Distinguishing noisy boson sampling from classical simulations

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Giving a convincing experimental evidence of the quantum supremacy over classical simulations is a challenging goal. Noise is considered to be the main problem in such a demonstration, hence it is urgent to understand the effect of noise. Recently found classical algorithms can efficiently approximate, to any small error, the output of boson sampling with finite-amplitude noise. In this work it is shown analytically and confirmed by numerical simulations that one can efficiently distinguish the output distribution of such a noisy boson sampling from the approximations accounting for low-order quantum multiboson interferences, what includes the mentioned classical algorithms. The number of samples required to tell apart the quantum and classical output distributions is strongly affected by the previously unexplored parameter: density of bosons, i.e., the ratio of total number of interfering bosons to number of input ports of interferometer. Such critical dependence is strikingly reminiscent of the quantum-to-classical transition in systems of identical particles, which sets in when the system size scales up while density of particles vanishes.

1 Introduction

Quantum mechanics promises computational advantage over digital computers [1, 2]. Current technology is on the brink of building quantum devices with the promised advantage in some specific computational tasks, called the quantum supremacy [3], for which goal several quantum systems are considered [4, 5, 6, 7, 8] and a dramatic breakthrough was recently reported

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[9]. Can noise, always present in an experimental setup, compromise the quantum supremacy by allowing for an efficient classical simulation [10]? In this work we consider how a noisy boson sampling system can be distinguished from efficient classical approximations.

In boson sampling proposal of Aaronson & Arkhipov [4] the specific classically hard computational task is sampling from many-body quantum interference of N indistinguishable bosons on a unitary linear M -dimensional interferometer. At least in the so-called no-collision regime, when the output ports receive at most a single boson (i.e., for $M \gg N^2$ [11]), complexity-theoretic arguments have been found in Ref. [4] for the quantum advantage over classical simulations. In general, the output probabilities depend on the full many-body quantum interference of N bosons given by a sum of $N!$ quantum amplitudes, i.e., by the matrix permanents [12, 13], which are hard to compute [14, 15, 16, 17].

The boson sampling proposal had initiated efforts for experimental demonstration. Single photons [18, 19, 20, 21, 22] as well as Gaussian states [23, 24, 25] in optical interferometers, and the temporal-mode encoding [26, 27] were proposed and tested for experimental implementation with quantum optics. Experimental quantum optical platform has seen significant advances [28, 29, 30, 31] culminating recently in an experimental implementation with 20 photons on a 60-mode interferometer [32]. Alternative platforms include ion traps [33], superconducting qubits [34, 35], neutral atoms in optical lattices [36] and dynamic Casimir effect [37]. Initial estimate on the threshold size for demonstration of quantum supremacy with boson sampling was $N \approx 30$ bosons [4]. However, recent classical simulations [38, 39] pushed the threshold to $N \approx 50$ bosons. Inevitable experimental noise [35, 46, 40, 41, 42, 43, 44, 45] additionally opens possibilities for efficient classical approximation

algorithms [47, 48, 49, 50, 52, 53, 51]. Most importantly, if noise amplitudes do not vanish when the system size scales up, the recent algorithms of Refs. [48, 52, 53] approximate efficiently the output distribution of noisy boson sampling by low-order multi-boson interference.

Unavoidable noise has been accounted for in the boson sampling proposal [4] by allowing for an approximation error. Taking into account that the complexity-theoretic arguments for the quantum supremacy are asymptotic, whereas in an experiment one has a finite quantum device, one may wonder how small experimental error should be? In the spirit of Ref. [1], the experimental error problem can be formulated as follows: Can a classical algorithm efficiently sample from the output distribution of a quantum device in such a way that it would be impossible to tell from the sampling data whether we have the classical simulation or the quantum device? At the other extreme, the two output distributions can be efficiently distinguished. Since the promised computational advantage is due to the opening gap between exponential and polynomial computations in the size of a quantum system [3], it is reasonable to consider a method of distinguishing a quantum device from a classical simulation as efficient if it requires only a polynomial in the quantum system size number of samples.

As a partial answer to the above problem, in the present work it is shown that one can efficiently distinguish the output distribution of noisy boson sampling from output distributions of a wide range of classical algorithms, such as simulation with classical particles and the recent algorithms of Refs. [48, 52, 53]. We give an analytical expression for the lower bound on the total variation distance between the output distribution of noisy boson sampling and that of such a classical simulation and point that the probability of no particle counts in a subset of output ports can be used for distinguishing the quantum and classical distributions. The number of samples necessary for distinguishing the quantum and classical distributions critically depends on the density of bosons, defined as the ratio of the total number of interfering bosons to the number of input ports of interferometer, and not on the number of bosons or the number of ports of interferometer themselves. Our analytical results are valid asymptotically with the number of in-

terfering bosons, convergence to the asymptotic result is studied by using numerical simulations.

Previously the output distribution of boson sampling was shown [54] to be far in the total variation distance from the uniform distribution (argued to be an efficient approximation [55]), where a set of open problems was given. The present work partially resolves open problems (2) and (4)-(6) of Ref. [54] by considering a wide class of possible classical approximations to a noisy realization of boson sampling with arbitrary scaling of the interferometer size in the total number of interfering bosons, beyond the no-collision regime. In Ref. [56] it was shown how one can efficiently distinguish the output distribution of boson sampling from the simulation by classical particles. Statistical or pattern recognition techniques were also applied to assessment of boson sampling and distinguishing it from classical simulations [57, 58, 59, 60]. However, no analysis of the impact of realistic experimental noise was attempted previously, which is the main goal of the present work.

The text is organized as follows. In section 2 the boson sampling realization with linear optics and the classical approximations by low-order multi-boson interference are described and our results on distinguishing their respective output probability distributions are presented. In section 3 our principal findings are summarized and discussed. For better readability, the derivations and mathematical details are relegated to Appendices A-F.

2 Noisy boson sampling vs classical approximations

The boson sampling proposal of Ref. [4] considers the quantum interference of N identical single bosons on a unitary M -port in the no-collision regime $M \gg N^2$, here we consider it for arbitrary $M \geq N$. It turns out that distinguishing noisy boson sampling, as N scales up, crucially depends on density of bosons $\rho = N/M$, more precisely, on the scaling $\rho = \rho(N)$ (or, equivalently, $M = M(N)$). This fact allows us to consider all models with a given scaling $\rho = \rho(N)$ as a class of boson sampling, where N serves as the size parameter in the class. For example, for at least $\rho \sim 1/N$ we have the so-called no-collision regime, the main focus of Ref. [4].

Let us now mention known estimates on the number of classical computations required to simulate the boson sampling. For the ideal (noiseless) boson sampling, the fastest to date algorithms of Refs. [38, 39] can sample from its output distribution in $O(N2^N)$ computations. The number of computations depends on the density of bosons: for non-vanishing density of bosons the output probabilities can be given by the matrix permanents of rank-deficient matrices with reduced computational complexity [62, 63]. The following simple rule can be stated [61]: to sample from the boson sampling in arbitrary regime of density of bosons ρ by the algorithm of [39] requires at least as many classical simulations as for the no-collision boson sampling with $\mathcal{N}_\rho = N/(1 + \rho)$ bosons. The size parameter $\mathcal{N}_\rho$ gives also the expected number of output ports occupied by bosons as a function of the boson density.

Less is known about the scaling of classical computations for exact sampling from *noisy* boson sampling. There are several strong sources of noise: noisy interferometer [41, 43], partial distinguishability of bosons [42], and boson losses [44, 45]. It was shown that a finite number of lost bosons (or dark counts of detectors, or both) does not compromise the computational hardness of boson sampling in the no-collision regime. There are also results for the approximate sampling from noisy boson sampling, where the number of classical computations depends additionally on the scaling of the amplitude of noise in the system size [40, 48, 52, 53]. Efficient classical approximation algorithm of boson sampling in the no-collision regime with partially distinguishable bosons was found in Ref. [48], then extended to losses [52] and noise in interferometer [53]. For intermediate-size boson sampling devices the number of computations of the classical algorithm can be further optimized [64]. There is equivalence of different imperfections in their effect on classical hardness of boson sampling, e.g., noise in experimental platforms [22, 35] has a similar effect to that of the partial distinguishability of bosons [53] (below we will use the term noise for all imperfections).

The efficient classical algorithm of Refs. [48, 52, 53] can approximate the output distribution of N -boson sampling with a finite-amplitude noise by a smaller one, with K interfering bosons and $N - K$ classical particles, it becomes effi-

cient (i.e., polynomial in N, M) for bounded K , since the classical computations scale exponentially only in K [48]. Below we investigate if a noisy boson sampling with finite-amplitude noise can be efficiently distinguished from such a classical approximation, to answer the experimentally relevant problem posed in the previous section.

Probability distribution at output of noisy boson sampling

Let us now briefly describe an experimental implementation of boson sampling. We will use the popular example of linear optical setup with single photons, see fig. 1, and focus on two sources of noise: losses of photons and their partial distinguishability. Later on, we will include also the false random (a.k.a. dark) counts of detectors, i.e., the counts which are not triggered by photons. There is an equivalence of the losses compensated by dark counts with noise in the interferometer [53], thus our model takes into account the strongest sources of noise in linear optical experimental setup. The multi-photon component at input is neglected here. It can be a strong source of noise with the photon sources based on the parametric down conversion, e.g., when about N^2 sources are used to produce N single photons in the boson sampling from a Gaussian state [23, 24, 31]. With additional optical multiplexing of the sources one can significantly reduce the multiphoton component noise [65].

Figure 1: Noisy boson sampling vs classical approximation. Noisy N -boson quantum interference on a M -port (left). An adversary (right) tries to approximate the output distribution of N -boson quantum interference, $p(m_1, m_2, \dots, m_M)$, by the output distribution, $p^{(K)}(m_1, m_2, \dots, m_M)$, from a combination of K -boson interference (here $K = 3$) and $N - K$ distinguishable bosons.

Unitary linear optical interferometer U with

M input and output ports connects the optical mode $|k^{(in)}\rangle$ of input port k to the optical modes $|1^{(out)}\rangle, \dots, |M^{(out)}\rangle$ of the output ports: $|k^{(in)}\rangle = \sum_{l=1}^M U_{kl} |l^{(out)}\rangle$. Such an interferometer would take a Fock state of photons in the input ports into a superposition of the Fock states of the photons in the output ports according to the corresponding input-output relations between the boson creation operators in the input and output ports:

$$\hat{a}_k^\dagger = \sum_{l=1}^M U_{kl} \hat{b}_l^\dagger, \quad (1)$$

where $\hat{a}_k^\dagger$ ($\hat{b}_l^\dagger$) is the boson creation operator in input (output) port k (l). The (photon number-resolving) photon detection projects the output state of photons onto one of the Fock states in the output ports. A realistic optical setup, however, suffers from photon losses and not all the photons are detected. Ways to compensate for losses, such as by the post selection on a given number of photons [30], are currently under investigation. For boson sampling one can allow only a constant number of bosons to be lost [44]. On the other hand, if we assume that photons are lost independently of each other, there will be, on average, a proportional to N number of lost photons. The classical algorithm of Ref. [52] can efficiently approximate the output distribution with the proportional losses. The post selection strategy therefore can work only for a small-size boson sampling, since the probability to lose a constant number of photons vanishes exponentially with N . Thus photon losses are unavoidable source of noise in an optical realization of boson sampling. The input-output relation in the case of arbitrary lossy interferometer $\mathcal{U}$, can be cast in the following form [66]

$$\hat{a}_k^\dagger = \sum_{l=1}^M \mathcal{U}_{kl} \hat{b}_l^\dagger + \sum_{j=1}^M V_{kj} \hat{c}_j^\dagger, \quad (2)$$

where $\hat{c}_j^\dagger$ is the boson creation operator in loss mode j (e.g., photon absorption due to non-unitarity of an interferometer), and matrix V is such that the total unitarity is observed $\mathcal{U}\mathcal{U}^\dagger + VV^\dagger = I$ (see more details in appendix B).

