Autonomous Ticking Clocks from Axiomatic Principles

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There are many different types of time keeping devices. We use the phrase *ticking clock* to describe those which — simply put — “tick” at approximately regular intervals. Various important results have been derived for ticking clocks, and more are in the pipeline. It is thus important to understand the underlying models on which these results are founded. The aim of this paper is to introduce a new ticking clock model from axiomatic principles that overcomes concerns in the community about the physicality of the assumptions made in previous models. The ticking clock model in [1] achieves high accuracy, yet lacks the autonomy of the less accurate model in [2]. Importantly, the model we introduce here achieves the best of both models: it retains the autonomy of [2] while allowing for the high accuracies of [1]. What is more, [2] is revealed to be a special case of the new ticking clock model.

1 Introduction and basics

Clocks form part of our everyday lives. Understanding their fundamental limitations is an interesting and rich theoretical problem of study which may yield important design principles for improved future clocks. However, results pertinent to the performance of clocks are only of relevance if the theoretical clock models underpinning them capture the relevant properties. Therefore, understanding the clock models which underlie the results about clock performance, is as important as the results themselves.

Before discussing our findings and their motivation, let us first describe two different types of time keeping devices in order to set the scene for this work. We coin the phrase *ticking clocks* to refer to one type and call the other *stopwatches*. In the literature, both devices are often simply referred to as “clocks”, e.g. [1–12], yet it is worth introducing distinct names due to their different character. A stopwatch is a device which measures the elapsed time between two external events. There will be a starting time (e.g. the beginning of a race) and a stopping time (e.g. when the winner crosses the finish line). The stopwatch will attempt to measure the elapsed time. The earliest types of quantum clocks considered in the literature were of this form [13, 14]. These are also the types of clocks one often considers in a metrology setting, since it is

equivalent to measuring a phase. However, the action of measuring the stopwatch disturbs its internal dynamics, thus changing the outcome statistics of later time measurements.

On the other hand, one can consider a ticking clock which, roughly speaking, is a device which emits ticks at approximately regular intervals. A typical wall clock is a good classical example. Here one can listen or watch the clock face and will know in real-time when it ticks. Analogously to the above example, we will want our mathematical formulation of the ticking clock to allow continuous observations of whether it has ticked or not; without affecting its internal dynamics. Since there is no such requirement imposed on stopwatches, ticking clocks and stopwatches require very different mathematical formulations.

Ticking clocks and stopwatches are also physically very distinct objects. The following two examples illustrate this point quite nicely. Firstly, consider a race and measuring the elapsed time between the winner leaving the starting line and crossing the finishing line with a stopwatch. This task can also be carried out by a ticking clock, at least to an accuracy to within plus or minus the time between two consecutive ticks. However, what about if you arranged to meet a friend at a given location at, say, 13:00h tomorrow? If you were only equipped with a stopwatch with no other time reference, you would hopelessly fail to be on time. The reason for this negative predicament, is that you would have no external signal (like the winner crossing the finish line in the previous example) to know *when* to stop your stopwatch — when you eventually press the “stop button”, it may indicate that only 1 second has passed or maybe one week. One may hope to remedy this predicament by resetting their stopwatch immediately after it was stopped; and trying again while keeping a record of the previous outcome. However, this would only lead to a finite number of completely irregular instances when you would know what the time was. Consequently, you would almost surely be very late for the meeting with your friend.

The above hypothetical example involving the stopwatch, while conveying an important point, is a bit far fetched from our everyday experience since we do, in fact, always have access to ticking clocks — albeit bad ones — such as the visual difference between day and night. To study such scenarios, one could investigate a different type of time keeping device formed by

combining a stopwatch and a ticking clock to take advantage of the best properties of both. Atomic clocks are a good example of such devices. We will leave their study to future work.

If either the stopwatch or ticking clock is quantum mechanical in nature, then from a mathematical perspective, both of these devices output information from the clock on Hilbert space $\mathcal{H}_C$ to the “outside”. In the case of a ticking clock, it is advantageous to make this information transfer explicit within the model by including a register. At a conceptual level, the temporal information stored in the register is accessible via measurements which, ideally, should not disturb the dynamics of the clockwork. This latter property is important since it safe-guards against inadvertently hiding potential disturbances to the clockwork in measurements on the register. In the case of a stopwatch, this information retrieval from the clock via a POVM is passive, i.e. its retrieval is triggered by an *external* signal, and the clock reacts passively to the measurement [5]. This is in contrast to the information transfer to the register in the case of a ticking clock, in which the information transfer is triggered *internally* by the clock mechanism itself, with no help from external triggers. Both stopwatches and ticking clocks can be modelled by multipartite Hilbert spaces. The most two common elements we will discuss concern the bipartition $\mathcal{H}_C \otimes \mathcal{H}_{R_T}$. In keeping with the terminology of [7], we will refer to the space $\mathcal{H}_C$ as the *clockwork* while $\mathcal{H}_{R_T}$ will be called the *register*. In both cases, one can define a one-parameter channel $\mathcal{M}_{C \rightarrow C}^t : L(\mathcal{H}_C) \rightarrow L(\mathcal{H}_C)$, where $t \in \mathcal{S}_{ct}$ is *coordinate time*. The exact nature of the register depends heavily on the particulars of the model, and will become less opaque in the coming sections.

In these models, one should think of coordinate time as some unknown parameter which increases as time advances. It is used as a bookkeeping parameter, i.e. it is assumed that the channels $\mathcal{M}_{C \rightarrow C}^t(\rho_C)$, $\mathcal{M}_{C \rightarrow C}^t(\rho'_C)$ of two distinct stopwatches or ticking clocks correspond to the state of the clocks at the same “time”. In these models we assume that the ticking clock or stopwatch is initiated at a particular time, (i.e. that there is a minimum t for which the channel is defined for; which w.l.o.g., we can set to zero). The motivation is that, as we will see, ticking clocks emit temporal information to the outside in an irreversible fashion. Furthermore, time may be fundamentally continuous or discrete. These two separate cases are conveniently modelled by defining $\mathcal{M}_{C \rightarrow C}^t$ for $t \in \mathcal{S}_{ct} = [0, \infty)$ or $t \in \mathcal{S}_{ct} = (0, \delta, 2\delta, \dots)$ respectively. Here $\delta > 0$ is some fixed parameter which allows (if desired) to define a continuous time clock from a discrete one, by taking the limit $\delta \rightarrow 0^+$ in an appropriate way. These two cases will be referred to as discrete coordinate time and coordinate time respectively. Finally, observe that we should not think

of the coordinate time as being physical, in the sense that we could define a new coordinate time through a change of variable $t' := f(t)$ for some strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ so long as the ticking clocks when parametrised by t' rather than t satisfy the to-be-defined in section 4, axiomatic definition of a ticking clock.

For concreteness, the rest of this paper will concern the nature of ticking clocks. Conceptually, the goal of the clockwork is to provide the timing; changing the state of the register at the right moments — analogously to how the clockwork in a wall clock is the mechanism which moves the clock hands to produce ticks. As such, it should not need any timing from the “outside”. This physical requirement has been captured mathematically in previous models [1, 7] by requiring that the clockwork be Markovian (also known as a divisible channel), meaning

$$\mathcal{M}_{C \rightarrow C}^{t_1+t_2} = \mathcal{M}_{C \rightarrow C}^{t_1} \circ \mathcal{M}_{C \rightarrow C}^{t_2}, \quad (1)$$

for all $t_1, t_2 \in \mathcal{S}_{ct}$. This condition has been justified by considering the opposite scenario: suppose that $\mathcal{M}_{C \rightarrow C}^t$ were not divisible, i.e. eq. (1) does not hold for some $t_1, t_2 \in \mathcal{S}_{ct}$. Then, the channel being applied per unit of coordinate time (discrete or continuous), would have to depend on knowledge of the value of coordinate time itself. In other words, the device may need an additional time reference external to the setup. However, by definition the clockwork is supposed to contain all sources of timing necessary for the ticking clock to function. Requirement eq. (1) can be verified using the techniques discussed in [15]. This equation is discussed further in section 3.1.

