r)]$ and $b \in [2]$,

$$P_{p,r}(a, b|x, y) = P_{\hat{A}}(a, b|x - n(r), y).$$

- When $x = y \in [n(r)]$,

$$P_{p,r}(0, 0|x, x) + P_{p,r}(1, 1|x, x) = 1.$$

- When $x, y \in \{0, 1, n(r) + m(r), n(r) + m(r) + 1, n(r) + m(r) + 2\}$, define $f : \{0, 1, n(r) + m(r), n(r) + m(r) + 1, n(r) + m(r) + 2\} \rightarrow [5]$ by

$$f(x) = \begin{cases} x + 1 & \text{if } x = 0, 1, \\ x - n(r) - m(r) & \text{if } x = n(r) + m(r), \\ x + 2 - n(r) - m(r) & \text{otherwise.} \end{cases}$$

Then, if $a, b \in [3]$

$$P_{p,r}(a, b|x, y) = \hat{P}_{-\pi/p}(a, b|f(x), f(y)).$$

Proposition 4.2. *If a strategy*

$$S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{\{P_x^{(a)}|a \in [8]\}|x \in [n(r) + m(r) + 3]\}, \{\{Q_y^{(b)}|b \in [8]\}|y \in [n(r) + m(r) + 3]\})$$

induces $P_{p,r}$, then there exist unitaries $U_A \in \mathcal{U}(\mathcal{H}_A)$ and $U_B \in \mathcal{U}(\mathcal{H}_B)$ and a subnormalized state $|\psi_1\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\begin{aligned} \|\psi_1\|^2 &= \frac{1}{p-1} \\ |\psi\rangle &= \sum_{j=1}^{(p-1)} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle \end{aligned}$$

where $\log_r j$ is the discrete log.

Before we prove the proposition, we fix some notations used in the proofs of this proposition and the next theorem. We relabel $P_0^{(a)}$ and $Q_0^{(b)}$ as $P_{o_1}^{(a)}$ and $Q_{o_1}^{(b)}$, and relabel $P_1^{(a)}$ and $Q_1^{(b)}$ as $P_{o_2}^{(a)}$ and $Q_{o_2}^{(b)}$. We can also identify variables of $\hat{A}x = 0$ corresponding to the generators u_1 and u_2 , so we label the corresponding Alice and Bob's projectors as $P_{u_1}^{(a)}, P_{u_2}^{(a)}$ and $Q_{u_1}^{(b)}, Q_{u_2}^{(b)}$. For $s \in \{u_1, u_2, o_1, o_2\}$, we further define

$$P_s := P_s^{(0)} - P_s^{(1)} \quad Q_s := Q_s^{(0)} - Q_s^{(1)},$$

and

$$\begin{aligned} O_A &:= P_{o_1} P_{o_2} & O_B &:= Q_{o_1} Q_{o_2}, \\ U_A &:= P_{u_1} P_{u_2} & U_B &:= Q_{u_1} Q_{u_2}. \end{aligned}$$

Proof. Define $P_{n(r)+m(r)+1} := P_{n(r)+m(r)+1}^{(0)} - P_{n(r)+m(r)+1}^{(1)}$, $P_{n(r)+m(r)+2} := P_{n(r)+m(r)+2}^{(0)} - P_{n(r)+m(r)+2}^{(1)}$, and

$$|\psi_1\rangle = \frac{1}{2} (P_{n(r)+m(r)+1}^{(0)} + P_{n(r)+m(r)+1}^{(1)} - iP_{n(r)+m(r)+2} P_{n(r)+m(r)+1}) |\psi\rangle$$

Since S can induce $\hat{P}_{-\pi/p}$, by Proposition 3.3, we know $\|\psi_1\| = 1/\sqrt{p-1}$, and

$$O_A |\psi_1\rangle = \omega_p^{-1} |\psi_1\rangle, \quad O_B |\psi_1\rangle = \omega_p |\psi_1\rangle.$$

Since S can also induce $P_{\hat{A}}$, and $P_{\hat{A}}$ with Alice and Bob's roles flipped, by Proposition 2.4, we know

$$\begin{aligned} U_A O_A U_A^\dagger |\psi\rangle &= O_A^r |\psi\rangle, \\ U_B O_B U_B^\dagger |\psi\rangle &= O_B^r |\psi\rangle. \end{aligned}$$