Another strong source of noise is partial distinguishability of photons, affecting the output state of photons on a linear interferometer [67]. Partial distinguishability can be associated with different

internal states of photons (such as temporal profile of a photon in case of Ref. [67]), which are unaffected by the input-output relation in Eq. (2) and not resolved in an experiment. In this case, the boson operators in the input-output relations of Eq. (2) refer to the same internal state of boson on the input as well as on the output of the interferometer. Partial distinguishability of N bosons can be accounted for by introducing a function on permutations $\sigma \in S_N$, acting on the internal states of bosons [42, 68]. This function is defined as follows. For pure internal states of bosons, $\hat{\rho}_k = |\psi_k\rangle\langle\psi_k|$, $k = 1, \dots, N$, it reads

$$J(\sigma) = \prod_{k=1}^N \langle\psi_{\sigma(k)}|\psi_k\rangle. \quad (3)$$

For mixed internal states $J(\sigma)$ is a convex sum of the products as in Eq. (3). Note that $J(\sigma)$ of Eq. (3) factorizes $J(\sigma) = J(\mu_1)J(\mu_2)\dots J(\mu_q)$ according to the disjoint cycle decomposition $\sigma = \mu_1\mu_2\dots\mu_q$ of permutation σ [69], where each cycle $\mu: k_1 \rightarrow k_2 \rightarrow \dots \rightarrow k_{|\mu|} \rightarrow k_1$ ($|\mu|$ denotes the cycle length) contributes a factor $J(\mu)$ given by a similar expression as the right hand side of Eq. (3) with σ replaced by μ . In general, each cycle-factor $J(\mu)$ accounts for a specific $|\mu|$ -boson interference process [70] and the factorization of J occurs when bosons are uncorrelated, i.e., when their internal state is factorized $\hat{\rho}_1 \otimes \dots \otimes \hat{\rho}_N$. For instance, in the famous 2-photon interference experiment [67], for the transposition of the photons (i.e., 2-cycle $1 \rightarrow 2 \rightarrow 1$) we have $J = |\langle\psi_1|\psi_2\rangle|^2$, with $|\psi_k\rangle$ being the temporal shape of photon k . This value enters the probability of the coincidence count at the output of a unitary linear 2-port: $p(1,1) = (1 - J)/2$, i.e., the Mandel dip in Ref. [67] (more on the partial distinguishability can be found in Ref. [68]). Explicit analytical results below are obtained for the uniform partial distinguishability, controlled by a single distinguishability parameter $0 \leq \xi \leq 1$. This model applies when boson k with probability ξ is in some pure internal state $|\phi_0\rangle$ and with probability $1 - \xi$ in an unique internal state $|\phi_k\rangle$, orthogonal to all other internal states. Either pure $\hat{\rho}_k^{(a)}$ or mixed $\hat{\rho}_k^{(b)}$ state of boson k may correspond to such a

case:

$$\begin{aligned}\hat{\rho}_k^{(a)} &= |\psi_k\rangle\langle\psi_k|, \quad |\psi_k\rangle = \sqrt{\xi}|\phi_0\rangle + \sqrt{1-\xi}|\phi_k\rangle, \\ \hat{\rho}_k^{(b)} &= \xi|\phi_0\rangle\langle\phi_0| + (1-\xi)|\phi_k\rangle\langle\phi_k|, \\ \langle\phi_k|\phi_j\rangle &= \delta_{kj}, \quad k, j \in \{0, 1, \dots, N\}.\end{aligned}\quad (4)$$

For bosons in internal states $\hat{\rho}_1^{(a,b)}, \dots, \hat{\rho}_N^{(a,b)}$ of Eq. (4), each μ -cycle for $|\mu| \geq 2$ contributes the factor $\xi^{|\mu|}$ to the distinguishability function in Eq. (3). The sum of all the cycle lengths in a permutation σ satisfies $\sum |\mu| = N - c_1(\sigma)$, where $c_1(\sigma)$ is the number of fixed points (1-cycles), hence $J(\sigma) = \xi^{N-c_1(\sigma)}$. If photon detectors do not resolve the internal states of photons, the same distinguishability function corresponds to pure-state or mixed-state model in Eq. (4). In an experiment, there is always noise in photon parameters, such as random fluctuations of photon arrival times, thus the internal states of photons

are always mixed. Whatever is the model of photon detection, the distinguishability of photons due to mixed states will persist, since even by completely resolving the internal states of photons one is not able to affect the distinguishability coming from the fluctuations of photon parameters [71]. Thus partial distinguishability of photons is a source of unavoidable noise in boson sampling experiments with photons.

We can now give the output distribution of our noisy boson sampling model with boson losses and partially distinguishable bosons. When N bosons, with the distinguishability function $J(\sigma)$, are sent to the input ports $k = 1, \dots, N$ of a lossy interferometer $\mathcal{U}$, the probability to detect exactly n of them, $0 \leq n \leq N$, at the output ports in a configuration $\mathbf{m} = (m_1, \dots, m_M)$, $m_1 + \dots + m_M = n$, i.e., when m_l bosons are detected at output port l , reads (see appendix B for details)

$$p(\mathbf{m}) = \frac{1}{\mathbf{m}!} \sum_{\sigma \in S_N} J(\sigma) \sum_{\mathbf{k}} \sum_{\tau \in S_n} \prod_{\alpha=1}^n \mathcal{U}_{\sigma\tau(k_\alpha), l_\alpha}^* \mathcal{U}_{\tau(k_\alpha), l_\alpha} \prod_{\alpha=n+1}^N (I - \mathcal{U}\mathcal{U}^\dagger)_{k_\alpha, \sigma(k_\alpha)}, \quad (5)$$

where the sum over $\mathbf{k} = (k_1, \dots, k_n)$ stands for summation over all n -dimensional subsets of $\{1, \dots, N\}$, the multi-set $l_1, \dots, l_n$ contains the output ports with bosons (one for each detected boson) and $\mathbf{m}! = m_1! \dots m_M!$.

When only $n < N$ bosons are detected at the output, there is still some interference between the $N - n$ lost bosons and the n detected ones, reflected in exchange of bosons between the output and the loss modes by permutation σ in Eq. (5). Only for a diagonal loss matrix $I - \mathcal{U}\mathcal{U}^\dagger$ no such boson exchange occurs (in this case $\sigma(k_\alpha) = k_\alpha$ for all $n+1 \leq \alpha \leq N$) and the residual interference disappears. In the latter case, boson losses can be considered to occur, e.g., at the input of the interferometer [50] and there is such $D = \text{diag}(\eta_1, \dots, \eta_M)$, $0 \leq \eta_k \leq 1$, that $\mathcal{U} = \sqrt{D}\mathcal{U}'$ for some $\mathcal{U}'\mathcal{U}'^\dagger = I$. Parameter η_k in this case is the probability that boson sent to input port k actually passes the interferometer (i.e., $1 - \eta_k$ is the probability that boson k is lost). Below, some of our explicit results are given for the uniform loss model with $\mathcal{U} = \sqrt{\eta}U$, $U^\dagger U = I$, where η is the uniform transmission of

such an interferometer.

Approximations by K interfering bosons and $N - K$ classical particles

In the no-collision regime, the output distribution of noisy N -boson sampling model can be approximated, to any small error, by a similar noisy model with bosonic particles, where there are only K interfering bosons and the rest $N - K$ particles are distinguishable bosons (i.e., classical particles) [48, 52, 53]. The parameter K is chosen according to the required approximation error. Below we adopt such a classical approximation as the classical adversary, see fig. 1.

The simplest way to introduce the above described classical approximation is via the respective distinguishability function $J^{(K)}(\sigma)$, which is equal to $J(\sigma)$ for permutations σ with at least $N - K$ fixed points and to zero otherwise [53]:

$$J^{(K)}(\sigma) \equiv \begin{cases} J(\sigma), & c_1(\sigma) \geq N - K, \\ 0, & c_1(\sigma) < N - K. \end{cases} \quad (6)$$

With such a distinguishability function, for $K >$

1, $N - K$ randomly chosen bosons contribute to the output probability in the same way as classical particles, whereas the rest K bosons are allowed to interfere. By setting $K = 1$ or $K = 0$ we get N classical particles (since a permutation with $N - 1$ fixed points is the identity permutation).

Our goal is to estimate the total variation distance between the output distributions of noisy boson sampling (p) and a classical approximation ($p^{(K)}$), where both distributions are given by Eq. (5) with the corresponding distinguishability functions $J(\sigma)$ and $J^{(K)}(\sigma)$. The total variation distance is defined as follows

$$\mathcal{D}(p, p^{(K)}) = \frac{1}{2} \sum_{\mathbf{m}} |p(\mathbf{m}) - p^{(K)}(\mathbf{m})|, \quad (7)$$

where the sum runs over all possible configurations $\mathbf{m} = (m_1, \dots, m_M)$ of bosons in the output ports. The key observation is that for any subset Ω of the configurations of bosons in the output ports we have

$$\mathcal{D}(p, p^{(K)}) \geq |P_\Omega - P_\Omega^{(K)}|, \quad P_\Omega \equiv \sum_{\mathbf{m} \in \Omega} p(\mathbf{m}). \quad (8)$$

Observe that the equality is necessarily achieved for a certain subset Ω_* depending on $\mathcal{U}$ and other parameters of the setup.

Analytical analysis below is carried out for the difference in probability, $\Delta P_L = P_{\Omega_L} - P_{\Omega_L}^{(K)}$, of all the output bosons to be detected in a subset of $M - L$ output ports, or, equivalently, no particle counts in the complementary subset of the output ports $\Omega_L \equiv \{l_1, \dots, l_L\}$. Let us assume that such a subset is chosen once for a given setup (below we will return to this point in more detail). Since we consider arbitrary (or randomly chosen multiports) we will use $\Omega_L = \{1, \dots, L\}$ in our analytical and numerical considerations below.

At this stage, the dark counts of detectors can be easily accounted for. Dark counts of a detector follow Poisson distribution $p_d(n) = \frac{\nu^n}{n!} e^{-\nu}$, where we assume a uniform rate ν for all detectors. Hence, the dark counts contribute the factor $e^{-L\nu}$ to the the probability of no counts in L output ports. Denoting $\Delta J = J - J^{(K)}$ we get from Eqs. (5)-(6) the following expression for the difference in probability of no counts in L output

ports (see details in appendices A and B)

$$\Delta P_L = e^{-L\nu} \sum_{\sigma \in S_N} \Delta J(\sigma) \prod_{k=1}^N A_{k, \sigma(k)}, \quad (9)$$

$$A_{kj} \equiv \delta_{kj} - \sum_{l \in \Omega_L} \mathcal{U}_{kl} \mathcal{U}_{jl}^*.$$

In the limit of large number of interfering bosons $N \gg 1$, the lower bound on the total variation distance between a noisy boson sampling and the classical approximation by K interfering bosons, Eq. (8), depends on the density of bosons $\rho = N/M$, and not on the number of interfering bosons N and the size M of interferometer themselves. We will prove this in the case of interferometer with uniform transmission η , $\mathcal{U} = \sqrt{\eta}U$, $U^\dagger U = I$, and uniform distinguishability ξ , i.e., with for the distinguishability function $J(\sigma) = \xi^{N-c_1(\sigma)}$. For $L = 1$ the difference in probability becomes

$$\Delta P_1 = e^{-\nu} \sum_{n=K+1}^N (-\eta)^n \sum_{\mathbf{k}} \prod_{\alpha=1}^n |U_{k_\alpha, 1}|^2$$

$$\times \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m,$$

$$d_m \equiv m! \sum_{s=0}^m \frac{(-1)^s}{s!}, \quad (10)$$

where the summation over $\mathbf{k} = (k_1, \dots, k_n)$ runs over all n -dimensional subsets of $1, \dots, N$ (the derivation can be found in appendix C). In Eq. (10) the first two sums are due to expansion of the loss matrix $A_{kj} = \delta_{kj} - \eta U_{k1} U_{j1}^*$ of Eq. (9) in powers of η , while the third sum gives the partial distinguishability of the n -boson interference: there are $\binom{n}{m} d_m$ amplitudes of m -boson interferences [71], weighted by ξ^m , where d_m is the total number of m -dimensional derangements (permutations with no fixed points) [69] and $\binom{n}{m}$ is the number of m dimensional subsets of n bosons. Such interferences up to the order $m \leq K$ are accounted for by our classical model, hence the summations in Eq. (10) start from $K + 1$.

Eq. (10) and the known average [72] over the Haar-random interferometer U ,

$$\langle |U_{k_1, 1}|^2 \dots |U_{k_n, 1}|^2 \rangle = \frac{1}{M^{(n)}},$$

where $M^{(n)} = M(M+1) \dots (M+n-1)$, allow

us to easily find the average value

$$\langle \Delta P_1 \rangle = e^{-\nu} \sum_{n=K+1}^N \binom{N}{n} \frac{(-\eta)^n}{M^{(n)}} \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m. \quad (11)$$

Similarly, one can derive an expression for the variance $\langle (\Delta P_1)^2 \rangle - \langle \Delta P_1 \rangle^2$ (see appendix C).

For $K \ll \sqrt{N}$ the difference in probability in Eq. (11) becomes a function of K , the amplitudes of noise, ξ, η, ν and the density of bosons $\rho = N/M$. To extract the dependence on the setup parameters from Eq. (11), we use that $d_{K+i} = (K+i)! (e^{-1} + R_{K+i})$ with $|R_m| < 1/(m+1)!$ (see appendix F) and, assuming that K is sufficiently large, drop the remainder R_{K+i} . By expanding the resulting expression in powers of K^2/N and retaining only the leading order terms we obtain (see appendix C):

$$|\langle \Delta P_1 \rangle| \approx \frac{(\xi\eta\rho)^{K+1}}{1 + \xi\eta\rho} e^{-1-\nu-\eta\rho} \equiv W_1, \quad (12)$$

$$\frac{\langle (\Delta P_1)^2 \rangle - \langle \Delta P_1 \rangle^2}{W_1^2} \approx \frac{(1-\rho)(K+1)^2}{N}.$$

Furthermore, for the class of interferometers having one balanced output port $|U_{k,l}| = \frac{1}{\sqrt{M}}$ one can show that the difference in probability of no counts in the balanced output port satisfies

$$|\Delta P_1| \geq W_1 \left(1 - \left| O \left(\frac{K^2}{N} \right) \right| \right), \quad (13)$$

where the minus sign indicates a negative correction (see details in appendix C). Such interferometers contain a wide class: $U = \mathcal{F}(1 \oplus V)$ with the Fourier interferometer $\mathcal{F}_{kl} = \frac{1}{\sqrt{M}} e^{2i\pi \frac{kl}{M}}$ and an arbitrary $(M-1)$ -dimensional unitary interferometer V .