Demanding eq. (1) has some immediate consequences, the most important of which is that the clockwork is fully determined at all times by the “smallest coordinate time step”. In the case of continuous coordinate time, under appropriate continuity assumptions, this reads:

$$\mathcal{M}_{C \rightarrow C}^t = \lim_{\delta \rightarrow 0^+} (\mathcal{M}_{C \rightarrow C}^\delta)^{\circ \frac{t}{\delta}} \quad t/\delta \in \mathbb{Z} \quad (2)$$

for all $t \in \mathcal{S}_{ct}$. In the case of discrete coordinate time, eq. (2) holds if one does not take the limit $\delta \rightarrow 0^+$. The $\circ$ in the power in eq. (2) represents composition of the channel $\mathcal{M}_{C \rightarrow C}^\delta$ with itself t/δ times. We will use this notation throughout this paper. The authors of [7] construct a ticking clock model by describing how the clockwork they introduce interacts with a register.

Paper overview: We review the model [7] in section 2 followed by discussing its downsides. Then in section 3 we discuss two important principles for ticking clocks, followed by the axiomatic definition of a new ticking clock model in section 4. This new model satisfies the conditions introduced in section 3, while overcoming the shortcomings discussed in section 2.

We then discuss some important properties and definitions for the new ticking clock before moving on to section 5, where we formulate a channel representation for the axiomatically defined ticking clock. We then show how the ticking clock representation admits an autonomous implementation, and prove some properties for its clockwork. In section 6 we show how previous examples of clocks from the literature are either special cases of the new formulation or can easily be adapted to fit into it. In section 7 we discuss measures of accuracy for the clock and conclude with a summary of the highlights and outlook in section 8.

2 The ticking clock model of [7]

2.1 Model description

The authors start by describing their model for a discrete coordinate time ticking clock.¹ The continuous coordinate time ticking clock, is then realised by allowing the discrete time step parameter δ to tend to zero while demanding a certain continuity condition.

We start with their notion of the tick register with Hilbert space $\mathcal{H}_{R_T}$ for a ticking clock. This is a memory to which the temporal information coming from the clockwork is recorded. The register is formed by a tensor product space, $\mathcal{H}_{R_T} = \mathcal{H}_{R_1} \otimes \mathcal{H}_{R_2} \otimes \mathcal{H}_{R_3} \otimes \dots$ over local registers $\mathcal{H}_{R_i}$ which are all isomorphic to a fixed register $\mathcal{H}_{R_1}$.

The authors define a channel $\mathcal{M}_{C \rightarrow CR_1}^\delta : L(\mathcal{H}_C) \rightarrow L(\mathcal{H}_C \otimes \mathcal{H}_{R_1})$ for some fixed $\delta > 0$. This gives rise to the state of the register after N applications of the channel (N discrete coordinate time steps); denoted $\rho_{R_T}(N) = \rho_{R_1 R_2 R_3 \dots}$. Its local states are given by

$$\rho_{R_l} := \begin{cases} \text{tr}_C [\mathcal{M}_{C \rightarrow CR_1}^{l\delta} (\rho_C^0)], & \text{for } l = 1, 2, \dots, N, \\ \rho_{R_l}^0, & \text{for } l = N + 1, \dots \end{cases} \quad (3)$$

where $\rho_{R_l}^0$ is the l^{th} initial local state, and $\mathcal{M}_{C \rightarrow CR_1}^{l\delta}$ is defined recursively by applying the channel of the clockwork l times: $\mathcal{M}_{C \rightarrow CR_1}^{l\delta} := \mathcal{M}_{C \rightarrow CR_1}^\delta \left(\text{tr}_{R_1} \left[\mathcal{M}_{C \rightarrow CR_1}^{(l-1)\delta} (\rho_C^0) \right] \right)$, $l \in \mathbb{N}_{>0}$.

As discussed in section 1, every application of the channel $\mathcal{M}_{C \rightarrow CR_1}^\delta$ needed in the construction of eq. (3) corresponds to one time step of discrete coordinate time. However, eq. (3) does not yet constitute the ticking clock model in [7]; it requires one more ingredient — a so-called *gear system*. It has a corresponding channel, called the gear system channel, denoted $G_{R_T \rightarrow R_T}$, which moves every local register site to the

left by one local site in-between every application of the clockwork channel; see fig. 1.

As depicted in fig. 1, we can think of the clockwork initially interacting with the 1st register, with all the other registers to the right. It is convenient to only keep track of the register which the clockwork is currently acting on and those which are to its right. Keeping to this convention, the gear system channel $G_{R_T \rightarrow R_T}$ when applied to a product register state

$$\rho_{R_T} = \rho_{R_1}^{(1)} \otimes \rho_{R_2}^{(2)} \otimes \rho_{R_3}^{(3)} \otimes \dots \quad (4)$$

m times achieves $G_{R_T \rightarrow R_T}^{om}(\rho_{R_T}) = \rho_{R_{1+m}}^{(1+m)} \otimes \rho_{R_{2+m}}^{(2+m)} \otimes \rho_{R_{3+m}}^{(3+m)} \otimes \dots$

The gear system is an integral part of the ticking clock, since without it, the clockwork would not be able to access all the local registers; see fig. 1. It may be a mechanical system such as a rack and pinion, or non mechanical such as a kinetic degree of freedom associated with the register, turning it into flying qubits on a line in rectilinear motion.

Thus the gear system moves the register along as if it were on a conveyor belt in the following sense: for the 1st application of the channel $\mathcal{M}_{C \rightarrow CR_1}^\delta$, the clockwork interacts with the 1st register $\mathcal{H}_{R_1}$ and the registers are then instantaneously moved to the left by one local register site via the gear system so that the clockwork now interacts with the 2nd register $\mathcal{H}_{R_2}$ for the second application of the channel $\mathcal{M}_{C \rightarrow CR_1}^\delta$. The gear system then moves all the register sites by one site to the left as before and the process is repeated indefinitely. The local states of the register $\text{tr}_C [\mathcal{M}_{C \rightarrow CR_1}^{l\delta} (\rho_C^0)]$, generated via the l^{th} application of the channel, determine whether a tick occurred or not. The intuition is that the clockwork is releasing some temporal information at every application of the channel, so that after a sufficiently large number of applications of the channel it will contain too little temporal information to be useful and the register states ρ_{R_l} for sufficiently large l , contain very little temporal information. The registers could contain tick/no-tick information by having the channel write “0” to the l^{th} register, $\text{tr}_C [\mathcal{M}_{C \rightarrow CR_1}^{l\delta} (\rho_C^0)] = |0\rangle\langle 0|_{R_l}$, in the case of no-tick, or a “1” in the case of a tick, $\text{tr}_C [\mathcal{M}_{C \rightarrow CR_1}^{l\delta} (\rho_C^0)] = |1\rangle\langle 1|_{R_l}$. Here $|0\rangle, |1\rangle$ are two orthogonal states.

The authors then introduce a continuous coordinate ticking clock by demanding that the channel $\text{tr}_{R_1} [\mathcal{M}_{C \rightarrow CR_1}^\delta]$ satisfies an ϵ -continuity condition. That is to say, there exists $\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ such that,

$$\|\text{tr}_{R_1} [\mathcal{M}_{C \rightarrow CR_1}^\delta] - \mathcal{I}_C\|_\diamond \leq \epsilon(\delta), \quad (5)$$

where $\epsilon(\delta) \rightarrow 0^+$ as $\delta \rightarrow 0^+$. $\mathcal{I}_C$ is the identity channel on the clockwork, and $\|\cdot\|_\diamond$ is the diamond norm. The authors then specify that one applies the channel $\mathcal{M}_{C \rightarrow CR_1}^\delta$ a number of times which is proportional to

¹The authors of [7] do not use the terminology “ticking clocks” and instead refer to their devices as “clocks”. In keeping with the terminology introduced in this paper, we will use the former denomination.

Figure 1: In figures a) to c), square boxes indicate the local register sites while the clockwork acting on a local register site is indicated via a blue arrow.

a) Depiction of the ticking clock for $t = \delta$. The clockwork channel is applied once and the output on the register, ρ_{R_1} , written to the 1st register site.

b) Depiction of the ticking clock for $t = 2\delta$. Observe that the register has been shifted by one local register site to the left via an application of the gear system channel and $\rho_{R_2}^0$ has been transformed into ρ_{R_2} via one application of the clockwork channel.

c) Depiction of the ticking clock for $t = N\delta$. Observe that the register has been shifted by N local register sites when compared with a), the first N local register sites have been written to and the clockwork channel has been applied N times in total.

$\delta\epsilon$ with δ of order $1/\epsilon$, so as to achieve non-trivial ticking clock dynamics. The continuum limit case was further studied in [1] with a few additional physically motivated constraints introduced. See fig. 2 a) for a depiction of the combined clockwork and gear system channels.