By substitution and Lemma 2.5, we know

$$\begin{aligned} O_A(U_A^\dagger)^j|\psi\rangle &= (U_A^\dagger)^j O_A^{r^j}|\psi\rangle, \\ O_B(U_B^\dagger)^j|\psi\rangle &= (U_B^\dagger)^j O_B^{r^j}|\psi\rangle, \end{aligned}$$

for $j \in [p-1]$. Then,

$$\begin{aligned} O_A(U_A^\dagger)^j|\psi_1\rangle &= (U_A^\dagger)^j O_A^{r^j}|\psi_1\rangle = \omega_p^{-r^j} (U_A^\dagger)^j|\psi_1\rangle, \\ O_B(U_B^\dagger)^j|\psi_1\rangle &= (U_B^\dagger)^j O_B^{r^j}|\psi_1\rangle = \omega_p^{r^j} (U_B^\dagger)^j|\psi_1\rangle, \end{aligned}$$

where we use the fact that $|\psi_1\rangle$ can be expressed using Alice's projectors and Bob's projectors as shown in Equations (23) and (27). Hence, $\{(U_A^\dagger U_B^\dagger)^j|\psi_1\rangle | j \in [p-1]\}$ is an orthogonal set and

$$\left\| \sum_{j \in [p-1]} (U_A^\dagger U_B^\dagger)^j|\psi_1\rangle \right\| = 1.$$

The last step is to check that $\langle\psi| \sum_{j \in [p-1]} (U_A^\dagger U_B^\dagger)^j|\psi_1\rangle = 1$. Using the fact that $U_A U_B |\psi\rangle = P_{u_1} Q_{u_1} P_{u_2} Q_{u_2} |\psi\rangle = |\psi\rangle$, we can see that

$$\langle\psi| \sum_{j \in [p-1]} (U_A^\dagger U_B^\dagger)^j|\psi_1\rangle = (p-1)\langle\psi|\psi_1\rangle.$$

Hence, the problem is reduced to calculate $\langle\psi|\psi_1\rangle$, which is

$$\begin{aligned} & \frac{1}{2} \langle\psi| (P_{n(r)+m(r)+1}^{(0)} + P_{n(r)+m(r)+1}^{(1)} \\ & \quad - iP_{n(r)+m(r)+2} P_{n(r)+m(r)+1}) |\psi\rangle \\ &= \frac{1}{p-1} - \frac{i}{2} \langle\psi| Q_{n(r)+m(r)+2} P_{n(r)+m(r)+1} |\psi\rangle \\ &= \frac{1}{p-1}, \end{aligned}$$

where $\langle\psi| Q_{n(r)+m(r)+2} P_{n(r)+m(r)+1} |\psi\rangle = 0$ comes from the correlation. Then the proposition follows. \square

Theorem 4.3. *Let S be an inducing strategy of $P_{p,r}$ with a shared state $|\psi\rangle$. Then there exist an isometry $\Phi_A \otimes \Phi_B$ and a state $|junk\rangle$ such that*

$$\begin{aligned} \Phi_A \otimes \Phi_B(|\psi\rangle) &= |junk\rangle \otimes |\tilde{\psi}\rangle \\ \Phi_A \otimes \Phi_B(O_A|\psi\rangle) &= |junk\rangle \otimes [(O_1 O_2) \otimes \mathbb{I}] |\tilde{\psi}\rangle \\ \Phi_A \otimes \Phi_B(O_B|\psi\rangle) &= |junk\rangle \otimes [\mathbb{I} \otimes (O_1 O_2)] |\tilde{\psi}\rangle \\ \Phi_A \otimes \Phi_B(U_A|\psi\rangle) &= |junk\rangle \otimes [(U_1 U_2) \otimes \mathbb{I}] |\tilde{\psi}\rangle \\ \Phi_A \otimes \Phi_B(U_B|\psi\rangle) &= |junk\rangle \otimes [\mathbb{I} \otimes (U_1 U_2)] |\tilde{\psi}\rangle, \end{aligned}$$

where $|\tilde{\psi}\rangle$ is defined in Equation (20) and O_1, O_2, U_1, U_2 are defined in Equations (16) to (19) respectively.

The isometry $\Phi_A \otimes \Phi_B$ is given in the figure below. It is designed based on the swap isometry proposed in [30].

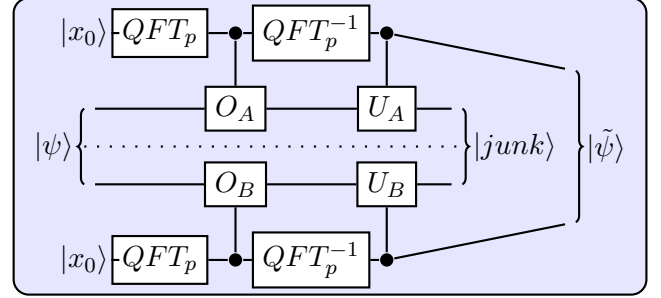


Figure 1: The isometries $\Phi_A \otimes \Phi_B$.