By considering more than one output port for the same setup one can maximize $|\langle \Delta P_L \rangle|$ as a function of L (i.e., the lower bound on the total variation distance can be optimized). Indeed, for noiseless boson sampling, $\xi = \eta = 1$ and $\nu = 0$, for classical particles ($K = 1$) and $L \ll M$ we have $\langle \Delta P_L \rangle = (1 + \rho)^{-L} - e^{-L\rho}$ [73], where the average is over the Haar-random unitary interferometer, maximized when $L = \text{int}(1/\rho) = \text{int}(M/N)$.

The analytical results of Eqs. (10)-(13) were verified by numerical simulations of Eq. (9) for the difference in probability of no counts in $L \geq 1$ output ports (the numerical approach is

Figure 2: The lower bound on the total variation distance. Uniform boson losses, with the transmission $\eta = 0.8$, serve as noise. In panel (a) the classical simulation is by distinguishable bosons ($K = 1$), whereas in panel (b) by the third-order multi-boson interferences ($K = 3$). In panel (a) and (b) the solid lines give the asymptotic analytical estimate W_1 , the squares ($N = 12$) and circles ($N = 24$) give the difference in probability of no counts in a single output port, obtained by statistical averaging over random multiports. In panel (a) the dashed line gives the difference in probability of no counts in a balanced output port and the hexagons give the statistical average of the difference in probability of no counts in $L = \text{int}(M/N)$ output ports. In panel (b) the dotted lines compare the analytically obtained average with the statistical average for the same N , the dashed line gives the analytical average for $N = 48$, and the dash-dotted that for $N = 84$.

described in appendix E). The results are plotted in fig. 2. In the simulations the uniform boson losses with the transmission $\eta = 0.8$, $\mathcal{U} = \sqrt{\eta}U$, is the only source of noise (i.e., bosons are com-

pletely indistinguishable, $\xi = 1$, and there no dark counts of detectors, $\nu = 0$). Averaging over the unitary matrix U by selecting it uniformly randomly from the unitary group we obtain what is called in fig. 2 the “statistical average”. Our method allows such averaging only for small number of bosons $N \leq 24$, due to necessity of computing a large number of N -dimensional matrix permanents. Therefore, for $N \leq 24$ and $L = 1$ we check against the statistical average the analytical expression of Eq. (11), called in fig. 2 the “analytical average”, which is then used for larger numbers of bosons.

The numerical results confirm the scale-invariance predicted by Eq. (12), when the applicability condition $K \ll \sqrt{N}$ is satisfied: in fig. 2(a) (where $K = 1$) the statistical average values $|\langle \Delta P_1 \rangle|$ with $N = 12$ and $N = 24$ interfering bosons are almost indiscernible, whereas in fig. 2(b) ($K = 3$) the numerical results gradually approach our asymptotic estimate W_1 as the number of interfering bosons N scales up from $N = 12$ to $N = 84$. Similar convergence of $|\langle \Delta P_1 \rangle|$ to W_1 as N scales up was observed numerically for other values $K > 3$. The opening gap between W_1 and $|\langle \Delta P_1 \rangle|$ in fig. 2(a) results from the approximation for the number of derangements $d_{K+i} \approx e^{-1}(K+i)!$ (see Eq. (10)) used in Eq. (12), not justified for $K = 1$.

Finally, our results agree with the *upper bound* on the total variation distance in the no-collision regime. A noisy boson sampling setup of arbitrary large size $N \gg 1$ can be approximated in the no-collision regime by the distribution $p^{(K)}$ [48, 52, 53] to the total variation distance $\mathcal{D}(p, p^{(K)}) = O((\xi\sqrt{\eta})^{K+1})$. Whereas the upper bound is independent of N in this regime, our asymptotic *lower bound* W_1 of Eq. (12) vanishes due to dependence on the density of bosons $W_1 \sim \rho^{K+1}$, since in the no-collision regime $\rho \ll 1/N$. No upper bound is known beyond the no-collision regime.

The set of efficient distinguishers

The above results not only show that one *can* efficiently distinguish the output distribution of a noisy boson sampling device from that of a classical approximation, such as those used in Refs. [48, 52, 53], but provides a large set of such distinguishers. The probability P_L of detecting zero counts in L output ports, used in Eq. (9), is

such a distinguisher. Note that there are $\binom{M}{L}$ of ways to select a subset Ω_L with L ports from the total M of them. There is enough subsets Ω_L (with different L and Ω_L) with independent probabilities $P_L = P_L(\Omega_L)$ ¹ for inversion of the relation $P_L = \sum_{\Omega_L} p(\mathbf{m})$, where $p(\mathbf{m})$ is the output distribution of boson sampling, thus there is a strong correlation between the set of probabilities $P_L(\Omega_L)$ and the output distribution.

For uniform distinguishability $J(\sigma) = \xi^{N-c_1(\sigma)}$, uniform detector dark counts rate ν , and arbitrary lossy interferometer $\mathcal{U}$ we get P_L as a single matrix permanent of N -dimensional positive-semidefinite Hermitian matrix

$$P_L = e^{-L\nu} \text{per} A(\xi), \quad A_{kj}(\xi) \equiv \begin{cases} A_{kk}, j = k, \\ \xi A_{kj}, j \neq k, \end{cases} \quad (14)$$

where A is from Eq. (9). Such permanents can be efficiently approximated by a quantum-optics inspired algorithm [74, 75].

There is also an analytical expression for the average probability of no counts P_1 in a single output port of our noisy boson sampling with uniform distinguishability, uniform losses and dark counts (details in appendix D),

$$\langle P_1 \rangle = e^{-\nu} \sum_{n=0}^N \frac{(N)_n}{M^{(n)}} (-\eta)^n \sum_{s=0}^n \frac{\xi^{n-s} (1-\xi)^s}{s!}, \quad (15)$$

where $(N)_n = N(N-1)\dots(N-n+1)$. Moreover, for $N \gg 1$ and $\rho\eta \ll 1$ one can approximate (see appendix D):

$$\langle P_1 \rangle \approx \frac{\exp(-\nu - \rho\eta[1-\xi])}{1 + \xi\rho\eta}, \quad (16)$$

$$\frac{\langle (P_1 - \langle P_1 \rangle)^2 \rangle}{\langle P_1 \rangle^2} \approx \frac{(\eta\rho)^2(1-\rho)}{N} \left(1 - \xi + \frac{\xi}{1 + \rho\eta\xi} \right)^2.$$

In an experiment, there is no need to actually compute P_L : one has only to choose L output ports and estimate P_L from the number of output data with zero counts in them. Let us analyze an example. According to the standard estimation theory (e.g., Ref. [76]), the probability P_1 can be estimated, with α -confidence level, as $P_1^{(E)} = (T_S + z_\alpha^2/2)/(T + z_\alpha^2)$, where T is the

¹For example, one sufficient number of subsets is obtained by fixing $L < M - N$ and varying Ω_L . Note, however, that one cannot sample from the output distribution $p(\mathbf{m})$ using the sufficient set of P_{Ω_L} .

total number of samples, T_S is the number of successes (samples with zero counts in the selected output port) and z_α is the quantile of the standard normal distribution (e.g., for $\alpha = 0.05$, 95%-confidence, $z_\alpha^2 \approx 4$). For

$$T = P_1(1 - P_1) \left(\frac{1 - \alpha/2}{\epsilon W_1} \right)^2$$

samples the relative confidence interval for P_1 becomes $\frac{|P_1 - P_1^{(E)}|}{W_1} = \epsilon$. Setting $\epsilon = \alpha$, with the same confidence level as that of the estimate on P_1 , by experimentally estimating P_1 one can tell the output of the quantum device from that of the classical simulation. For $K = O(1)$ and bounded amplitudes of noise, the number of samples sufficient for telling apart the two outputs depends on N only through the density of bosons $\rho = N/M$ (see Eq. (12)). Hence, for any scale $M \sim N^\kappa$, $\kappa \geq 1$, it is polynomial in N , which by our definition means efficiency. Interestingly, the required number of samples remains bounded as N scales up if density of bosons remains finite, i.e., for $M \sim N$.

Approximations accounting for *all* K -boson interferences

One important observation on the choice of our classical model, Eq. (6), is in order. The highest order of quantum multiboson interference accounted by the classical approximation in Eq. (6) is obviously K . However, by allowing only K bosons to interfere, such a model does not account for the multi-boson interference where groups of up to K bosons interfere between themselves. Such an interference corresponds to a subset of permutations in the permutation group, which decompose into the disjoint cycles of length up to K . Thus, if one wants to take into account *all* the multiboson interferences up to the K th order, one has to use a different model [70], say $J_+^{(K)}$, obtained by setting to zero the distinguishability function $J(\sigma)$ on all permutations σ having cycles of length $n \geq K + 1$ in the disjoint cycle decomposition:

$$J_+^{(K)}(\sigma) \equiv \begin{cases} J(\sigma), & c_{K+i}(\sigma) = 0, \forall i \geq 1, \\ 0, & \exists c_{K+i}(\sigma) > 0, \end{cases} \quad (17)$$

where $c_n(\sigma)$ is the number of cycles of length n in the cycle decomposition.

However, switching to the model of Eq. (17), brings insignificant changes to the results, e.g., changes only the K -independent factor in W_1 of Eq. (12) (see details in appendix F). Our model with $J^{(K)}$ of Eq. (6) simplifies the calculations and allows for a simple numerical algorithm (see appendix E), whereas producing essentially the same lower bound on the total variation distance as the model in Eq. (17). This fact leads to two important conclusions: (i) our approach can distinguish the output distribution of a noisy boson sampling device from that of an approximation even if the latter accounts for all multiboson interferences up to order $K \ll \sqrt{N}$ and (ii) the probability P_L (14) essentially depends on multiboson interferences of the orders above K . These conclusions are very important in discussion of the assessments methods of boson sampling by verifying only some low-order correlations at the output distribution, considered below.

Tests based on low-order correlators

The second-order correlations in output distribution were previously used for assessment of boson sampling [57, 58]. Current experiments on boson sampling have reached a stage when the output probability space is so large that direct verification by comparison with the output probabilities $p(\mathbf{m})$ Eq. (5) is out reach (both computationally and due to excessively large number of samples for such an assessment). Only the low-order correlations are therefore checked, such as in the recent benchmark demonstration [32], where only up to 4th-order correlations have been verified. Such and other similar assessments are, however, insufficient, since some approximations using a smaller number of interfering bosons, such those of Refs. [48, 52, 53], can pass them.

Let us consider a whole class of tests based on r th-order correlations, e.g.,

$$C_r = \langle \prod_{l=1}^r \hat{b}_l^\dagger \hat{b}_l \rangle - \prod_{l=1}^r \langle \hat{b}_l^\dagger \hat{b}_l \rangle, \quad (18)$$

where $r = O(1)$. In the ideal case of boson sampling with no noise, for $N \gg 1$ the Haar measure, $\text{Prob}(\dots)$, of unitary interferometers where all the occupations $m_1, \dots, m_M$ of the output ports remain bounded by some number s reads [11]

$$\text{Prob}(\max(m_l) \leq s) \approx \left[1 - \left(\frac{\rho}{1 + \rho} \right)^{s+1} \right]^M. \quad (19)$$

Fixing $\delta \ll 1$, one can therefore state that in a randomly selected unitary interferometer, with high probability $\text{Prob} \approx 1 - \delta$, the maximal boson bunching count at the output reads $s = \ln\left(\frac{N}{\rho\delta}\right) / \ln\left(\frac{1+\rho}{\rho}\right)$. With the same probability, the number of bosons $\mathcal{K}$ detected in r output ports of the ideal boson sampling scales at most as $\mathcal{K} = O(rs) = O(r \ln N)$. This means that with $1 - \delta$ chances any test based on the r -order correlations will not succeed to tell the output distribution of even the ideal boson sampling from that of our classical model with $K = O(r \ln N)$. Indeed, the probability to detect up to K bosons in r output ports depends on multiboson interferences only up to the order K [70]. Due to this fact and the above estimate on the number of bosons detected in r output ports, our model of Eq. (17) with $K = O(r \ln N)$ would faithfully account for all the correlations of order r , e.g., such as in Eq. (18), passing any test based on them. In contrast, the probability of no counts in a subset of output ports allows to distinguish the output distribution produced by such a classical simulation from that of the boson sampling, with or without noise, since the applicability condition of our results, such as in Eq. (12), is satisfied: $K = O(r \ln N) \ll \sqrt{N}$ as N scales up.

3 Discussion of the results

In the spirit of Ref. [1], we have asked if a quantum system realizing imperfect/noisy boson sampling [4] can be efficiently and faithfully simulated classically as the system size scales up. To this goal, we have investigated whether it is possible to efficiently distinguish the output distribution of a noisy realization of boson sampling from that of the classical approximations that take into account only low-order multiboson interferences, where the term “low-order” means an order $K \ll \sqrt{N}$, where N is the total number of interfering bosons. Our choice of the classical approximations was dictated by the recent Refs. [48, 52, 53], which can approximate the output distribution of boson sampling with any finite noise amplitudes and to any given error in the total variation distance by adapting the order K of the accounted multiboson interference.