2.2 Drawbacks

Unlike the clockwork channel, for which the resources required for its implementation such as energy or dimensionality are studied, the gear channel is always considered to be a free resource whose implementation is not studied. However, if such a gear system channel existed, rather than applying it in tandem with the clockwork channel $\mathcal{M}_{C \rightarrow CR_1}^\delta$, one could arguably apply it in tandem with a much simpler channel — bypassing the clockwork altogether — and achieve an *idealised ticking clock*. In this instance, this means a ticking clock for which $\text{tr}[\rho_{R_T}(t)\rho_{R_T}(t')] = \hat{\delta}(t - t')$ for all $t, t' \in \mathcal{S}_{ct}$, where $\hat{\delta}(\cdot)$ is the Dirac-delta func-

Figure 2: a) Illustration of the ticking clock model in [7]: All the local register site qubits $\sigma_{R_1}, \sigma_{R_2}, \dots$ are initially set to $|0\rangle|0\rangle$. The channel $\tilde{\mathcal{M}}_{CR_T \rightarrow R_T}^\delta := G_{R_T \rightarrow R_T}(\text{tr}_C[\mathcal{M}_{C \rightarrow CR_1}^\delta(\rho_C^0)] \text{tr}_{R_1}[\cdot] \otimes \mathcal{I}_{R_1}(\cdot) \otimes \mathcal{I}_{R_1}(\cdot) \otimes \dots)$, where $\mathcal{I}_{R_1}(\cdot)$ is the identity channel, is then applied repeatedly at times $t = \delta, 2\delta, 3\delta, \dots$. The state of the register at some fixed time $t > 0$ is obtained by setting $N = t/\delta$ and taking limit $\delta \rightarrow 0^+$. The number of 1's corresponds to the number of ticks which have occurred in time interval $[0, t]$. b) Same scenario as in a) but now swapping the clockwork channel $\text{tr}_C[\mathcal{M}_{C \rightarrow CR_1}^\delta(\rho_C^0)] \text{tr}_{R_1}[\cdot]$ with the Pauli X channel $\sigma_X(\cdot)$ which maps $|0\rangle|0\rangle$ to $|1\rangle|1\rangle$. The register now records the time with zero error, even though the Pauli X channel produces no temporal information — unlike the channel it replaced. We thus see that all the temporal information comes solely from the gear system $G_{R_T \rightarrow R_T}$. Alternatively, the same observation holds when using $|1\rangle|1\rangle_{R_1} \text{tr}_{R_1}[\cdot]$, rather than $\sigma_X(\cdot)$. Even in scenario a), the gear system is functioning as a perfect stopwatch: by counting the number of zeros between ticks, one can determine precisely the coordinate time interval between ticks.

tion in the continuous coordinate time limit, and a Kronecker-delta in the discrete coordinate time case. Physically speaking, this means that the state of the register at any two different coordinate times are orthogonal to each other. Consequently, the coordinate time at any given instance can be determined exactly from the registers via measurement. See fig. 2 b) for an illustration of this drawback.

This highlights the 1st drawback with such a model: the gear system channel — while it is not supposed to contain temporal information — actually can function like an idealised ticking clock. While this argument is rather indirect, the following argument is direct in the sense that it applies to all ticking clocks of the authors using the gear system in conjunction with the clockwork as intended.

If one has a ticking clock, and no other resource, then they should arguably *not* be able to determine the precise time between ticks — to do that, one would need an additional time keeping device, such as a very precise stopwatch. However, by simply counting the number of zeros between the ones in the register, one can determine precisely the time between ticks. This is a direct consequence of the gear system moving the register along by one qubit sequentially in perfect tandem with the passing of coordinate time. Observe also that this holds independently of how regular the ticks are — it could be a ticking clock which is very accurate and the ticks occur at highly regular intervals, or very imprecise with ticks occurring randomly with respect to coordinate time. It is also important in this argument that one can measure all the local registers to access all the zeros and ones. This is an assumption in their model. In the continuum limit, this would require accessing an infinite amount of information. How one would do so, is another open question.

One might hope to remedy these drawbacks by simply removing the gear system altogether and allowing the clockwork to always write to the same initial qubit register at all times. While this indeed means that the number of zeros (“no-ticks”) emitted by the clockwork between the ones (“ticks”) is now not recorded in the register as one would like, the state of the register in the instances when ticks occur will now be the same regardless of how many times the clock has ticked. Likewise, the register will also be in the same state ($|0\rangle\langle 0|_{R_I}$) between any two consecutive ticks. As such, in the continuous time limit, the register will be in the state $|1\rangle\langle 1|_{R_I}$ a measure zero amount of time, and in the state $|0\rangle\langle 0|_{R_I}$ almost always. This behaviour is clearly problematic and at odds with that of familiar ticking clocks, such as a wall clock.

Ideally, one satisfactory option would be to assign just one qubit of memory to which all the zeros (“no ticks”) information is written to, and a new register qubit to be allocated to the output of the clockwork

every time the ticking clock ticks. Therefore, at any instance, one would be able to determine the time by simply reading the number of ones in the register, but since the no tick information is always overwritten, one would not be able to determine how much coordinate time has passed between ticks. This hypothetical solution is unfortunately not possible, since it would require the gear system channel $G_{R_T \rightarrow R_T}$ to know when the clockwork is going to tick, yet since it acts solely on the register, it cannot do so. This highlights the difficulty of removing the gear system or altering its behaviour in a beneficial way. In section 4, we will show, via explicit construction, a satisfactory solution.

The above described drawback holds true for both the discrete and continuous time coordinate ticking clocks. In the continuous time coordinate case, there is also another drawback related to its memory storage. We will leave its discussion to section 3.2.

Note that the authors do suggest that a high precision gear system is not needed. Their argument is based on assuming that the gear system can fail with some probability p , where fail means that the gear channel $G_{R_T \rightarrow R_T}$, is replaced with the channel $\tilde{G}_{R_T \rightarrow R_T} = p\mathcal{I}_{R_T} + (1 - p)G_{R_T \rightarrow R_T}$, with $\mathcal{I}_{R_T}$ the identity channel on $\mathcal{H}_{R_T}$. However, their reasoning is based on the fact that replacing the channel $G_{R_T \rightarrow R_T}$ with this one, incurs (at most) an irrelevant change in the state of the clockwork at arbitrary coordinate times $t \in \mathcal{S}_{ct} = (0, \delta, 2\delta, \dots)$. However, there are several issues with this approach. On the one hand, no study of the induced change in the register is produced; yet the accuracy of the ticking clocks according to their measure (the Alternative Ticks Game), is solely a function of the register states on $\mathcal{H}_{R_T}$ in the large coordinate time limit. Second, in the continuous ticking clock limit ($\delta \rightarrow 0^+$), the gear system moves the register continuously, and any physically motivated gear system may produce errors which are irreconcilable with the error model described above. Furthermore, errors in the gear system are cumulative, and if the gear system writes the tick to an incorrect location in the register, it is possible to change the outcome of the alternative ticks game, thus changing its accuracy according to this measure.

3 Two basic principles for ticking clocks models: Finite running memory and Self-timing

We now present two basic principles to physically motivate descriptions of ticking clocks. The first is conceptually desirable, but arguably not necessary, while the second is more essential.

Figure 3: A depiction of a clock's register satisfying the finite running memory condition. The grid represents a section of the (possibly infinite dimensional) space $\mathcal{H}_{R_T}$. The dotted lines indicate that the grid may continue ad infinitum, while each square in the grid represents one qubit of memory. If the condition eq. (7) can be satisfied, then one can find a subspace in which $\hat{P}$ projects onto (the blue squares in the figure) which includes all — up to an arbitrarily small amount ϵ — of the changes produced in the register during the time interval $[0, t]$, such that $\hat{P}_\perp \rho_{R_T}(0) \hat{P}_\perp \approx \hat{P}_\perp \rho_{R_T}(t) \hat{P}_\perp$, i.e. that there has been effectively no change in the rest of the register (depicted by the white squares).

3.1 Self-timing

Understanding the underlying timing resources of a ticking clock is an important task. Otherwise, any physical implementation of it may require unaccounted for timing resources. Therefore identifying and quantifying such resources is important. A simple counter-example where the timing resources are unaccounted for, is a clock model with unitary dynamics governed by a time-dependent Hamiltonian over the clockwork and register.