The isometry $\Phi_A \otimes \Phi_B$ has the following steps:

1. Append control register $|x_0\rangle_{A'}$ on Alice's side and $|x_0\rangle_{B'}$ on Bob's side, where $|x_0\rangle$ is orthogonal to \mathbb{C}^{p-1} ;
2. Apply Quantum Fourier Transform (QFT_p) to Alice and Bob's control registers;
3. Apply Controlled- $O_{A/B}$ operations (i.e. if the control register is in state $|x_k\rangle_{A'/B'}$, apply $O_{A/B}^k$);
4. Apply inverse Quantum Fourier Transform (QFT_p^{-1}) to the control registers;
5. Apply Controlled- $U_{A/B}$ operations (i.e. If Alice's control register is in state $|x_j\rangle$, she applies $U_A^{\log_r(p-j)}$. If Bob's control register is in state $|x_j\rangle$, he applies $(U_B)^{\log_r j}$).

Proof. Proposition 4.2 implies that

$$\Phi_A \otimes \Phi_B(|\psi\rangle) = \Phi_A \otimes \Phi_B\left(\sum_{j=1}^{(p-1)} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle\right).$$

Then the state $|\psi\rangle$ is evolved by the isometry as

follows

$$\begin{aligned}
& \sum_{j=1}^{p-1} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle |x_0\rangle_{A'} |x_0\rangle_{B'} \\
& \xrightarrow{QFT_p} \frac{1}{p} \sum_{k_1, k_2 \in [p]} \sum_{j=1}^{p-1} (U_A^\dagger U_B^\dagger)^{\log_r j} \\
& \quad \times |\psi_1\rangle |x_{k_1}\rangle_{A'} |x_{k_2}\rangle_{B'} \\
& \xrightarrow{\text{Controlled-}O_{A/B}} \frac{1}{p} \sum_{k_1, k_2 \in [p]} \sum_{j=1}^{p-1} O_A^{k_1} (U_A^\dagger)^{\log_r j} \\
& \quad \times O_B^{k_2} (U_B^\dagger)^{\log_r j} |\psi_1\rangle |x_{k_1}\rangle_{A'} |x_{k_2}\rangle_{B'} \\
& = \frac{1}{p} \sum_{k_1, k_2 \in [p]} \sum_{j=1}^{p-1} (U_A^\dagger U_B^\dagger)^{\log_r j} \\
& \quad \times \omega_p^{(k_2 - k_1)j} |\psi_1\rangle |x_{k_1}\rangle_{A'} |x_{k_2}\rangle_{B'} \\
& \xrightarrow{QFT_p^{-1}} \frac{1}{p^2} \sum_{j=1}^{p-1} \sum_{l_1, l_2 \in [p]} (U_A^\dagger U_B^\dagger)^{\log_r j} \\
& \quad \times \left(\sum_{k_1 \in [p]} \omega_p^{k_1(p-j-l_1)} \right) \\
& \quad \times \left(\sum_{k_2 \in [p]} \omega_p^{k_2(j-l_2)} \right) |\psi_1\rangle |x_{l_1}\rangle_{A'} |x_{l_2}\rangle_{B'} \\
& = \sum_{j=1}^{p-1} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle |x_{p-j}\rangle_{A'} |x_j\rangle_{B'} \\
& \xrightarrow{\text{Controlled-}U_{A/B}} \sum_{j=1}^{p-1} U_A^{\log_r j} (U_A^\dagger)^{\log_r j} U_B^{-\log_r j} (U_B^\dagger)^{\log_r j} \\
& \quad \times |\psi_1\rangle |x_{p-j}\rangle_{A'} |x_j\rangle_{B'} \\
& = |\psi_1\rangle \otimes \sum_{j=1}^{p-1} |x_{p-j}\rangle_{A'} |x_j\rangle_{B'},
\end{aligned}$$

where we use the fact that for $l \in [p]$,

$$\sum_{k \in [p]} \omega_p^{k(j-l)} = \begin{cases} p & \text{if } l = j \\ 0 & \text{otherwise.} \end{cases}$$

The derivations of the other four equations follow very similar argument, so instead of giving the full proof, we give the key steps for them. For O_A and O_B , we use the fact that

$$\begin{aligned}
O_A \sum_{j=1}^{p-1} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle &= \sum_{j=1}^{p-1} \omega_p^{-j} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle, \\
O_B \sum_{j=1}^{p-1} (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle &= \sum_{j=1}^{p-1} \omega_p^j (U_A^\dagger U_B^\dagger)^{\log_r j} |\psi_1\rangle.
\end{aligned}$$