Our main result is that one can (and we point exactly how) efficiently distinguish noisy boson sampling with finite amplitudes of noise from the

classical approximations accounting for multiboson interferences up to an order $K = O(1)$. The latter class of approximation coincides with the efficient classical approximations of Refs. [48, 52, 53].

It is also found that the required number of runs of a noisy boson sampling for the purpose of distinguishing its output distribution from the considered class of approximations shows critical dependence on the scaling with N of the density of bosons $\rho = N/M$, performing a transition from a polynomial in N number of runs for the vanishing density of bosons, e.g., in the no-collision regime [4] (when $\rho \sim 1/N$ at the least), to an N -independent number of runs for a finite density of bosons.

Our results pose some questions. In the no-collision regime, the lower bound on the total variation distance between the output distribution of noisy boson sampling and that of the approximation by low-order multiboson interferences vanishes when the number of bosons scales up. On the other hand, in Refs. [48, 52, 53] an upper bound on the total variation distance was found, which does not vanish with the total number of bosons. Are there better efficient classical approximations to the noisy boson sampling, which account for the multi-boson interferences to the same low order and narrow the gap between the upper and lower bounds? Is it possible to find a better way to distinguish the two output distributions in the no-collision regime, i.e., requiring much smaller number of samples? These and related questions are left for the future work.

We conclude by noting that the critical dependence on the density of particles, and not on the number of particles themselves, is strikingly reminiscent of the quantum-to-classical transition in systems of identical particles, occurring when the total number of particles scales up whereas the density of particles vanishes. Such a transition results in the mean-field approximation to a large system of identical bosons, the most prominent example being the Bose-Einstein condensation of weakly interacting bosons [77], experimentally achieved with dilute gases [78, 79] and approximated by the mean-field order parameter obeying the Gross-Pitaevskii equation [80, 81].

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A Lower bound for uniformly lossy interferometer

Here we derive the lower bound on the total variation distance between the output distribution of N -boson sampling, affected by boson losses, partial distinguishability of bosons and dark counts of detectors, and that of the classical approximation by K interfering bosons and $N - K$ classical particles (distinguishable bosons), below called the K -reduced model. Here we consider the case of uniformly lossy interferometer $\mathcal{U} = \sqrt{\eta}U$, $UU^\dagger = I$, with the transmission $0 < \eta \leq 1$, whereas bosons can be in an arbitrary state of partial distinguishability described by a function $J(\sigma)$ on permutations $\sigma \in S_N$, defined in the main text.

Let us fix the notations. We will use $\mathcal{U}[k_1, \dots, k_n | l_1, \dots, l_n]$ for the submatrix of $\mathcal{U}$ on the rows $k_1, \dots, k_n$ (input ports with a boson) and a multi-set of columns $l_1, \dots, l_n$ (the output ports with bosons) corresponding to occupations $\mathbf{m} = (m_1, \dots, m_M)$, $|\mathbf{m}| \equiv m_1 + \dots + m_M = n$ (referred below as the output configuration).

Let us first consider the case of no dark counts of detectors and give the probability of detecting n bosons at the output of a lossy boson sampling device. For uniform losses, we can assume boson losses to occur at the input [50], hence, with the probability $\eta^n(1 - \eta)^{N-n}$ only n out of N bosons are sent through interferometer U . Then the probability $p_0(\mathbf{m})$ to detect n bosons in an output configuration $\mathbf{m}$ reads [42, 68]

$$p_0(\mathbf{m}) = \eta^n(1 - \eta)^{N-n} \sum_{\mathbf{k}} p(\mathbf{l}|\mathbf{k}), \quad (20)$$

$$p(\mathbf{l}|\mathbf{k}) \equiv \frac{1}{\mathbf{m}!} \sum_{\sigma_{1,2} \in S_n} J_{\mathbf{k}}(\sigma_1 \sigma_2^{-1}) \prod_{\alpha=1}^n U_{k_{\sigma_1(\alpha)}, l_\alpha}^* U_{k_{\sigma_2(\alpha)}, l_\alpha},$$

where $\mathbf{k} = (k_1, \dots, k_n)$, a subset of the input ports $1, \dots, N$, from which the detected bosons

originate, $p(\mathbf{l}|\mathbf{k})$ is the probability of n partially distinguishable bosons from input ports $\mathbf{k}$, whose state of distinguishability is described by the distinguishability function $J_{\mathbf{k}}(\sigma)$, the output ports $\mathbf{l} = (l_1, \dots, l_n)$ correspond to the configuration $\mathbf{m}$, and the summation runs over the permutations $\sigma_{1,2} \in S_n$ of n bosons in the input ports $\mathbf{k}$.

Now let us describe the effect of the dark counts of detectors. Assume that there are $\mathcal{N}$ total detector clicks in an output configuration $\mathbf{s} = (s_1, \dots, s_M)$, thus additionally to n detected bosons there are $\mathcal{N} - n$ dark counts (generally, multiple) corresponding to an output configuration $\mathbf{r} = (r_1, \dots, r_M)$, $|\mathbf{r}| = \mathcal{N} - n$ and $\mathbf{r} \subset \mathbf{s}$ (meaning that $r_l \leq s_l$ for all $l = 1, \dots, M$). The corresponding probability $p(\mathbf{s})$ becomes

$$p(\mathbf{s}) = \sum_{n=0}^{\mathcal{N}} \eta^n(1 - \eta)^{N-n} \sum_{\substack{\mathbf{r} \subset \mathbf{s} \\ |\mathbf{r}| = \mathcal{N} - n}} \frac{\nu^{\mathcal{N} - n}}{\mathbf{r}!} e^{-M\nu} \sum_{\mathbf{k}} p(\mathbf{l}|\mathbf{k}), \quad (21)$$

where the output ports $\mathbf{l} = (l_1, \dots, l_n)$ correspond to the output configuration $\mathbf{m} = \mathbf{s} - \mathbf{r}$, $|\mathbf{m}| = n$. The expression of Eq. (21) is complicated, however, as we need to sum the probabilities in Eq. (21) corresponding to no counts in a subset of output ports, the effect of dark counts will be given by a simple factor.

We consider how close is the output probability distribution of Eq. (21) to that of the K -reduced model obtained by replacing $N - K$ partially distinguishable bosons (from randomly selected inputs) by completely distinguishable ones (i.e., classical particles). As explained in the main text, this model corresponds to the probability distribution $p^{(K)}(\mathbf{m})$ similar to that of Eq. (21) but with the following distinguishability function

$$J^{(K)}(\sigma) \equiv \begin{cases} J(\sigma), & c_1(\sigma) \geq N - K, \\ 0, & c_1(\sigma) < N - K. \end{cases} \quad (22)$$

Consider a lower bound on the total variation distance $\mathcal{D} = \mathcal{D}(p, p^{(K)})$ between our noisy boson sampling and its K -reduced model. We have $\mathcal{D} = \sum_n \mathcal{D}_n$, where $\mathcal{D}_n$ corresponds to exactly n detected bosons at the output. The lower bound is given by the absolute value of the difference in probability to detect all the output bosons in a fixed subset of output ports, below set to be $L + 1 \leq l \leq M$, i.e., no counts in output ports $\Omega_L = \{1, \dots, L\}$. In this case, the effect of dark counts can be accounted in a very simple manner.

Being independent of the input state of bosons, they contribute the factor $e^{-L\nu}$ to the probability of zero counts in a subset of L output ports. Therefore, we have

$$\begin{aligned} \mathcal{D} &= \sum_{n=0}^N \mathcal{D}_n \geq |P_L - P_L^{(K)}|, \\ P_L &\equiv e^{-\nu L} \sum_{n=0}^N \sum_{|\mathbf{m}|=n} ' p_0(\mathbf{m}), \\ P_L^{(K)} &\equiv e^{-\nu L} \sum_{n=0}^N \sum_{|\mathbf{m}|=n} ' p_0^{(K)}(\mathbf{m}), \end{aligned} \quad (23)$$

where the summations with prime run over all output configurations $\mathbf{m} = (0, \dots, 0, m_{L+1}, \dots, m_M)$ with no counts in the first L output ports. Denoting $\Delta P_L = P_L - P_L^{(K)}$ (and similar for other quantities of the two models, e.g. $\Delta J = J - J^{(K)}$, etc) and using Eq. (20) after some algebra we obtain

$$\begin{aligned} \Delta P_L &= e^{-\nu L} \sum_{n=0}^N \eta^n (1 - \eta)^{N-n} \sum_{\mathbf{k}} \sum_{|\mathbf{m}|=n} ' \Delta p(\mathbf{l}|\mathbf{k}) \\ &= e^{-\nu L} \sum_{n=0}^N \eta^n (1 - \eta)^{N-n} \sum_{\mathbf{k}} \sum_{\sigma \in S_n} \Delta J_{\mathbf{k}}(\sigma) \prod_{\alpha=1}^n H_{k_{\alpha}, k_{\sigma(\alpha)}} \\ &= e^{-\nu L} \sum_{\tau \in S_N} \Delta J(\tau) \prod_{k=1}^N A_{k, \tau(k)}, \end{aligned} \quad (24)$$

where H and A are N -dimensional positive semidefinite Hermitian matrices, defined as follows:

$$\begin{aligned} H_{kj} &= \sum_{l=L+1}^M U_{kl} U_{jl}^*, \\ A_{kj} &= (1 - \eta) \delta_{kj} + \eta H_{kj} = \delta_{kj} - \eta \sum_{l=1}^L U_{kl} U_{jl}^*. \end{aligned} \quad (25)$$

To arrive at the result in Eqs. (24) and (25) we have performed the following steps. We have used an identity for the sum of output probabil-

ities [56, 73]

$$\begin{aligned} \sum_{|\mathbf{m}|=n} ' \Delta p(\mathbf{l}|\mathbf{k}) &= \sum_{l_1=L+1}^M \dots \sum_{l_n=L+1}^M \frac{\mathbf{m}!}{n!} \Delta p(\mathbf{l}|\mathbf{k}) \\ &= \frac{1}{n!} \sum_{\sigma_{1,2} \in S_n} \Delta J_{\mathbf{k}}(\sigma_1 \sigma_2^{-1}) \prod_{\alpha=1}^n H_{k_{\sigma_2(\alpha)}, k_{\sigma_1(\alpha)}} \\ &= \frac{1}{n!} \sum_{\sigma_{1,2} \in S_n} \Delta J_{\mathbf{k}}(\sigma_1 \sigma_2^{-1}) \prod_{\alpha=1}^n H_{k_{\alpha}, k_{\sigma_1 \sigma_2^{-1}(\alpha)}} \\ &= \sum_{\sigma \in S_n} \Delta J_{\mathbf{k}}(\sigma) \prod_{\alpha=1}^n H_{k_{\alpha}, k_{\sigma(\alpha)}}, \end{aligned}$$

where $\sigma \equiv \sigma_1 \sigma_2^{-1}$, the following identity

$$\prod_{\alpha=1}^n H_{k_{\sigma_2(\alpha)}, k_{\sigma_1(\alpha)}} = \prod_{\alpha=1}^n H_{k_{\alpha}, k_{\sigma_1 \sigma_2^{-1}(\alpha)}}, \quad (26)$$

replaced the sum over all output configurations $\mathbf{m}$ by n independent sums over output ports $l_1, \dots, l_n$, using that for any symmetric function $f(l_1, \dots, l_n)$,

$$\sum_{|\mathbf{m}|=n} f(l_1, \dots, l_n) = \sum_{l_1=L+1}^M \dots \sum_{l_n=L+1}^M \frac{\mathbf{m}!}{n!} f(l_1, \dots, l_n), \quad (27)$$

and observed that for any permutation $\tau \in S_N$

$$\begin{aligned} \prod_{k=1}^N A_{k, \tau(k)} &= \sum_{n=0}^N \eta^n (1 - \eta)^{N-n} \\ &\times \sum_{\mathbf{k}} \prod_{\alpha=1}^n H_{k_{\alpha}, \tau(k_{\alpha})} \prod_{\alpha=n+1}^N \delta_{k_{\alpha}, \tau(k_{\alpha})} \\ &= \sum_{n=0}^N \eta^n (1 - \eta)^{N-n} \sum_{\mathbf{k}} \prod_{\alpha=1}^n H_{k_{\alpha}, k_{\sigma(\alpha)}}, \end{aligned} \quad (28)$$

where $\sigma \in S_n$, acting on the subindex in k_{α} , is defined by the relation $k_{\sigma(\alpha)} = \tau(k_{\alpha})$ (we can introduce such σ , since τ permutes elements in $\mathbf{k}$ between themselves).

To derive Eq. (24) we have used here that the interferometer has uniform transmission η . However, as proven below, the resulting expression applies for an arbitrary (non-uniformly) lossy interferometer $\mathcal{U}$.