One should distinguish the concept of self-timing from that of autonomy. An autonomous ticking clock can be thought of as one in which all resources for the clock to run can be explicitly accounted for. An example of an autonomous clock is [2]. Such ticking clocks are clearly also self-timing but the contrary is not necessarily true. The extent to which the ticking clock model presented in this paper is autonomous, will be discussed in sections 5 and 8.1.

We say that the (continuous coordinate time) ticking clock is *self-timing* if its one-parameter channel on the clockwork and register, $\mathcal{M}_{CR_T \rightarrow CR_T}^t$, is divisible:

$$\mathcal{M}_{CR_T \rightarrow CR_T}^{t_1+t_2} = \mathcal{M}_{CR_T \rightarrow CR_T}^{t_1} \circ \mathcal{M}_{CR_T \rightarrow CR_T}^{t_2} \quad (6)$$

for all $t_1, t_2 \geq 0$. The reasoning is that if this were not the case, then one could use systems alien to the register and clockwork to provide timing. In fact, after the identification of the register space $\mathcal{H}_{R_T}$, the smallest additional space one needs to include so that eq. (6)

is satisfied (if such a space exists), is a means with which to identify a clockwork space.

One may wonder why we do not demand this divisibility requirement directly for the clockwork channel, since indeed, the point of the clockwork is to provide all the timing — the register should be a passive element. However, we will see in proposition 3 that while the clockwork will indeed be divisible in most circumstances, there will be others in which it may not, yet the register will not be providing a source of timing in these cases.

This definition of self-timing differs from that of self-containment from [7]. In particular, the output in eq. (6) is on the entire register R_T . Therefore the self-contained ticking clocks from [7] are not necessarily self-timing according to the above definition. One can of course make them self-timing by including explicitly the channel for the gear system together with that of the clockwork; however, the estimates of the ticking clock's precision in [1, 7] are solely based on the properties of the clockwork alone, and thus do not take into account the properties of the gear system, yet the gear system itself can be used as an idealised clock (as discussed in section 2).

3.2 Finite running memory

A requirement for any realistic model of a ticking clock is that it only utilises finite resources per unit of coordinate time. In this section we introduce a definition which captures this notion for the clock's register by demanding that the clockwork can only invoke a finite change on it per unit of coordinate time.

We say that a ticking clock requires *finite running memory* if for every tuple $(\epsilon > 0, t > 0, \rho_{R_T}(0) \in \mathcal{S}(\mathcal{H}_{R_T}))$ there exists a projector $\hat{P}$ onto a *finite* dimensional subspace $\mathcal{H}_P \subseteq \mathcal{H}_{R_T}$ such that

$$\begin{aligned} & \|\rho_{R_T}(t) - \hat{P}\rho_{R_T}(t)\hat{P} - (\hat{P}_\perp \rho_{R_T}(t)\hat{P} + h.c.) \\ & - \hat{P}_\perp \rho_{R_T}(0)\hat{P}_\perp\|_1 \leq \epsilon, \end{aligned} \quad (7)$$

where $\rho_{R_T}(t) := \text{tr}_C [\mathcal{M}_{CR_T \rightarrow CR_T}^t(\rho_{CR_T})]$ is the state of the register at coordinate time t , and $\hat{P}_\perp := \mathbb{1}_{R_T} - \hat{P}$. See fig. 3 for a graphical illustration. Note that it is important that the condition holds for all initial register states on R_T , even if some initial register states are not relevant for the functioning of the ticking clock. This is because such states are physical and if the ticking clock is a physically realistic model, it should satisfy the finite running memory requirement even in these scenarios — regardless of whether the register is correctly encoding the information from the clockwork in such cases. This reasoning is analogous to why quantum information theorists demand quantum channels be completely positive rather than just positive — even if they do not intend to apply their channels on entangled states.

Figure 4: a) Register from [7]: in the continuous coordinate time limit ($\delta \rightarrow 0$), during any finite time interval (e.g. between two consecutive ticks), the clockwork needs to interact with an infinite number of qubit registers R_i — one at a time and sequentially. The blue triangle indicates the location of the clockwork relative to the register site it is writing to. The grey bar represents the infinite number of register sites. b) Depiction of a digital wrist watch. Here between seconds (or “ticks”) the digital display does not change, and during any finite coordinate time interval, the display exhibits a finite number of distinguishable states. This is analogous to the register in the new model presented here: between ticks, the register does not change.

Observe that the register in the ticking clock model of [7] does not satisfy the finite running memory requirement since the output on the register is independent of the initial register state, and the clockwork has to interact with infinitely many copies of the register subspace R_i in any finite proper interval of coordinate time; see fig. 4 a).

One may feel that infinite dimensional registers are physical since, indeed, spaces with continuous spectrum are physical. Consider for example the case in which the register is “a particle in a box” $\mathcal{H}_{R_T} = L^2[0, 1]$. One could in principle store an infinite amount of information in the box by partitioning it into infinitesimally small orthogonal compartments. However, due to technological constraints, such information would not be retrievable nor writable, and a more realistic setup would be to store only a finite amount of information in finitely many partitions — each one, containing an infinite number of orthogonal states. Any resolvable reader would then consist of a projective measure $\{\hat{P}_l\}_{l \in \mathbb{N}}$, where each $\hat{P}_l$ projects onto one compartment of the register. This

way, while each $\hat{P}_l$ may project onto an infinite dimensional subspace, one can never discern between different orthogonal states on the subspace. Under such a condition, the finite running memory condition given by eq. (7) should still hold when $\hat{P}$ is replaced with any linear combination of a *finite* number of projectors $\hat{P}_l$. What is more, we would also require that the ticking clock cannot *write* an infinite amount of information to every register subspace. One way to ensure this, would be to require that the ticking clock channel written in Kraus form, $\mathcal{M}_{C_{RT} \rightarrow C_{RT}}^t(\cdot) = \sum_n K_n(t)(\cdot)K_n^\dagger(t)$, has Kraus operators which admit an expression $K_n(t) = \sum_{l \in \mathbb{N}} \hat{\gamma}_{l,n}(t)\hat{P}_l$, for some operators $\hat{\gamma}_{l,n}(t) \in \mathcal{B}(\mathcal{H}_{C_{RT}})$.

For simplicity, the model we introduce in the following section will satisfy the former finite running memory condition; eq. (7). This is to say, the projectors $\hat{P}_l$ will project onto finite dimensional spaces. It could however be generalised to contain a register satisfying the latter condition also.

4 New ticking clock model

We now propose a ticking clock model through a set of physically motivated axioms. It will be self-timing and of finite running memory. Unlike the model discussed in section 2, it will be a continuous coordinate time model from the outset. We discuss its accuracy in section 7.

The following describes the extension of a clockwork channel $\mathcal{M}_{C \rightarrow C}^t$ to include the interaction with the register R_T . All the conditions regarding how the ticking clock functions will be laid-out in this section. While some of these will be similar to those of section 2, the new model will not assume any of the conditions nor setup from said section. For example, it will not require a gear system.

The tick register here is also different to that described in section 2. It consists of $N_T + 1$ orthonormal states ($|0\rangle_{R_T}, |1\rangle_{R_T}, |2\rangle_{R_T}, \dots, |N_T\rangle_{R_T}$) representing no tick, 1 tick, 2 ticks, $\dots$, $N_T \in \mathbb{N}_{>0}$ ticks respectively. While it is clear that any ticking clock with a finite dimensional tick register satisfies the finite running memory condition of section 3.2; in this case, the fulfilment of this condition is not inherently related to its finite dimensionality. Indeed, if one takes the infinite dimensional limit $N_T \rightarrow \infty$ in the ticking clock model in section 5 which results from the axioms of the current section, the resulting ticking clock *also* satisfies the finite running memory condition of section 3.2.

To start with, we describe a *periodic* register which resembles the familiar clock which repeats itself whenever the memory is full, e.g. every 12 or 24 hours. It naturally satisfies $|n\rangle_{R_T} = |n \bmod N_T + 1\rangle_{R_T}$, for $n \in \mathbb{Z}$. Later in this section we will consider a variant

of this.

In order for a device to be considered a ticking clock, it should satisfy some conditions on its clockwork and tick registers. After introducing the following shorthand notation, we discuss 5 such conditions. Let

$$\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\cdot) := \mathcal{M}_{CR_T \rightarrow CR_T}^t((\cdot) \otimes |k\rangle\langle k|_{R_T}) : \quad (8)$$

$$\mathcal{B}(\mathcal{H}_C) \mapsto \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{R_T}),$$

for $k = 0, 1, \dots, N_T$ denote the ticking clock channel with a finite dimensional clockwork when acting on the k^{th} register state.