For U_A and U_B , we use the fact that

$$\begin{aligned}
O_A (U_A^\dagger)^{\log_r j-1} |\psi_1\rangle &= \omega_p^{-jr^{-1}} (U_A^\dagger)^{\log_r j-1} |\psi_1\rangle, \\
O_B (U_B^\dagger)^{\log_r j-1} |\psi_1\rangle &= \omega_p^{jr^{-1}} (U_B^\dagger)^{\log_r j-1} |\psi_1\rangle,
\end{aligned}$$

for all $1 \leq j \leq p-1$. \square

Theorem 4.3 implies that, for any odd prime number p whose primitive root is r , there exists a correlation of size $\Theta(r^2)$ that can self-test a maximally entangled state of local dimension $p-1$. Since there exists $r \in \{2, 3, 5\}$ such that there are infinitely many prime numbers whose smallest primitive root is r [21], we can apply Theorem 4.3 to the set D of all such odd prime numbers and obtain Theorem 1.

5 Conclusion and discussions

We have shown that there exists a family of constant-sized correlations such that each correlation of this family can self-test a maximally entangled state of a different dimension.

Curious readers may wonder if we can self-test $|\tilde{\psi}\rangle \otimes |EPR\rangle^{\otimes 2}$ since the inducing strategy uses it. Indeed, we can self-test it, but the proof of the self-test requires more details of the representation of $\Gamma(\hat{A})$ and modifying $\hat{A}x = 0$ to introduce equations of the Magic square game. Since the techniques to self-test $|EPR\rangle^{\otimes 2}$ is standard in the literature, we refer to [11, Section 6], which is the previous arXiv version of this paper, for details of self-testing $|\tilde{\psi}\rangle \otimes |EPR\rangle^{\otimes 2}$.

The other question to ask is if we can self-test O_1 and O_2 individually. We don't have an answer for this question. The progress of answering this question is summarized here. Following the calculation in the proof of Proposition 3.3, we can see that

$$\begin{aligned}
(P_1 P_2) P_2 |\psi_1\rangle &= \omega_p P_2 |\psi_1\rangle, \\
(Q_1 Q_2) Q_2 |\psi_1\rangle &= \omega_p^{-1} Q_2 |\psi_1\rangle.
\end{aligned}$$

Then, to self-test O_1 and O_2 on Alice's side, we need to modify the controlled- $U_{A/B}$ step so that when the control-register is in the state $|x_j\rangle$, both Alice and Bob apply $U_{A/B}^{\log_r j}$. If we denote the modified isometry by $\Phi'_A \otimes \Phi'_B$,

$$\Phi'_A \otimes \Phi'_B (P_{o_x} |\psi\rangle) = |junk\rangle \otimes (O_x \otimes \mathbb{I}) |\tilde{\psi}\rangle,$$

with $x = 1, 2$. What is interesting is that if we want to self-test O_1 and O_2 on Bob's side, we need

to modify the controlled- $U_{A/B}$ step again, so that when the control-register is in the state $|x_j\rangle$, both Alice and Bob apply $U_{A/B}^{\log_r(p-j)}$. If we denote the modified isometry by $\Phi''_A \otimes \Phi''_B$,

$$\Phi''_A \otimes \Phi''_B(Q_{o_y}|\psi\rangle) = |junk\rangle \otimes (\mathbb{I} \otimes O_y)|\tilde{\psi}\rangle,$$

with $y = 1, 2$. Since $\Phi'_A \otimes \Phi'_B$ and $\Phi''_A \otimes \Phi''_B$ are different from $\Phi_A \otimes \Phi_B$, we cannot conclude that $P_{p,r}$ is a self-test of $|\tilde{\psi}\rangle$ and the binary observables O_1 and O_2 . On the other hand, it has been shown projectors of arbitrary rank can be self-tested [17]. If we can self-test O_1 , O_2 and even U_1 and U_2 , we can get a second self-test of projectors of arbitrary rank.

The last question about $P_{p,r}$ is about its robustness. In the arXiv version [11], we have shown that if a quantum strategy S can induce a correlation P' such that

$$\max_{x,y,a,b} |P'(a,b|x,y) - P_{p,r}(a,b|x,y)| \leq \epsilon,$$

then,

$$\|\Phi_A \otimes \Phi_B(|\psi\rangle) - |junk\rangle \otimes |\tilde{\psi}\rangle\| = O(r^p \epsilon^{1/8}).$$

The derivation of the robustness is very long and technical, so we choose the journal version to focus on illustrating the basic ideas. Improving the robustness bound is another open problem about $P_{p,r}$.

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