B Lower bound for arbitrary lossy interferometer

Recall that an arbitrary lossy linear interferometer with M input ports can be imbedded into a

unitary one with $2M$ input ports [66] with introduction of M auxiliary boson modes describing losses. One such $2M$ -dimensional unitary interferometer reads [56]

$$\hat{\mathcal{U}} = \begin{pmatrix} \mathcal{U} & V \\ -V^\dagger F & \sqrt{D} \end{pmatrix}, \quad (29)$$

where $\mathcal{U} = \sqrt{B}F$, with $B = \mathcal{U}\mathcal{U}^\dagger$ and a unitary matrix F , V is such that $VV^\dagger = I - \mathcal{U}\mathcal{U}^\dagger$, and $D = \text{diag}(\eta_1, \dots, \eta_M)$ with η_l being eigenvalue of B (a singular value of $\mathcal{U}$). Bosons at the output ports $M+1 \leq l \leq 2M$ correspond to losses. Below we will only need to consider the $M \times 2M$ -dimensional block $(\mathcal{U}, V)$. For zero dark counts, the probability that n out of input N partially distinguishable bosons are detected at the output of a lossy interferometer $\mathcal{U}$ in an

output configuration $\mathbf{m} = (m_1, \dots, m_M)$ can be obtained by summation of the general formula for output probability $p(\mathbf{m})$ in the unitary interferometer $\hat{\mathcal{U}}$ [68], similar as in Eq. (20), over all possible output configurations $\mathbf{r} = (r_1, \dots, r_M)$, $|\mathbf{r}| = N - n$, of bosons in the output ports $M+1 \leq l \leq 2M$. Since the probability is symmetric in the input/output ports, we designate output ports $l_1, \dots, l_n$ to be the detected bosons, i.e., $1 \leq l_k \leq M$, $1 \leq k \leq n$. Using a summation identity similar to that of Eq. (27),

$$\begin{aligned} & \sum_{|\mathbf{r}|=N-n} f(l_1, \dots, l_N) \\ &= \sum_{l_{n+1}=M+1}^{2M} \dots \sum_{l_N=M+1}^{2M} \frac{\mathbf{r}!}{(N-n)!} f(l_1, \dots, l_N), \end{aligned} \quad (30)$$

and recalling that $VV^\dagger = I - \mathcal{U}\mathcal{U}^\dagger$ we obtain

$$\begin{aligned} p(\mathbf{m}) &= \sum_{|\mathbf{r}|=N-n} \frac{1}{\mathbf{m}!\mathbf{r}!} \sum_{\sigma_{1,2} \in S_N} J(\sigma_1\sigma_2^{-1}) \left[\prod_{k=1}^n \mathcal{U}_{\sigma_1(k),l_k}^* \mathcal{U}_{\sigma_2(k),l_k} \right] \prod_{k=n+1}^N V_{\sigma_1(k),l_k}^* V_{\sigma_2(k),l_k} \\ &= \frac{1}{(N-n)!\mathbf{m}!} \sum_{\sigma_{1,2} \in S_N} J(\sigma_1\sigma_2^{-1}) \left[\prod_{k=1}^n \mathcal{U}_{\sigma_1(k),l_k}^* \mathcal{U}_{\sigma_2(k),l_k} \right] \prod_{k=n+1}^N (I - \mathcal{U}\mathcal{U}^\dagger)_{\sigma_2(k),\sigma_1(k)} \\ &= \frac{1}{\mathbf{m}!} \sum_{\sigma_R \in S_N} J(\sigma_R) \sum_{\tau \in S_n} \sum_{\mathbf{k}} \left[\prod_{\alpha=1}^n \mathcal{U}_{\sigma_R\tau(k_\alpha),l_\alpha}^* \mathcal{U}_{\tau(k_\alpha),l_\alpha} \right] \prod_{\alpha=n+1}^N (I - \mathcal{U}\mathcal{U}^\dagger)_{k_\alpha,\sigma_R(k_\alpha)}, \end{aligned} \quad (31)$$

where we have introduced subsets $\mathbf{k} = (k_1, \dots, k_n)$ of the input ports of $1, \dots, N$, where $k_\alpha \equiv \sigma_2(\alpha)$, used an identity similar to that of Eq. (26) to recast the summation over $\sigma_2 \in S_N$ as that over the subsets $\mathbf{k}$ and permutations $\tau \in S_n$ in each such subset, and introduced the relative permutation $\sigma_R = \sigma_1\sigma_2^{-1}$. Note that the r.h.s. of Eq. (31) does not depend on the el-

ements of the auxiliary unitary matrix $\hat{\mathcal{U}}$ other than $\mathcal{U}$.

Consider now the probability $P_L(n)$ of no counts in L output ports $l = 1, \dots, L$ with exactly n bosons detected at the output of $\mathcal{U}$. For zero dark counts, by using the summation identity of Eq. (27) we get

$$\begin{aligned} P_L(n) &= \sum_{l_1=L+1}^M \dots \sum_{l_n=L+1}^M \frac{\mathbf{m}!}{n!} p(\mathbf{m}) = \frac{1}{n!} \sum_{\sigma \in S_N} J(\sigma) \sum_{\tau \in S_n} \sum_{\mathbf{k}} \left[\prod_{\alpha=1}^n \mathcal{H}_{\sigma\tau(k_\alpha),\tau(k_\alpha)} \right] \prod_{\alpha=n+1}^N (I - \mathcal{U}\mathcal{U}^\dagger)_{\sigma(k_\alpha),k_\alpha} \\ &= \sum_{\sigma \in S_N} J(\sigma) \sum_{\mathbf{k}} \left[\prod_{\alpha=1}^n \mathcal{H}_{\sigma(k_\alpha),k_\alpha} \right] \prod_{\alpha=n+1}^N (I - \mathcal{U}\mathcal{U}^\dagger)_{\sigma(k_\alpha),k_\alpha}, \end{aligned} \quad (32)$$

where we have introduced a positive semi-definite Hermitian matrix

$$\mathcal{H}_{kj} = \sum_{l=L+1}^M \mathcal{U}_{kl} \mathcal{U}_{jl}^* \quad (33)$$

and used an identity for permutation τ , similar as in Eq. (26).

Thanks to the summation in Eq. (32) over all products of matrix elements on exactly n input ports $\mathbf{k} = (k_1, \dots, k_n)$ selected from the total N of them, the result of Eq. (32) can be rewritten as the n th order derivative in an auxiliary variable x at $x = 0$ of the N th order polynomial:

$$P_L(n) = \frac{1}{n!} \frac{d^n}{dx^n} \sum_{\sigma \in S_N} J(\sigma) \prod_{k=1}^N \left(x \mathcal{H} + I - \mathcal{U} \mathcal{U}^\dagger \right)_{k, \sigma(k)}. \quad (34)$$

Eq. (34) is very convenient for the derivation of the expression for the total probability of no counts in L output ports $P_L = \sum_{n=0}^N P_L(n)$. Indeed, since the polynomial in Eq. (34) is of order N , the latter sum is the Taylor series about $x = 0$ with $\Delta x = 1$, with the result being the value of the polynomial at $x = 1$. Taking into account also the random counts factor $e^{-\nu L}$, we get

$$P_L = e^{-\nu L} \sum_{\sigma \in S_N} J(\sigma) \prod_{k=1}^N \left(\mathcal{H} + I - \mathcal{U} \mathcal{U}^\dagger \right)_{k, \sigma(k)}. \quad (35)$$

Finally, introducing a positive semidefinite Hermitian matrix A , similar as in Eq. (25), we obtain from Eq. (35):

$$\begin{aligned} \Delta P_L &= e^{-L\nu} \sum_{\sigma \in S_N} \Delta J(\sigma) \prod_{k=1}^N A_{k, \sigma(k)}, \\ A_{kj} &\equiv \mathcal{H}_{kj} + \delta_{kj} - \sum_{l=1}^M \mathcal{U}_{kl} \mathcal{U}_{jl}^* = \delta_{kj} - \sum_{l=1}^L \mathcal{U}_{kl} \mathcal{U}_{jl}^*, \end{aligned} \quad (36)$$

where $\Delta J = J - J^{(K)}$.

C Estimating ΔP_1

Here we derive an analytical estimate on the difference in probability in Eq. (36) for $L = 1$ in the simplest case of a uniform partial distinguishability ξ and a uniform transmission η , $\mathcal{U} = \sqrt{\eta} U$. In this case, as explained in the main text, the distinguishability function for N bosons reads

$J(\sigma) = \xi^{N-c_1(\sigma)}$, where $c_1(\sigma)$ is the number of fixed points in permutation σ . Setting the output port with no counts to be $l = 1$ and expanding the product in Eq. (36) by using the expression $A_{kj} = \delta_{kj} - \eta U_{k1} U_{j1}^*$, we obtain

$$\begin{aligned} \Delta P_1 &= e^{-\nu} \sum_{\tau \in S_N} \Delta J(\tau) \prod_{k=1}^N \left(\delta_{k, \tau(k)} - \eta U_{k,1} U_{\tau(k),1}^* \right) \\ &= e^{-\nu} \sum_{n=0}^N (-\eta)^n \sum_{\sigma \in S_n} \Delta J_n(\sigma) \sum_{\mathbf{k}} \prod_{\alpha=1}^n |U_{k_\alpha,1}|^2 \end{aligned} \quad (37)$$

where we have partitioned the set $1, \dots, N$ into two subsets $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{k}' = (k_{n+1}, \dots, k_N)$, such that $\mathbf{k}'$ contains the fixed points of τ , taken into account that $\prod_{\alpha=1}^n U_{k_\alpha,1} U_{\tau(k_\alpha),1}^* = \prod_{\alpha=1}^n |U_{k_\alpha,1}|^2$, introduced permutation $\sigma \in S_n$ such that $\tau(k_\alpha) = k_{\sigma(\alpha)}$ for $\alpha = 1, \dots, n$, and used that $J(\tau) = \xi^{N-c_1(\sigma)} \equiv J_n(\sigma)$ for $\tau = I \otimes \sigma$ with $\sigma \in S_n$ and $I \in S_{N-n}$ (the identity permutation), in this case $c_1(\tau) = N - n + c_1(\sigma)$. By Eq. (22) $\Delta J_n(\sigma) = J_n(\sigma) - J_n^{(K)}(\sigma)$ becomes

$$\Delta J_n(\sigma) = \begin{cases} 0 & c_1(\sigma) \geq n - K, \\ \xi^{n-c_1(\sigma)} & c_1(\sigma) < n - K. \end{cases} \quad (38)$$

Since $\Delta J_n(\sigma)$ depends only on the number of fixed points $c_1(\sigma)$, we can perform summation in Eq. (37) over the permutations

$$\sum_{\sigma \in S_n} \Delta J_n(\sigma) = \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m, \quad (39)$$

where we have used an identity for permutations in S_n having $n - m$ fixed points [69] (see also Eq. (82) in appendix F)

$$\sum_{\sigma \in S_n} \delta_{c_1(\sigma), n-m} = \binom{n}{m} d_m, \quad d_m \equiv m! \sum_{s=0}^m \frac{(-1)^s}{s!}. \quad (40)$$

Using Eq. (39) into Eq. (37) and observing that by Eq. (38) nonzero terms have $n \geq K + 1$ we obtain

$$\begin{aligned} \Delta P_1 &= e^{-\nu} \sum_{n=K+1}^N (-\eta)^n \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m \\ &\quad \times \sum_{\mathbf{k}} \prod_{\alpha=1}^n |U_{k_\alpha,1}|^2. \end{aligned} \quad (41)$$

C.1 Balanced output port: lower bound on $|\Delta P_1|$

Let us first estimate the r.h.s. of Eq. (41) for the interferometer with a balanced output port $|U_{k,1}|^2 = 1/M$. In this case we can simplify the sum in Eq. (41) as follows

$$\begin{aligned}
& \sum_{n=K+1}^N (-\eta)^n \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m \sum_{\mathbf{k}} \frac{1}{M^n} \\
&= \sum_{n=K+1}^N \binom{N}{n} \left(-\frac{\eta}{M}\right)^n \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m \\
&= \sum_{m=K+1}^N d_m \xi^m \sum_{n=m}^N \binom{N}{n} \binom{n}{m} \left(-\frac{\eta}{M}\right)^n \\
&= \sum_{m=K+1}^N d_m \binom{N}{m} \left(-\frac{\xi\eta}{M}\right)^m \left(1 - \frac{\eta}{M}\right)^{N-m}, \tag{42}
\end{aligned}$$

where we have used the binomial identity in the sum over $n-m$. To further simplify the result we will use that $(-1)^m d_m > (-1)^m e^{-1} m!$. The latter inequality is obvious for odd m , when in the sum defining d_m Eq. (40) the consecutive pairs of addends add a positive value, whereas for even m there is an extra positive term as compared to the odd case. We have from Eqs. (41) and (42)

$$\begin{aligned}
\Delta P_1 &\geq e^{-1-\nu} \sum_{m=K+1}^N \frac{N!}{(N-m)!} \left(-\frac{\xi\eta}{M}\right)^m \left(1 - \frac{\eta}{M}\right)^{N-m} \\
&= e^{-1-\nu} \left(1 - \frac{\eta}{M}\right)^N \sum_{m=K+1}^N \frac{N!}{(N-m)!} (-X)^m, \tag{43}
\end{aligned}$$

where we have introduced

$$X \equiv \frac{\xi\eta}{M} \left(1 - \frac{\eta}{M}\right)^{-1} = \frac{\rho\xi\eta}{N - \rho\eta}. \tag{44}$$