1) *Time invariance symmetry condition:*²

$$\text{tr}_{R_T} \left[\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\rho_C) |k+l\rangle\langle k+l|_{R_T} \right] = \text{tr}_{R_T} \left[\mathcal{M}_{C \rightarrow CR_T}^{t,k'}(\rho_C) |k'+l\rangle\langle k'+l|_{R_T} \right] \quad (9)$$

for all $t \geq 0$, $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ and $l \in \mathbb{Z}$, $k, k' \in (0, 1, \dots, N_T)$ such that $k+l, k'+l \in (0, 1, \dots, N_T)$. Physically, this condition means that the dynamics of the clockwork is invariant under translation of the input and output states of the register by the same amount. This is what one expects from a ticking clock; e.g. the probability of a ticking clock ticking 2 hours in the future according to coordinate time (and the state of the clockwork at this time), given that the clock's register was initiated to 3pm, should be the same as if it were initiated to 6pm.

The instances of eq. (9) for which l is negative correspond to the state of the clockwork when the register is found to have “un-ticked”, i.e. that one finds the register to be in a state corresponding to an *earlier* time than it was initiated to, while coordinate time has increased.

2) *The self-timing condition:* For all $t \geq 0$, the ticking clock channel is Markovian:

$$\mathcal{M}_{CR_T \rightarrow CR_T}^{t_1+t_2} = \mathcal{M}_{CR_T \rightarrow CR_T}^{t_2} \circ \mathcal{M}_{CR_T \rightarrow CR_T}^{t_1} \quad (10)$$

for all $t_1, t_2 \geq 0$. This condition implies that no temporal information can come from systems alien to the clockwork and register. See section 3.1 for more details.

3) *The zeroth order condition:*

$$\mathcal{M}_{CR_T \rightarrow CR_T}^0 = \mathcal{I}_{CR_T} \quad (11a)$$

$$\lim_{t \rightarrow 0^+} \|\mathcal{M}_{CR_T \rightarrow CR_T}^t - \mathcal{I}_{CR_T}\| = 0. \quad (11b)$$

This condition simply states that if no time has passed, then no change is permitted in the ticking clock.

²In the following and throughout this paper, we will often omit tensor products with the identity operator on either C or R_T for brevity.

³This equation is known as uniform continuity in the semi-group literature [16].

The next condition concerns the probabilities of ticks. We denote the probability that the register is in the state corresponding to the l^{th} tick, at coordinate time t , given the $|k\rangle\langle k|_{R_T}$ register state as input, by

$$\tilde{p}_l^{(k)}(t) := \text{tr} \left[\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\rho_C) |l\rangle\langle l|_{R_T} \right]. \quad (12)$$

4) *The leading order condition:*

$$\lim_{t \rightarrow 0^+} \frac{\sum_{l=0}^{N_T} \tilde{p}_l^{(k)}(t)}{\tilde{p}_{f(k)}^{(k)}(t)} = 0, \quad (13)$$

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ where $f(k) = k+1 \bmod N_T + 1$. This condition imposes the constraint that the clock cannot “skip a tick”. More precisely, between a coordinate time at which k ticks have occurred, and a later coordinate time at which $l > k$ ticks have occurred, the probability that ticks $k+1, k+2, \dots, l-1$ have occurred is one.

Conditions 1) to 4) provide the necessary ingredients to define a ticking clock with a periodic register, but before doing so, it is advantageous to consider a distinct scenario which we call *cut-off register*.

In this scenario, the register will stop changing when it is full. For $N_T = 60$, an analogous wall clock would be one which you start at 8:00 and it stops ticking one hour later at 9:00. Both cut-off and periodic register models have clear advantages and disadvantages: While the periodic ticking clock will never stop ticking, one can only determine the time up to multiples of its period (although this can be circumvented by counting the ticks in real-time); but while this issue does not arise in the cut-off case, it is only useful for keeping track of time until it runs out of memory. Both types of register exhibit some common characteristics, see fig. 4 b).

While conditions similar to 2) and 4) can be defined for the cut-off register, due to the asymmetry in its boundary conditions, it is complicated to do so. A more direct and intuitive requirement is the following:

5) *The cut-off register condition:* for every ticking clock channel with a cut-off register $\mathcal{M}_{CR_T \rightarrow CR_T}^t$, there exists a ticking clock channel with a periodic register denoted $\tilde{\mathcal{M}}_{CR_T \rightarrow CR_T}^t$ and satisfying conditions 1) to 4), such that in the $t \rightarrow 0^+$ limit:

$$\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\rho_C) = \tilde{\mathcal{M}}_{C \rightarrow CR_T}^{t,k}(\rho_C) + o(t) \quad (14a)$$

for $k = 0, 1, \dots, N_T - 1$ and

$$\text{tr}_C \left[\mathcal{M}_{C \rightarrow CR_T}^{t,N_T}(\rho_C) \right] = |N_T\rangle\langle N_T|_{R_T} + o(t), \quad (14b)$$

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ and where $o(\cdot)$ is little-o notation. This requirement captures some of the behaviour of the ticking clock with a periodic register, while enforcing the condition that given the last register state

$|N_T\rangle\langle N_T|$ as input, it can no longer invoke a change in the register — i.e. it “stops ticking”.

After these general remarks, we are now ready to state the technical definition of a ticking clock:

Definition 1 (Ticking clock). *A ticking clock is a pair $(\rho_{\text{CR}_T}^0, (\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)_{t \geq 0})$, with $\rho_{\text{CR}_T}^0 \in \mathcal{S}(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_T})$ the state of the clockwork and register at coordinate time $t = 0$, and where the interaction between the clockwork and register, governed by the channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t$, satisfies conditions 1) to 4) in the case of a periodic register, and conditions 2) and 5) in the case of a cut-off register.*

Observe that for a ticking clock with a cut-off register, the state of the clockwork given the final state of the register $|N_T\rangle\langle N_T|$ as input, is of no relevance, since it does not affect the state of the register (and hence the tick statistics) anymore. We emphasize this point with the following definition.

Definition 2 (Clockwork equivalence). *Two ticking clocks with a cut-off register are said to be clockwork equivalent if the two following conditions are both satisfied:*

- a) *Their underlying ticking clock channels with a periodic register in eq. (14a), namely $\tilde{\mathcal{M}}_{\text{CR}_T \rightarrow \text{CR}_T}^t$, are the same in both cases.*
- b) *The states of their clockwork when inputting the register state $|N_T\rangle\langle N_T|$, namely*

$$\text{tr}_{\text{R}_T} \left[\mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t, N_T}(\rho_C) \right], \quad (15)$$

differ for some $t \geq 0$ and $\rho_C \in \mathcal{S}(\mathcal{H}_C)$.

At present the registers are a collection of $N_T + 1$ orthonormal states without spatial structure. One may furthermore demand that local structure is given to the register states via some distance measure $\text{Dis}(|l\rangle_{\text{R}_T}, |m\rangle_{\text{R}_T}) \geq 0$ that satisfies $\text{Dis}(|l\rangle_{\text{R}_T}, |m\rangle_{\text{R}_T}) < \infty$ for all $l, m = 0, 1, \dots, N_T$. This requirement is physically motivated by noting that when it is imposed, and the register has a local structure, condition 4) (eq. (13)) implies that the clockwork does not have to “travel” an *infinite* distance in finite time to write the next tick to the register — which would be unphysical. Furthermore, one can minimise the “speed of sound” in the register by arranging the local sites on the register so that $\text{Dis}(|l\rangle_{\text{R}_T}, |m\rangle_{\text{R}_T}) = g(|l - m|)$ for some monotonically increasing function g in the case of a cut-off register.⁴ The simplest example of such a set-up is when the register is embedded into $\mathcal{H}^{\otimes N_T+1}$ where $\mathcal{H}$ is the space of a qubit spanned by $|0\rangle, |1\rangle$ and we identify $|n\rangle_{\text{R}_T} = |1\rangle^{\otimes n} \otimes |0\rangle^{\otimes N_T+1-n}$, and $\text{Dis}(|l\rangle_{\text{R}_T}, |m\rangle_{\text{R}_T}) = |l - m|$; $l, m = 0, 1, \dots, N_T$. In

⁴In the periodic register case, one would use a period version of this.

this case, at the instance when the $(k+1)^{\text{th}}$ tick occurs, the clockwork “flips” the qubit “next to” the k^{th} qubit.

Temporal information emitted from a clock in the standard treatment is classical in nature. Therefore, one should foremost consider *classical registers* R_T . These are registers for which the action of the clockwork on them does not produce coherence in the fixed basis $\{|0\rangle_{\text{R}_T}, |1\rangle_{\text{R}_T}, \dots, |N_T\rangle_{\text{R}_T}\}$, when the register input is diagonal in this basis; namely dynamics of the form:

$$\mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t, k}(\rho_C^0) = \sum_{n=0}^{N_T} \rho_C^{(n; k)}(t) \otimes |n\rangle\langle n|_{\text{R}_T}, \quad (16)$$

for all $k = 0, 1, \dots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$, where $\rho_C^{(n; k)}(t)$ are arbitrary subnormalised states on the clockwork. Considering dynamics of this form is also equivalent to assuming that the register is measured in the basis $\{|0\rangle_{\text{R}_T}, |1\rangle_{\text{R}_T}, \dots, |N_T\rangle_{\text{R}_T}\}$ associated with ticks, since in this case any coherence will be destroyed and does not affect the measurement. Therefore, requesting dynamics of the form eq. (16) is not to be confused with demanding that the register is itself classical, but moreover can be interpreted as requiring that we are only allowed to extract classical information from it. We will therefore take eq. (16) as the defining property of a classical register:

Definition 3 (Classical register). *A ticking clock $(\rho_{\text{CR}_T}^0, (\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)_{t \geq 0})$ has a classical register if the channels $\mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t, k}$ are of the form eq. (16) for all $t \geq 0$, $k = 0, 1, \dots, N_T$.*

Classical registers have the advantageous property that they can be “continuously observed” without changing the properties of the ticking clock — analogously to how one can continuously look at a wall clock or listen for its ticks without disturbing the dynamics of its clockwork (and hence accuracy). This may come as a surprise for two reasons: For one, clearly the state of the ticking clock in eq. (16), before and after measuring the register in the basis $\{|n\rangle_{\text{R}_T}\}$ is different. Secondly, the Zeno effect [17–19] dictates that if a quantum system is continuously measured, then it will stop evolving altogether — which is clearly at-odds with the desired properties of a ticking clock.

The solution to the 1st apparent problem is to recall that the state of the register is a probabilistic mixture and thus the change in the state due to the register’s measurement is due to a change in our knowledge about which state the register is in. This is analogous to the description of any purely classical ticking clock which is not perfectly accurate: while in every run of the ticking clock in which it is continuously observed, the state of the register will always be known exactly; in order to calculate the statistics associated with its accuracy, one will need the ensemble of all possible

ticking clock runs weighted by the probability that each trajectory occurs. The 2nd apparent problem is resolved by showing that the Zeno effect does not apply to continuous measurements of the register when it is of the form eq. (16).

The following proposition formalises the previous two remarks by showing that one can continuously measure the register without affecting the statistics associated with the probabilistic distribution of ticks. Before stating it, we need to introduce some notation and definitions:

Let $\mathcal{P}_l(\cdot) := |l\rangle\langle l|_{\mathcal{H}_T}(\cdot)|l\rangle\langle l|_{\mathcal{H}_T} / \text{tr}[(\cdot)|l\rangle\langle l|_{\mathcal{H}_T}] : \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{R}_T}) \rightarrow \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{R}_T})$ denote the channel which takes any ticking clock state and outputs the state of a ticking clock after measuring the register in the register basis $\{|0\rangle_{\mathcal{R}_T}, |1\rangle_{\mathcal{R}_T}, \dots, |N_T\rangle_{\mathcal{R}_T}\}$ and finding the register to be in the state $|l\rangle_{\mathcal{R}_T}$.

Definition 4 (Measured channels). *Given a ticking clock $(\rho_{\mathcal{C}\mathcal{R}_T}^0, (\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t)_{t \geq 0})$, we call the following channel $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{R}_T}) \rightarrow \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{R}_T})$ a measured channel:⁵*

$$\mathcal{C}\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t[\mathbf{s}_N](\cdot) := \bigcirc_{n=1}^N (\mathcal{P}_{l_n} \circ \mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^{t_n})(\cdot), \quad (17)$$

where $\mathbf{s}_N := (l_n, t_n)_{n=1}^N$ is the sequence of measurement outcome indices $l_n = 0, 1, \dots, N_T$ and times $t_n \geq 0$; s.t. $\sum_{n=1}^N t_n = t$. In the case that $\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t$ has a classical register we call the channel a classical register measured channel.

The channel eq. (17) corresponds to the state of the ticking clock at coordinate time t when the free evolution of the ticking clock was interrupted at times t_n by register measurements with outcomes $|l_n\rangle\langle l_n|_{\mathcal{R}_T}$. Let $\text{Prob}[\mathbf{s}_N]$ be the probability that the sequence of outcomes with indices $l_1, l_2, \dots, l_N$ at times $t_1, t_2, \dots, t_N$ occurs. We denote the set of all sequences of outcomes at times $(t_1, t_2, \dots, t_N) =: \mathbf{t}$ by

$$\mathbf{S}_N(\mathbf{t}) := \{(l_n, t_n)_{n=1}^N : l_n \in \{0, 1, \dots, N_T\}\}. \quad (18)$$

Proposition 1 (Measured register equivalence). *For all coordinate times $t_n \geq 0$ s.t. $\sum_{n=1}^N t_n = t$ and for all $N \in \mathbb{N}_{>0}$, the dynamics of any ticking clock with a classical register is equal to that of the ensemble of classical register measured channels, where the ensemble is weighted by the probability of the classical register measured channel occurring:*

$$\begin{aligned} & \mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t(\rho_C \otimes |k\rangle\langle k|_{\mathcal{R}_T}) \\ &= \sum_{\mathbf{s}_N \in \mathbf{S}_N(\mathbf{t})} \text{Prob}[\mathbf{s}_N] \mathcal{C}\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t[\mathbf{s}_N](\rho_C \otimes |k\rangle\langle k|_{\mathcal{R}_T}) \end{aligned} \quad (19)$$

⁵We use the notation $\bigcirc_{n=1}^N f_n(\cdot) := f_N \circ f_{N-1} \circ \dots \circ f_1(\cdot)$.

for all $(t_n)_{n=1}^N$, $N \in \mathbb{N}_{>0}$, $t \geq 0$, $k = 0, 1, \dots, N_T$, and $\rho_C \in \mathcal{S}(\mathcal{H}_C)$.

See section B.1 for proof. A direct consequence of proposition 1 is that if we choose $t_n = t/N$ followed by taking the limit $N \rightarrow \infty$ for fixed t on the r.h.s. of eq. (19), we are in the regime of continuous measurements proposed in the Zeno effect. However, in this continuous measurement case, proposition 1 certifies that the ticking clock channel $\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t(\rho_C \otimes |k\rangle\langle k|_{\mathcal{R}_T})$ still adequately describes the statistics. Indeed, if the register starts in the state $|k\rangle\langle k|_{\mathcal{R}_T}$, the probability $\text{Prob}[\mathbf{s}_N]$ specialised to the case of finding the register in the state $|k\rangle\langle k|_{\mathcal{R}_T}$ for all time $t \in [0, \tau]$, for some $\tau > 0$ would have to be one if Zeno's mechanism were to hold. We will later see that it is however only true for some irrelevant trivial clocks.

For later purposes, it is useful to introduce a notion of a “classical ticking clock”. This notion of classicality is effectively the same as the one introduced in [1] but stated for the ticking clock introduced in this paper (definition 1).

Definition 5 (Classical ticking clock). *We call a ticking clock $(\rho_{\mathcal{C}\mathcal{R}_T}^0, (\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t)_{t \geq 0})$ classical, if there exists a basis $\{|l\rangle_l\}_l$ spanning the clockwork Hilbert space $\mathcal{H}_C$, for which the clockwork remains incoherent in this basis at all coordinate times:*

$$\text{tr}_{\mathcal{R}_T} [\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t(\rho_{\mathcal{C}\mathcal{R}_T}^0)] = \sum_l p_l(t) |l\rangle\langle l|_C, \quad \forall t \geq 0. \quad (20)$$

Likewise, we call a ticking clock a *quantum ticking clock* if it does not satisfy the classical ticking clock criterion. Thus unless otherwise specified, a ticking clock may be quantum or classical.