The sum on the r.h.s. of Eq. (43) can be rewritten as an integral

$$\begin{aligned}
\sum_{m=K+1}^N \frac{(-X)^m}{(N-m)!} &= \frac{(-X)^{K+1}}{(N-K-1)!} \\
&\times \int_0^\infty dt e^{-t} (1-Xt)^{N-K-1}. \tag{45}
\end{aligned}$$

We estimate the integral in Eq. (45) by dividing it into two parts for some $a = N^{1-\delta}$ with $0 <$

$\delta < 1$ to be specified below, such that $a \geq 2$. Denoting for simplicity $n \equiv N - K - 1$, we have

$$\begin{aligned}
& \int_0^\infty dt e^{-t} (1-Xt)^n = \int_0^{\frac{1}{aX}} dt e^{-t} (1-Xt)^n \\
&+ \int_{\frac{1}{aX}}^\infty dt e^{-t} (1-Xt)^n \equiv \mathcal{I}_1 + \mathcal{I}_2 \geq |\mathcal{I}_1 - \mathcal{I}_2|. \tag{46}
\end{aligned}$$

We need to estimate the integral $\mathcal{I}_1$ from below and $\mathcal{I}_2$ from above. Since $0 \leq Xt \leq 1/a$ in the integral $\mathcal{I}_1$, we can perform the Taylor expansion of $\ln(1-Xt)$ in Xt and obtain a lower bound:

$$\begin{aligned}
(1-Xt)^n &= \exp\{n \ln(1-Xt)\} \\
&= \exp\left\{-n \sum_{s=1}^\infty \frac{(Xt)^s}{s}\right\} \\
&= \exp\left\{-nXt \left(1 + \sum_{s=1}^\infty \frac{(Xt)^s}{s+1}\right)\right\} \\
&\geq \exp\left\{-nXt \left(1 + \sum_{s=1}^\infty \frac{(Xt)^s}{s}\right)\right\} \\
&\geq \exp\{-n\beta_a Xt\},
\end{aligned}$$

where we have set $\beta_a \equiv 1 - \ln\left(1 - \frac{1}{a}\right)$ and used that $\ln(1-Xt) \geq \ln(1-1/a)$ for $Xt \leq 1/a$. We obtain

$$\begin{aligned}
\mathcal{I}_1 &\geq \int_0^{\frac{1}{aX}} dt \exp\{-(1 + \beta_a[N-K-1]X)t\} \\
&= \frac{1 - \exp\{-(1 + \beta_a[N-K-1]X)/(aX)\}}{1 + \beta_a[N-K-1]X} \\
&\geq \frac{1 - \exp\{-1/(aX)\}}{1 + \beta_a[N-K-1]X}, \tag{47}
\end{aligned}$$

where we have used that $\beta_a(N-K-1) \geq 0$.

Consider now the integral $\mathcal{I}_2$ of Eq. (46). By changing the integration variable $\tau = t + 1/(aX)$, using the same partition into two integrals as in

Eq. (46) and that $a \geq 2$ we get

$$\begin{aligned}
\mathcal{I}_2 &= e^{-\frac{1}{aX}} \int_0^\infty d\tau e^{-\tau} \left(1 - \frac{1}{a} - X\tau\right)^{N-K-1} \\
&\leq e^{-\frac{1}{aX}} \left[\mathcal{I}_1 + \int_{\frac{1}{aX}}^\infty d\tau e^{-\tau} \left|X\tau + \frac{1}{a} - 1\right|^{N-K-1} \right] \\
&\leq e^{-\frac{1}{aX}} \left[\mathcal{I}_1 + \int_{\frac{1}{aX}}^\infty d\tau e^{-\tau} \left|X\tau - \frac{1}{a}\right|^{N-K-1} \right] \\
&= e^{-\frac{1}{aX}} \left[\mathcal{I}_1 + e^{-\frac{1}{aX}} X^{N-K-1} \int_0^\infty dt e^{-t} t^{N-K-1} \right] \\
&= e^{-\frac{1}{aX}} \left[\mathcal{I}_1 + e^{-\frac{1}{aX}} (N-K-1)! X^{N-K-1} \right], \tag{48}
\end{aligned}$$

where we obtain $\mathcal{I}_1$ by dropping $-1/a$ in the integral (hence, the inequality). Using Eq. (44) and Stirling's approximation we get $(N-K-1)! X^{N-K-1} \sim (\rho\xi\eta/e)^{N-K}$ as $N-K \gg 1$ (recall that $\xi, \eta, \rho \leq 1$), thus the second term in the square brackets on the r.h.s of Eq. (48) is exponentially small. Using Eq. (44), the lower bound from Eq. (47), the upper bound from Eq. (48), retaining only the leading order as $N \rightarrow \infty$, we obtain

$$\begin{aligned}
&\int_0^\infty dt e^{-t} (1-Xt)^{N-K-1} \\
&\geq \left| \left(1 - e^{-\frac{1}{aX}}\right) \mathcal{I}_1 - e^{-\frac{2}{aX}} (N-K-1)! X^{N-K-1} \right| \\
&\geq \frac{1 - O\left(e^{-\frac{N^\delta}{\rho\xi\eta}}\right)}{1 + (N-K-1)X[1 - \ln(1 - N^{-1+\delta})]} \\
&= \frac{1}{1 + \rho\xi\eta} \left[1 - \left| O\left(\frac{1}{N^{1-\delta}} + \frac{K}{N}\right) \right| \right], \tag{49}
\end{aligned}$$

where we have expanded the denominator in the final step (the minus sign at $O(\dots)$ means that a small positive error is subtracted). The parameter δ will be determined below.

What is left is to estimate the front factors in Eq. (43) and (45). We have

$$\left(1 - \frac{\eta}{M}\right)^N = \exp(-\rho\eta) \left[1 - \left| O\left(\frac{1}{N}\right) \right| \right]. \tag{50}$$

Using the definition Eq. (44) and the following identity (derived by using Euler's summation for-

mula, similarly as in Ref. [73])

$$\prod_{l=1}^K \left(1 - \frac{l}{N}\right) = \exp\left(-\frac{K(K+1)}{2N}\right) \left(1 + O\left(\frac{K^3}{N^2}\right)\right),$$

we obtain

$$\begin{aligned}
\frac{N! X^{K+1}}{(N-K-1)!} &= \left(\frac{\rho\xi\eta}{1 - \rho\eta/N}\right)^{K+1} \prod_{l=1}^K \left(1 - \frac{l}{N}\right) \\
&= \left(\rho\xi\eta e^{-\frac{K/2+\xi\eta}{N}}\right)^{K+1} \left[1 + O\left(\frac{K^3}{N^2}\right) \right] \\
&= (\rho\xi\eta)^{K+1} \left[1 - \left| O\left(\frac{K^2}{N}\right) \right| \right]. \tag{51}
\end{aligned}$$

Recall that $K = O(1)$, as $N \rightarrow \infty$, since K is the fixed cut-off order on quantum many-body interference accounted for by the efficient classical algorithms of Refs. [48, 50, 53]. However, here we only need to require *much more relaxed* condition, obtained by requirement that the discarded terms in Eqs. (43), (45), (49), (50) and (51) are all vanishing with an optimized rate, which occurs for $K^2 \sim N^\delta$ (for the same δ as in Eq. (49)). Recalling also the sign factor $(-1)^{K+1}$, due to $(-X)^{K+1}$ in Eq. (45), we obtain the following asymptotic result

$$\begin{aligned}
(-1)^{K+1} \Delta P_1 &\geq W_1 \left[1 - \left| O\left(\frac{K^2}{N}\right) \right| \right] \\
W_1 &\equiv \frac{(\eta\xi\rho)^{K+1}}{1 + \eta\xi\rho} e^{-1-\nu-\eta\rho}. \tag{52}
\end{aligned}$$

Below will show that the leading order of W_1 from Eq. (52) is also the leading order of the lower bound in a Haar-random interferometer under a similar condition $K^2 \ll N$, whereas the variance scales as $O(K^2/N)$.

C.2 The average $\langle \Delta P_1 \rangle$ in the Haar-random interferometer

We will use the following known average in a Haar-random unitary interferometer U [72]

$$\left\langle \prod_{\alpha=1}^n |U_{k_\alpha, 1}|^{2s_k} \right\rangle = \frac{s_1! \dots s_n!}{M^{(n)}}, \tag{53}$$

where $m^{(n)} \equiv m(m+1) \dots (m+n-1)$ (Pochhammer symbol for the rising factorial).

Consider the average lower bound in a Haar-random interferometer obtained by averaging Eq.

(41). Observing that $\langle |x| \rangle \geq |\langle x \rangle|$, we have $\langle \mathcal{D} \rangle \geq |\langle \Delta P_1 \rangle|$. We obtain

$$\begin{aligned} \langle \Delta P_1 \rangle &= e^{-\nu} \sum_{n=K+1}^N (-\eta)^n \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m \sum_{\mathbf{k}} \frac{1}{M^{(n)}} \\ &= e^{-\nu} \sum_{n=K+1}^N \binom{N}{n} \frac{(-\eta)^n}{M^{(n)}} \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m, \end{aligned} \quad (54)$$

where the only difference from the similar expression in Eq. (42) is the Pochhammer symbol $M^{(n)}$ instead of M^n in the denominator. The factors in the first sum in Eq. (54) have the following upper bound

$$\left| \binom{N}{n} \frac{(-\eta)^n}{M^{(n)}} \right| < \frac{(\rho\eta)^n}{n!}, \quad (55)$$

thus they decrease exponentially fast with n . Due to the expansion (derived by using Euler's summation, similar as in Ref. [73])

$$M^{(n)} = M^n e^{\frac{n^2}{2M}} \left(1 + O\left(\frac{n^3}{M^2}\right) \right), \quad (56)$$

we can neglect the difference between $(\eta/M)^n$ and $\eta^n/M^{(n)}$ for $n^2 \ll M$. These two observations allows us to conclude that, without much of error, one can substitute the Pochhammer symbol $M^{(n)}$ by M^n on the r.h.s. of Eq. (54), thus obtaining the same estimate on the average lower bound as in Eq. (52), if for the few first terms in the sum $n^2 \ll M = N/\rho$, which necessitates that $K^2 \ll N/\rho$. Below we give an alternative derivation of the leading order of the average lower bound in Eq. (54).

Using Pochhammer symbol for the falling factorial, $(m)_n \equiv m(m-1)\dots(m-n+1)$, we get

$$\begin{aligned} &\sum_{n=K+1}^N \binom{N}{n} \frac{(-\eta)^n}{M^{(n)}} \sum_{m=K+1}^n \binom{n}{m} d_m \xi^m \\ &= \sum_{n=K+1}^N \frac{(N)_n}{M^{(n)}} (-\eta)^n \sum_{m=K+1}^n \frac{d_m}{m!} \frac{\xi^m}{(n-m)!} \\ &= \frac{(N)_{K+1}}{(M)_{K+1}} (-\eta\xi)^{K+1} \sum_{l=0}^{N-K-1} \frac{(N-K-1)_l}{(M+K+1)_l} (-\eta)^l \\ &\quad \times \sum_{s=0}^l \frac{d_{s+K+1}}{(s+K+1)!} \frac{\xi^s}{(l-s)!}, \end{aligned} \quad (57)$$

where we have introduced new indices l and s by $n = l + K + 1$ and $m = s + K + 1$. Due to the

above discussion, we can approximate the falling factorial in the reduced sum by the corresponding power, using that (obtained similar as Eq. (56))

$$(N)_n = N^n e^{-\frac{n^2}{2N}} \left(1 + O\left(\frac{n^3}{N^2}\right) \right). \quad (58)$$

Assuming that K is not small for the approximation $d_m/m! \approx e^{-1}$ for $m \geq K+1$ (see Eq. (40)) and further details in Eq. (82) of appendix F) and approximating the exponential series by the exponential functions for $N-K \gg 1$ we have

$$\begin{aligned} &\sum_{l=0}^{N-K-1} \frac{(N-K-1)_l}{(M+K+1)_l} (-\eta)^l \sum_{s=0}^l \frac{d_{s+K+1}}{(s+K+1)!} \frac{\xi^s}{(l-s)!} \\ &\approx e^{-1} \sum_{l=0}^{N-K-1} \left(-\frac{(N-K-1)\eta}{M+K+1} \right)^l \sum_{s=0}^l \frac{\xi^s}{(l-s)!} \\ &= e^{-1} \sum_{s=0}^{N-K-1} \xi^s \sum_{l=s}^{N-K-1} \frac{1}{(l-s)!} \left(-\frac{(N-K-1)\eta}{M+K+1} \right)^{l-s} \\ &= e^{-1} \sum_{s=0}^{N-K-1} \left(-\frac{(N-K-1)\eta}{M+K+1} \right)^s \sum_{l=s}^{N-K-1} \frac{1}{(l-s)!} \\ &\quad \times \left(-\frac{(N-K-1)\eta}{M+K+1} \right)^{l-s} \\ &\approx \sum_{s=0}^{N-K-1} \left(-\frac{(N-K-1)\eta}{M+K+1} \right)^s \exp \left\{ -1 - \frac{N-K-1}{M+K+1} \eta \right\} \\ &\approx \frac{\exp \left\{ -1 - \frac{N-K-1}{M+K+1} \eta \right\}}{1 + \frac{N-K-1}{M+K+1} \eta \xi} \\ &= \frac{\exp \left\{ -1 - \rho\eta \right\}}{1 + \rho\eta\xi} \left[1 + O\left(\frac{K}{N}\right) \right]. \end{aligned} \quad (59)$$

In the same way, we can also approximate the ratio of two falling factorials in the first factor in Eq. (57)

$$\frac{(N)_{K+1}}{(M)_{K+1}} (-\eta\xi)^{K+1} = (-\rho\xi\eta)^{K+1} \left[1 + O\left(\frac{K^2}{N}\right) \right].$$

Therefore, we obtain from Eqs. (54), (57) and (59)

$$(-1)^{K+1} \langle \Delta P_1 \rangle \approx \frac{(\eta\xi\rho)^{K+1}}{1 + \eta\xi\rho} e^{-1-\nu-\eta\rho} \quad (60)$$

where now the lower bound of Eq. (52) is an approximation to the average for sufficiently large values of K and $N \gg K^2$.

Below, using the same approximations as those used for derivation of Eq. (60), we show that the

relative variance of the difference in the probability in Eq. (41) is of the order $O(K^2/N)$, which means that except for a vanishing fraction of interferometers as N scales up, the average result in Eq. (60) is the leading order of an absolute lower bound.

C.3 Variance of ΔP_1 in the Haar-random interferometers

Consider the variance of ΔP_1 in Eq. (41). Due to Eq. (53) the variance $\mathcal{R} = \langle (\Delta P_1)^2 \rangle - \langle \Delta P_1 \rangle^2$ becomes

$$\begin{aligned} \mathcal{R} &= e^{-2\nu} \sum_{n_1=K+1}^N \sum_{n_2=K+1}^N (-\eta)^{n_1+n_2} \\ &\times \sum_{m_1=K+1}^{n_1} \binom{n_1}{m_1} d_{m_1} \xi^{m_1} \sum_{m_2=K+1}^{n_2} \binom{n_2}{m_2} d_{m_2} \xi^{m_2} \Theta_{n_1, n_2}, \end{aligned} \quad (61)$$

with

$$\begin{aligned} \Theta_{n_1, n_2} &\equiv \sum_{\mathbf{k}^{(1)}} \sum_{\mathbf{k}^{(2)}} \langle \prod_{i=1,2} \prod_{k \in \mathbf{k}^{(i)}} |U_{k,1}|^2 \rangle \\ &- \sum_{\mathbf{k}^{(1)}} \sum_{\mathbf{k}^{(2)}} \prod_{i=1,2} \langle \prod_{k \in \mathbf{k}^{(i)}} |U_{k,1}|^2 \rangle \\ &= \sum_{\mathbf{k}^{(1)}} \sum_{\mathbf{k}^{(2)}} \left[\frac{2^{|\mathbf{k}^{(1)} \cap \mathbf{k}^{(2)}|}}{M^{(n_1+n_2)}} - \frac{1}{M^{(n_1)}} \frac{1}{M^{(n_2)}} \right], \end{aligned} \quad (62)$$

where we have used that for a common element k in $\mathbf{k}^{(1,2)}$ (denoting this subset by $\mathbf{k}^{(1)} \cap \mathbf{k}^{(2)}$) there is a factor 2 in the r.h.s. of Eq. (53). Let us simplify the sum in Eq. (62). By assuming that $n_2 \geq n_1$ and using a dummy variable $a \in \{1, 2\}$ we get for the two terms in the square brackets in Eq. (62)

$$\begin{aligned} \sum_{\mathbf{k}^{(1)}} \sum_{\mathbf{k}^{(2)}} a^{|\mathbf{k}^{(1)} \cap \mathbf{k}^{(2)}|} &= \binom{N}{n_2} \sum_{s=0}^{n_1} \binom{n_2}{s} \binom{N-n_2}{n_1-s} a^s \\ &= \sum_{s=0}^{\min(n_1, n_2)} \frac{a^s}{s!(n_1-s)!(n_2-s)!} \frac{N!}{(N-n_1-n_2+s)!}. \end{aligned} \quad (63)$$

Since the expression in Eq. (63) is symmetric in n_1 and n_2 , it applies for any n_1 and n_2 . Now, using the Pochhammer notations for the falling

factorials, for $\theta_{n_1, n_2} \equiv n_1! n_2! \Theta_{n_1, n_2}$ we get

$$\begin{aligned} \theta_{n_1, n_2} &= \sum_{s=0}^{\min(n_1, n_2)} \frac{(n_1)_s (n_2)_s (N)_{n_1+n_2-s}}{s! M^{(n_1+n_2)}} \\ &\times \left(2^s - \frac{M^{(n_1+n_2)}}{M^{(n_1)} M^{(n_2)}} \right). \end{aligned} \quad (64)$$

By setting $n_i = l_i + K + 1$ and $m_i = s_i + K + 1$ in Eq. (61) we obtain using Eq. (64)

$$\begin{aligned} \mathcal{R} &= e^{-2\nu} \sum_{n_1=K+1}^N \sum_{n_2=K+1}^N (-\eta)^{n_1+n_2} \theta_{n_1, n_2} \\ &\times \sum_{m_1=K+1}^{n_1} \frac{d_{m_1}}{m_1!} \frac{\xi^{m_1}}{(n_1-m_1)!} \sum_{m_2=K+1}^{n_2} \frac{d_{m_2}}{m_2!} \frac{\xi^{m_2}}{(n_2-m_2)!} \\ &= (\eta \xi)^{2K+2} e^{-2\nu} \sum_{l_1, l_2=0}^{N-K-1} (-\eta)^{l_1+l_2} \theta_{l_1+K+1, l_2+K+1} \\ &\times \sum_{s_1=0}^{l_1} \frac{d_{s_1+K+1}}{(s_1+K+1)!} \frac{\xi^{s_1}}{(l_1-s_1)!} \\ &\times \sum_{s_2=0}^{l_2} \frac{d_{s_2+K+1}}{(s_2+K+1)!} \frac{\xi^{s_2}}{(l_2-s_2)!} \end{aligned} \quad (65)$$

As in the estimate of the average value $\langle \Delta P_1 \rangle$, only the lowest-order terms in the sums over $l_{1,2}$ contribute significantly to the result. Thus we can approximate the Pochhammer symbols by using the expansions in Eqs. (56) and (58). Moreover, for $K \ll \sqrt{N}$, it turns out to be sufficient for the leading-order approximation for θ_{n_1, n_2} of Eq. (64) to keep only the terms with $s = 0$ and $s = 1$ in the expansion. In this case one can simplify the variance $\mathcal{R}$ in Eq. (65). Using the following approximations for the ratios of Pochhammer symbols (setting $n = \max(n_1, n_2)$)

$$\begin{aligned} \frac{M^{(n_1+n_2)}}{M^{(n_1)} M^{(n_2)}} &= 1 + \frac{n_1 n_2}{M} + O\left(\frac{n^4}{M^2}\right), \\ \frac{(N)_{(n_1+n_2)}}{M^{(n_1)} M^{(n_2)}} &= \left(\frac{N}{M}\right)^{n_1+n_2} \left[1 + O\left(\frac{n^2}{N}\right) \right] \end{aligned}$$

we obtain for the first two terms ($s = 0, 1$) in Eq. (64)

$$\begin{aligned} & \sum_{s=0,1} \frac{(n_1)_s (n_2)_s (N)_{n_1+n_2-s}}{s! M^{(n_1+n_2)}} \left(2^s - \frac{M^{(n_1+n_2)}}{M^{(n_1)} M^{(n_2)}} \right) \\ &= \left(\frac{N}{M} \right)^{n_1+n_2} \left[1 + O\left(\frac{n^2}{N}\right) \right] \\ &\times \left\{ -\frac{n_1 n_2}{M} + O\left(\frac{n^4}{M^2}\right) + \frac{n_1 n_2}{N - n_1 - n_2 + 1} \right. \\ &\times \left. \left[1 - \frac{n_1 n_2}{M} + O\left(\frac{n^4}{M^2}\right) \right] \right\} \\ &= \rho^{n_1+n_2} (1 - \rho) \frac{n_1 n_2}{N} \left(1 + O\left(\frac{n}{N}\right) \right). \end{aligned}$$

The rest of the expansion in Eq. (64) is relatively much smaller for $n_{1,2} \leq K + 1 \ll \sqrt{N}$, since

$$\begin{aligned} & \sum_{s=2}^{\min(n_1, n_2)} \frac{(n_1)_s (n_2)_s (N)_{n_1+n_2-s}}{s! M^{(n_1+n_2)}} \left(2^s - \frac{M^{(n_1+n_2)}}{M^{(n_1)} M^{(n_2)}} \right) \\ &< \left(\frac{N}{M} \right)^{n_1+n_2} \sum_{s=2}^{\min(n_1, n_2)} \frac{(2^s - 1)(n_1)_s (n_2)_s}{s! N^s} \\ &< \rho^{n_1+n_2} \sum_{s=2}^n \frac{1}{s!} \left(\frac{2n^2}{N} \right)^s \\ &= 2\rho^{n_1+n_2} \frac{n^4}{N^2} \left[1 + O\left(\frac{n^2}{N}\right) \right] = O\left(\frac{n^4}{N^2}\right). \end{aligned}$$

Thus we have (recall that $n = \max(n_1, n_2)$)

$$\theta_{n_1, n_2} = \frac{(1 - \rho)}{N} n_1 n_2 \rho^{n_1+n_2} \left[1 + O\left(\frac{n^2}{N}\right) \right]. \quad (66)$$

For K sufficiently large, using the approximation $d_m/m! \approx e^{-1}$ for $m \geq K + 1$, we obtain from Eqs. (65) and (66) the leading order term of the variance as follows

$$\begin{aligned} \mathcal{R} &\approx \frac{(1 - \rho)}{N} (K + 1)^2 (\eta \xi \rho)^{2K+2} e^{-2[1+\nu]} \\ &\times \sum_{l_1, l_2=0}^{N-K-1} (-\eta \rho)^{l_1+l_2} \sum_{s_1=0}^{l_1} \frac{\xi^{s_1}}{(l_1 - s_1)!} \sum_{s_2=0}^{l_2} \frac{\xi^{s_2}}{(l_2 - s_2)!} \\ &= \frac{(1 - \rho)(K + 1)^2}{N} \left(\frac{e^{-1-\nu-\eta\rho} (\eta \xi \rho)^{K+1}}{1 + \xi \eta \rho} \right)^2, \end{aligned} \quad (67)$$

where we have taken into account that the four sums factorize into two double sums for (l_i, s_i) , $i = 1, 2$, with each factor being proportional to that evaluated in Eq. (59). Eq. (67) shows that,

as $N \rightarrow \infty$ and $K^2/N \rightarrow 0$, the average value $\langle \Delta P_1 \rangle$, given by Eq. (60), is the asymptotic difference in probability for almost all interferometers.

Obviously, there are interferometers which cannot satisfy Eq. (60). This is the class of almost trivial interferometers $\mathcal{U} = \sqrt{\eta} U$, $U U^\dagger = I$, where $U = \mathcal{P} V$ is a product of a permutation interferometer $\mathcal{P}$, exchanging the labels of the input ports, and an almost diagonal unitary interferometer V . More precisely, if $\max(|V_{k,l}|^2)$ for $k \neq l$ is much smaller than the average $\langle |U_{k1}|^2 \rangle = 1/M$. This class of interferometers corresponds to strongly concentrated output probability distribution, as N scales up, on the permutation of input ports by $\mathcal{P}$.

D Estimating the probability P_1

We consider the case of uniform losses and distinguishability: $\mathcal{U} = \sqrt{\eta} U$, $U^\dagger U = I$, and $J(\sigma) = \xi^{N-c_1(\sigma)}$. In the simplest case of $L = 1$ we can estimate the average value of P_1 of Eq. (35) for such a noisy boson sampling model. The easiest way to get the necessary expression for $\langle P_1 \rangle$ is by replacing $K + 1 \rightarrow 0$ in Eq. (54). We obtain

$$\begin{aligned} \langle P_1 \rangle &= e^{-\nu} \sum_{n=0}^N \binom{N}{n} \frac{(-\eta)^n}{M^{(n)}} \sum_{m=0}^n \binom{n}{m} d_m \xi^m \\ &= e^{-\nu} \sum_{n=0}^N \frac{(N)_n}{M^{(n)}} (-\eta)^n \sum_{s=0}^n \frac{\xi^{n-s} (1 - \xi)^s}{s!}, \end{aligned} \quad (68)$$

where we have used the summation identity (which can be obtained by the method of generating functions, see for instance, Ref. [69])

$$\sum_{m=0}^n \binom{n}{m} d_m \xi^m = n! \sum_{s=0}^n \frac{\xi^{n-s} (1 - \xi)^s}{s!}.$$

Using Eqs. (56) and (58) we obtain

$$\frac{(N)_n}{M^{(n)}} (-\eta)^n = (-\eta \rho)^n \left[1 + O\left(\frac{n^2}{N}\right) \right]. \quad (69)$$

Now we can perform an approximation assuming that $\rho \eta \ll 1$, such that only the powers $n \ll \sqrt{N}$ contribute significantly to the sum over n in Eq. (68), similar as in the computation of the average

difference in probability $\langle \Delta P_1 \rangle$. We get

$$\begin{aligned} \langle P_1 \rangle &\approx e^{-\nu} \sum_{n=0}^{\infty} (-\xi\eta\rho)^n \sum_{s=0}^n \frac{(1/\xi - 1)^s}{s!} \\ &= e^{-\nu} \sum_{s=0}^{\infty} \frac{(1/\xi - 1)^s}{s!} \sum_{n=s}^{\infty} (-\xi\eta\rho)^n \\ &= \frac{\exp(-\nu - \eta\rho[1 - \xi])}{1 + \xi\eta\rho}. \end{aligned} \quad (70)$$

Let us also estimate the variance $\mathcal{V} \equiv \langle P_1^2 \rangle - \langle P_1 \rangle^2$. We have from Eqs. (61) and (65) by using

Eqs. (54), (68), and (70)

$$\begin{aligned} \mathcal{V} &\approx \frac{(1 - \rho)}{N} \left[e^{-\nu} \sum_{n=0}^N n(-\rho\eta)^n \sum_{m=0}^n \frac{d_m}{m!} \frac{\xi^m}{(n - m)!} \right]^2 \\ &\approx \frac{(1 - \rho)}{N} \left(-\eta \frac{\partial}{\partial \eta} \langle P_1 \rangle \right)^2 \\ &\approx \frac{(1 - \rho)}{N} (\eta\rho)^2 \left(1 - \xi + \frac{\xi}{1 + \rho\eta\xi} \right)^2 \langle P_1 \rangle^2, \end{aligned} \quad (71)$$

where $\langle P_1 \rangle$ is given in Eq. (70).