5 Autonomous dynamics

In this section we formulate a representation of the ticking clock channel $\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t$ which holds if and only if the ticking clock satisfies the axiomatic definition 1, up to some stated equivalence. An alternative — more technical in nature — representation is left to appendix section A. The highly inquisitive reader may want to detour to section A before continuing here, while the more cursory reader may avoid section A.

Proposition 2 (Explicit ticking clock representation). *The pair $(\rho_C^0, (\mathcal{M}_{\mathcal{C}\mathcal{R}_T \rightarrow \mathcal{C}\mathcal{R}_T}^t)_{t \geq 0})$ form a ticking clock (definition 1) with a classical tick register (definition 3), up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator H as well as two finite sequences of operators $(L_j)_{j=1}^{N_L}$ and $(J_j)_{j=1}^{N_L}$ on $\mathcal{B}(\mathcal{H}_C)$; which are all t independent, such*

that for all $t \geq 0$ and $N_T \in \mathbb{N}_{>0}$,

$$\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t(\cdot) = e^{t\mathcal{L}_{\text{CR}_T}}(\cdot), \quad (21a)$$

$$\begin{aligned} \mathcal{L}_{\text{CR}_T}(\cdot) = & -i[\tilde{H}, (\cdot)] + \sum_{j=1}^{N_L} \tilde{L}_j(\cdot) \tilde{L}_j^\dagger - \frac{1}{2} \{ \tilde{L}_j^\dagger \tilde{L}_j, (\cdot) \} \\ & + \sum_{j=1}^{N_L} \tilde{J}_j^{(l)}(\cdot) \tilde{J}_j^{(l)\dagger} - \frac{1}{2} \{ \tilde{J}_j^{(l)\dagger} \tilde{J}_j^{(l)}, (\cdot) \}, \end{aligned} \quad (21b)$$

where the operators are $\tilde{H} = H \otimes \mathbb{1}_{\text{R}_T}$, $\tilde{L}_j = L_j \otimes \mathbb{1}_{\text{R}_T}$, $\tilde{J}_j^{(l)} = J_j \otimes O_{\text{R}_T}^{(l)}$, with

$$O_{\text{R}_T}^{(l)} := |1\rangle\langle 0|_{\text{R}_T} + |2\rangle\langle 1|_{\text{R}_T} + |3\rangle\langle 2|_{\text{R}_T} + \dots + |N_T\rangle\langle N_T - 1|_{\text{R}_T} + l |0\rangle\langle N_T|. \quad (21c)$$

In the cut-off register case $l = 0$, while $l = 1$ for the periodic register case.

See B.3 for a proof. It follows straightforwardly from a more technical representation discussed in appendix

Observe that the Lindbladian in eq. (21b) only requires local coupling between the orthogonal register states, according to the distance measure $\text{Dis}(\cdot, \cdot)$ introduced in section 4 after definition 1. The dynamics of the register also clearly satisfies the finite running memory condition in section 3.2.

While the ticking clock model in [7], in the case of continuous coordinate time t , does have a dynamical semigroup representation from the clockwork and individual tick registers to itself, $L(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_j}) \rightarrow L(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_j})$ for all $j \in \mathbb{N}$, (see [1]) a dynamical semigroup representation on the clockwork and the total tick register has not been shown to exist. As explained previously in section 2, since the dynamics of the register is dependent on the details of the gear system in their model, such a formulation would inevitably need to include a description of the gear system used, which, for any realistic gear system, would arguably lead to unaccounted for sources of error.

The following proposition shows that an effective Markovian dynamical semigroup for the clockwork can always be found. It therefore justifies associating the clockwork with the sole source of timing.

Proposition 3 (Clockwork representation). *Consider a ticking clock with a classical periodic register (definitions 1 and 3) written in the representation of proposition 2. Its clockwork channel, defined via*

$$\mathcal{M}_{C \rightarrow C}^t(\cdot) := \text{tr}_{\text{R}_T}[\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t((\cdot) \otimes |k\rangle\langle k|_{\text{R}_T})] \quad (22)$$

is k -independent for all $t \geq 0$ and of the form

$$\mathcal{M}_{C \rightarrow C}^t(\cdot) = e^{t\mathcal{L}_C}(\cdot), \quad (23)$$

with $\mathcal{L}_C$ equal to the r.h.s. of eq. (21b) under the replacements $\tilde{H} \mapsto H$, $\tilde{L}_j \mapsto L_j$ and $\tilde{J}_j^{(l)} \mapsto J_j$. What is more, for every ticking clock with a classical cut-off register written in the representation of proposition 2, there exists a ticking clock which is clockwork equivalent (definition 2), such that its clockwork is k -independent and given by eq. (23).

The proof is constructive and can be found in section B.4. In the case of the cut-off register, the representation used in the proof for which eq. (23) holds, has a ticking clock channel whose clockwork still produces ticks when the input register state is $|N_T\rangle\langle N_T|_{\text{R}_T}$, but does not write them to the register. This can be contrasted with the clockwork equivalent ticking clock with cut-off register representation in proposition 2. In this case, the clockwork stops producing ticks when the input register state is $|N_T\rangle\langle N_T|_{\text{R}_T}$.

Dynamical semigroups, such as the clockwork channel representation in eq. (23), do not have an inverse channel; and hence ticking clocks emit temporal information in an irreversible fashion.

Dynamical semigroups of the form eq. (21a) have been shown to have a microscopic description in which the system (which in the present case would constitute the clockwork and total register) interacts with an infinite dimensional environment on $\mathcal{H}_E$ via a time independent Hamiltonian under the appropriate limits. In particular, the Hamiltonian H_{tot} which leads to dynamics eqs. (21a) and (21b) is of the form

$$H_{\text{tot}} = H \otimes \mathbb{1}_{\text{R}_T E} - \tilde{H}_{\text{CR}_T} \otimes \mathbb{1}_E + \mathbb{1}_{\text{CR}_T} \otimes H_E + V, \quad (24)$$

where $\tilde{H}_{\text{CR}_T} \in \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_T})$ is tuned to counteract a shift in energy on the clockwork and register due to interactions with the environment with local Hamiltonian H_E , while V mediates the interaction between the register, clockwork, and environment. It takes on the form $V = \sum_n \mathbb{1}_{\text{R}_T} \otimes A_n^{(L)} \otimes B_n^{(L)} + O_{\text{R}_T}^{(l)} \otimes A_n^{(J)} \otimes B_n^{(J)}$ with $O_{\text{R}_T}^{(l)}$ given by eq. (21c). The $(A_n^{(L)})_n$ and $(A_n^{(J)})_n$ terms give rise to the operators $(L_j)_j$ and $(J_j)_j$ respectively while terms $(B_n^{(L)})_n, (B_n^{(J)})_n$ are suitably chosen local terms acting on the environment. There are two known types of limiting procedures which one can apply to eq. (24) to achieve dynamics of the form eqs. (21a) and (21b). One in which the time-scales of the environment are much shorter than those of the system — the so-called “weak coupling limit” — and the other where the time scales are reversed — called the “singular coupling limit”. See [20] for physical insight into these limits and [21] for how to re-scale the interactions to interchange between them. In [22] it is proven that all Lindblad operators in eq. (21b) are achievable via appropriate choice of the terms in eq. (24), in the singular coupling

limit. Finally, observe that the interaction term V in eq. (24) only requires local coupling to the register.

6 Ticking Clock Examples

In this section we will see how clocks from the literature are either a special case of the ticking clocks introduced here (definition 1), or that they can be easily adapted to be of this form. In all three examples, we provide the choice of H , $(L_j)_{j=1}^{N_L}$ and $(J_j)_{j=1}^{N_L}$ from proposition 2 for the cut-off register case. While the examples are also valid in the periodic case, they are distinct to those in the literature for this choice. The accuracy of these example clocks will be discussed in section 7.