E Numerical simulations of the lower bound

Here the method used to numerically simulate Eq. (9) of the main text is described. We assume, as in appendix C, a uniform overlap ξ over the internal states of bosons, thus $J(\sigma) = \xi^{N - c_1(\sigma)}$, where $c_1(\sigma)$ is the total number of fixed points of permutation σ . Denoting $n = c_1(\sigma)$ we get from Eqs. (22) and (24)

$$\begin{aligned} \Delta P_L &= e^{-\nu L} \sum_{\sigma \in S_N} \Delta J(\sigma) \prod_{k=1}^N A_{k, \sigma(k)} = e^{-\nu L} \sum_{n=0}^{N-K-1} \xi^{N-n} \sum_{\mathbf{k}} \sum_{\text{fix}(\sigma)=\mathbf{k}} \left[\prod_{\alpha=1}^n A_{k_\alpha, k_\alpha} \right] \prod_{\alpha=n+1}^N A_{k_\alpha, \sigma(k_\alpha)} \\ &= e^{-\nu L} \sum_{n=0}^{N-K-1} \sum_{\mathbf{k}} \sum_{\text{fix}(\sigma)=\mathbf{k}} \left[\prod_{\alpha=1}^n A_{k_\alpha, k_\alpha}^{(\xi)} \right] \prod_{\alpha=n+1}^N A_{k_\alpha, \sigma(k_\alpha)}^{(\xi)}, \end{aligned} \quad (72)$$

where $\mathbf{k} = (k_1, \dots, k_n)$, the sum with $\text{fix}(\sigma) = \mathbf{k}$ denotes the summation over all permutations with fixed points $\mathbf{k}$ (i.e., over all derangements of the complementary subset $(k_{n+1}, \dots, k_N)$), and $A_{kl}^{(\xi)} = \delta_{kl} A_{kk} + \xi(1 - \delta_{kl}) A_{kl}$. Let us define an additive function $f(\sigma)$ on the symmetric group S_N by setting

$$f(\sigma) = \prod_{k=1}^N A_{k, \sigma(k)}^{(\xi)}. \quad (73)$$

Then the sum in the expression on the r.h.s. of Eq. (72) can be rewritten as follows

$$\begin{aligned} &\sum_{n=0}^{N-K-1} \sum_{\mathbf{k}} \sum_{\text{fix}(\sigma)=\mathbf{k}} \left[\prod_{\alpha=1}^n A_{k_\alpha, k_\alpha}^{(\xi)} \right] \prod_{\alpha=n+1}^N A_{k_\alpha, \sigma(k_\alpha)}^{(\xi)} \\ &= \sum_{n=0}^{N-K-1} f(D_{N-n}), \end{aligned} \quad (74)$$

where $D_m \in S_N$ is the set of all permutations having exactly $N - m$ fixed points, i.e., the set of all derangements of m elements in the symmetric group S_N . Unfortunately, derangements are very difficult to handle numerically. Therefore, we need to rewrite the sum in Eq. (74) in terms of the matrix permanents (i.e., using all the permutations in a symmetric group). We will use the following summation identity (represent-

ing the generalised inclusion-exclusion principle) valid for any additive function f on the symmetric group S_N [82]

$$f(D_{N-n}) = \sum_{s=n}^N (-1)^{s-n} \binom{s}{n} f_s, \\ f_s \equiv \sum_{k_1 \dots k_s} f(S_{(k_{s+1}, \dots, k_N)}), \quad (75)$$

where $S_{(k_{s+1}, \dots, k_N)}$ is the symmetric subgroup S_{N-s} of all permutations of $(k_{s+1}, \dots, k_N)$. From Eq. (73) we obtain

$$f_s = \sum_{k_1 \dots k_s} \left[\prod_{\alpha=1}^s A_{k_\alpha, k_\alpha}^{(\xi)} \right] \sum_{\sigma \in S_{N-s}} \prod_{\alpha=s+1}^N A_{k_\alpha, \sigma(k_\alpha)}^{(\xi)} \\ = \sum_{k_1 \dots k_s} \left[\prod_{\alpha=1}^s A_{k_\alpha, k_\alpha}^{(\xi)} \right] \text{per} \left(A^{(\xi)}[k_{s+1} \dots k_N | k_{s+1} \dots k_N] \right), \quad (76)$$

i.e., we need to compute permanents of the positive semi-definite Hermitian matrices $A^{(\xi)}[k_{s+1} \dots k_N | k_{s+1} \dots k_N]$, obtained by the rows and columns $k_{s+1}, \dots, k_N$ of $A^{(\xi)}$. Using Eq. (76) can rewrite the sum in Eq. (74) as follows

$$\sum_{n=0}^{N-K-1} f(D_{N-n}) = \sum_{n=0}^{N-K-1} \sum_{s=n}^N (-1)^{s-n} \binom{s}{n} f_s \\ = \sum_{n=0}^{N-K-1} \left(\sum_{s=n}^{N-K-1} + \sum_{s=N-K}^N \right) (-1)^{s-n} \binom{s}{n} f_s \\ = \left(\sum_{s=0}^{N-K-1} \sum_{n=0}^s + \sum_{s=N-K}^N \sum_{n=0}^{N-K-1} \right) (-1)^{s-n} \binom{s}{n} f_s \\ = f_0 + (-1)^{N-K-1} \sum_{s=N-K}^N (-1)^s \binom{s-1}{N-K-1} f_s, \quad (77)$$

where we have used the summation identity (easily proven by induction in m)

$$\sum_{n=0}^m (-1)^n \binom{s}{n} = \begin{cases} (-1)^m \binom{s-1}{m}, & m < s \\ \delta_{s,0}, & m = s. \end{cases}$$

Therefore, we have derived the following result

$$\Delta P_L = e^{-\nu L} \left[f_0 - \sum_{s=N-K}^N (-1)^{N-K-s} \binom{s-1}{N-K-1} f_s \right], \quad (78)$$

where f_s is given by Eq. (76). This result allows one to simulate numerically ΔP_L by computing the matrix permanents.

F Comparing the models with distinguishability functions $J^{(K)}$ and $J_+^{(K)}$

Let us consider how switching from the approximation model with $J^{(K)}(\sigma)$ of Eq. (6) to that with $J_+^{(K)}(\sigma)$ of Eq. (17) would affect the results in Eqs. (9)-(12). The difference in probability ΔP_L in Eq. (9) depends on the difference of distinguishability functions, moreover, the expression for $\langle \Delta P_1 \rangle$ depends only on the sum $\sum_{\sigma \in S_n} \Delta J(\sigma)$ for $K+1 \leq n \leq N$ (see Eqs. (37)-(41) in appendix C), which for uniform distinguishability and losses, read:

$$\sum_{\sigma \in S_n} [J(\sigma) - J^{(K)}(\sigma)] = \sum_{m=K+1}^n \binom{n}{m} \xi^m d_m, \\ \sum_{\sigma \in S_n} [J(\sigma) - J_+^{(K)}(\sigma)] = \sum_{m=K+1}^n \binom{n}{m} \xi^m (d_m - d_m^{(K)}), \quad (79)$$

where we have introduced the number of derangements $m \equiv n - c_1(\sigma)$, have taken into account that derangements of m bosons are weighted by ξ^m , that the model with $J^{(K)}$ accounts only for the derangement with $m \leq K$, whereas that with $J_+^{(K)}$ accounts for those $(d_m^{(K)})$ containing only the l -cycles with $l \leq K$, see fig. 3.

Figure 3: Schematic view of the relation between the support domains in S_n of the original distinguishability function $J(\sigma)$, the model accounting for all K -boson interferences, $J_+^{(K)}(\sigma)$, and the model of K interfering bosons and $N-K$ classical particles, $J^{(K)}(\sigma)$.

To find an expression for $d_m^{(K)}$ one can use the generation function method, which allows to compute any cycle sum $Z_m(t_1, \dots, t_m)$ over permutation group S_m ,

$$Z_m(t_1, \dots, t_m) = \sum_{\sigma \in S_m} \prod_{l=1}^m t_l^{c_l(\sigma)}, \quad (80)$$

by using the explicit form of the generating function [69]

$$F(X) \equiv \sum_{m \geq 1} Z_m(t_1, \dots, t_m) X^m = \exp \left(\sum_{l=1}^{\infty} t_l \frac{X^l}{l} \right), \quad (81)$$

with $Z_m(t_1, \dots, t_m) = \left(\frac{d}{dX} \right)_{X=0}^m F(X)$. Let us first approximate d_m using this method. In this case $t_l = 1$ for $l \geq 2$ and $t_1 = 0$ in Eq. (80). From Eq. (81) we obtain

$$\begin{aligned} d_m &= \left(\frac{d}{dX} \right)_{X=0}^m \exp \left(\sum_{l=2}^{\infty} \frac{X^l}{l} \right) \\ &= \left(\frac{d}{dX} \right)_{X=0}^m \frac{e^{-X}}{1-X} \\ &= m! \sum_{s=0}^m \frac{1}{s!} \left(\frac{d}{dX} \right)_{X=0}^s \exp(-X) \\ &= m! \left\{ e^{-1} + R_m \right\}, \end{aligned} \quad (82)$$

where we have used the Leibniz formula for the m th derivative of a product and approximated the Taylor series of e^{-X} by its value at $X = 1$. The remainder R_m is bounded by the $(m+1)$ th term of the Taylor series maximized on the interval $[0, 1]$:

$$|R_m| \leq \frac{1}{(m+1)!}. \quad (83)$$

For $d_m^{(K)}$ one must set in $Z_m(t_1, \dots, t_m)$ of Eq. (80) $t_1 = t_{K+i} = 0$, for all $i \geq 1$, and $t_l = 1$ otherwise. Similar manipulations as in Eq. (82) give

$$\begin{aligned} d_m^{(K)} &= \left(\frac{d}{dX} \right)_{X=0}^m \exp \left(\sum_{l=2}^K \frac{X^l}{l} \right) \\ &= \left(\frac{d}{dX} \right)_{X=0}^m \frac{\exp \left(-X - \sum_{l=K+1}^{\infty} \frac{X^l}{l} \right)}{1-X} \\ &= m! \sum_{s=0}^m \frac{1}{s!} \left(\frac{d}{dX} \right)_{X=0}^s \exp \left(-X - \sum_{l=K+1}^{\infty} \frac{X^l}{l} \right) \\ &= m! \left\{ \exp \left(-1 - \sum_{l=K+1}^{Q_m} \frac{1}{l} \right) + R_{Q_m}^{(K)} \right\}, \end{aligned} \quad (84)$$

where for $m \geq K+1$ the optimal $Q_m \geq m$, minimizing the remainder $R_{Q_m}^{(K)}$ is unknown (i.e., the remainder is hard to estimate). This is the main difficulty of working out calculations in the model with $J_+^{(K)}$.

Let us now comment on what changes occur when our model with $J^{(K)}$ is substituted by that with $J_+^{(K)}$. In the expression for the average $|\langle \Delta P_1 \rangle|$ in appendix C the number of derangements, approximated as $d_m \approx m!e^{-1}$ (for $m \geq K+1$), has been used only once, to obtain the factor $\frac{\exp\{-1-\rho\eta\}}{1+\rho\eta\xi}$ in the expression for W_1 Eq. (12), see Eqs. (57) and (59) of appendix C. Hence, the only change in the result would be in the double sum in Eq. (57) with $d_m - d_m^{(K)}$ replacing d_m , whereas the rest of appendix C does not change. By Eqs. (82) and (84) the difference still satisfies a similar scaling $d_m - d_m^{(K)} = m!f(m, K)$, where $f(m, K)$ is a bounded function for all $m \geq K+1$. Therefore, instead of e^{-1} the new expression for W_1 would get in the case of the model with $J_+^{(K)}$ a more complicated factor.

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