6.1 Thermodynamic Ticking Clock

The clock in [2] is a ticking clock in which population is driven up a d dimensional “ladder” via the interaction with two thermal qubits maintained at thermal equilibrium at distinct hot and cold temperatures through their coupling to local thermal baths. A tick occurs when the population of the ladder reaches the top via spontaneous emission back to the ground state of the ladder. The tick is recorded in the tick register by a photo-detector. This register is of the same form as the one introduced here: it is classical in nature and is only written to when a tick occurs. It thus satisfies both the self-timing and finite running memory conditions of section 3. The thermodynamical clock in [2], is specified by choosing $N_L = 4$ with

$$L_1 = \sqrt{\gamma_h} \sigma_h, \quad L_2 = \sqrt{\gamma_h e^{-\beta_h E_h}} \sigma_h^\dagger, \quad (25a)$$

$$L_3 = \sqrt{\gamma_c} \sigma_c, \quad L_4 = \sqrt{\gamma_c e^{-\beta_c E_c}} \sigma_c^\dagger, \quad (25b)$$

$$J_1 = \sqrt{\Gamma} |0\rangle\langle d-1|_w, \quad J_2 = J_3 = J_4 = 0, \quad (25c)$$

where σ_h, σ_c are lowering operators for the hot and cold qubits respectively; $|0\rangle_w, |d-1\rangle_w$ are the ground and top states of the ladder, and the other coefficients are positive and defined in [2]. The Hamiltonian takes on the form $H = H_0 + H_{\text{int}}$ where H_0 is the local Hamiltonian for the qubits and ladder while H_{int} is the three-body interaction between them. Indeed, a simple calculation of the state of the clockwork $\rho_C^{(0)}(t)$ given that the 1st tick has not occurred in the time interval $[0, t]$, using eq. (36) and the above parameters yields $\rho_C^{(0)}(t) = e^{t\mathcal{L}_C^{(0)}}(\rho_C)$ with

$$\begin{aligned} \mathcal{L}_C^{(0)}(\cdot) = & i \left(\hat{H}_{\text{eff}}(\cdot) - (\cdot) \hat{H}_{\text{eff}}^\dagger \right) \\ & - \sum_{j=1}^4 \frac{1}{2} \left\{ L_j^\dagger L_j(\cdot) \right\} + L_j(\cdot) L_j^\dagger, \end{aligned} \quad (26)$$

where $\hat{H}_{\text{eff}} = H - i\Gamma |d-1\rangle\langle d-1|_w$. This is in exact correspondence with eq. B4 on page 8 of [2].

Figure 5: Qualitative plots for the Quasi-ideal ticking clock. Blue: magnitude of the amplitudes of the clockwork in the basis $(|t_j\rangle)_{j=1}^d$ at time $t = 0$. Orange: magnitude of amplitudes of the clockwork in the basis $(|t_j\rangle)_{j=1}^d$ around the time when the 1st tick is likely to occur. Green: coefficients $(V_j)_{j=1}^d$. Observe the the blue and green curves have very small overlap while the orange and green have a large overlap.

Since the clockwork is reset to its initial state after each tick, eq. (26) fully determines the statistics of all ticks as discussed in [2].

6.2 Quasi-ideal Ticking Clock

In [1] a ticking clock based upon the results from [8] was introduced in the context of the model from [7]. Here we consider the same clockwork but when its coupling to the register results from the axioms introduced in section 4, rather than the model [7]. It therefore satisfies definition 1 of a ticking clock. From [1] we have that $N_L = d$, the dimension of the clockwork and for all $j = 1, 2, \dots, d$:

$$L_j = 0, \quad J_j = \sqrt{2V_j} |\psi_C\rangle\langle t_j|, \quad (27)$$

where $(|t_j\rangle)_{j=1}^d$ is an orthonormal basis for $\mathcal{H}_C$. The state $\rho_C^0 = |\psi_C\rangle\langle\psi_C|$ is both the initial state of the clockwork and the state which it is reset to after each tick. It is called the Quasi-ideal clock state and follows a complex Gaussian distribution in the $(|t_j\rangle)_{j=1}^d$ basis. The coefficients $(V_j > 0)_{j=1}^d$ follow a peaked distribution; see [1] for details. The Hamiltonian H is a ladder Hamiltonian with equally spaced energy gaps and diagonal in the Fourier transform basis generated from $(|t_j\rangle)_{j=1}^d$.

The free dynamics of the clockwork according to H allows the complex Gaussian amplitude distribution to “move coherently” in the $(|t_j\rangle)_{j=1}^d$ basis until the peak of the distribution overlaps with the peak of the distribution $(V_j > 0)_{j=1}^d$, at which point a tick occurs and the clockwork is reset and starts again (see fig. 5). Note that the statistics of the 1st tick are invariant under the exchange of the Lindblad operators in eq. (27) with the simpler form $L_j = J_j = 0$ for $j = 1, 2, \dots, d-1$ and $L_d = 0, J_d = \sum_{j=1}^d \sqrt{2V_j} |j\rangle\langle t_j|$.

Here $\{|j\rangle \in \mathcal{H}_C\}_j$ is any orthonormal basis since in this case the clockwork does not need to be reset after it ticks — as can be seen formally from eq. (36).

6.3 Ladder Ticking Clock

This clock is a classical ticking clock (definition 5) which was defined in [23] and proven to be the most accurate classical continuous coordinate time clock in [1] in the context of the model [7]. As with the example of section 6.2, it can easily be adapted to the ticking clock model in definition 1. We find $N_L = d$, with

$$L_j = |c_{j+1}\rangle\langle c_j|, \quad J_j = 0, \quad (28a)$$

$$L_d = 0, \quad J_d = |c_1\rangle\langle c_d|, \quad (28b)$$

for some orthonormal basis $\{|c_j\rangle\}_{j=1}^d$. The clock's initial state is $\rho_C = |c_1\rangle\langle c_1|$ and since it is a classical clock, the Hamiltonian term vanishes, $H = 0$.

In an appropriate limit, this clock can also be approximated by the thermodynamic clock. See [2] for details.

7 Measures of accuracy

We call an *accuracy measure* of a ticking clock, any quantity which can be written solely as a function of measurement outcomes of the register on $\mathcal{H}_{R_T}$ at different coordinate times. It can only depend on the state of the clockwork indirectly, via its coupling to the register, but not on the state of the clockwork directly. This is important since with the accuracy measure one wants to capture how good the clockwork is at emitting temporal information to the “outside”. Good examples of accuracy measures are those which depend solely on the tick delay functions. The k^{th} tick *delay function* is the probability density that $k - 1$ ticks occur during the coordinate time interval $[0, t)$ while the k^{th} tick occurs exactly at coordinate time t . It is worth noting that while in the theoretical description of the ticking clock model one has knowledge of the state of the register *and* the corresponding value of coordinate time, in an actual physical implementation of any ticking clock, one can only infer coordinate time indirectly. As such, any measure of accuracy may be hard to calculate experimentally and may require many repeated experiments involving multiple copies of any given ticking clock in order to garner enough statistics. This constraint of the model, is a virtue not a weakness however, since by definition the only information we should have access to (i.e. which should be stored in a register), is the information about ticks — not coordinate time directly. The storage of coordinate time in the register was an inherent drawback of earlier ticking clock models.

We now consider the cut-off register model, since in this case, the k^{th} tick delay function (for $k =$

$1, 2, 3, \dots, N_T$) takes on exactly the same expression in terms of how it is related to dynamics on the clockwork as presented in [1] for the model [7] (see section A for details). The clock accuracy of the k^{th} tick R_k , introduced in [1, 2], is the square of the ratio between the mean and standard deviation of the k^{th} tick delay function with respect to coordinate time t . As such, there is a one-to-one relation between the ticking clocks in [1] and the ones introduced in this paper for $k = 1, 2, \dots, N_T$. Subsequently, all the theorems about the accuracy of ticking clocks in [1] also apply to those introduced here with a cut-off register. For instance, the most accurate classical ticking clock (see section 6.3) satisfies $R_k = kd$ for $k = 1, 2, \dots, N_T$; $d \in \mathbb{N}_{>0}$; where d is the Hilbert space dimension of the clockwork. The thermodynamic clock in section 6.1 can also achieve a similar accuracy; see [2]. On the other hand, there exists a quantum ticking clock (see section 6.2) whose accuracy is lower bounded by

$$R_k = kR_1, \quad R_1 \geq d^{2-\varepsilon} + o(d^{2-\varepsilon}), \quad (29)$$

for all $k = 1, 2, \dots, N_T$ and for all fixed $\varepsilon > 0$ in the large d limit [1]. Recently it has been shown that this bound is essentially tight for ticking clocks which only tick once [24]. It remains an open question whether a quantum ticking clock which ticks more than once can have ticks which have a higher accuracy.

The alternative ticks game measure of accuracy [7, 23], is applicable to the ticking clock models developed here, with the difference that the referee will need to play the game on-the-fly, rather than comparing register states at the end of the protocol (as proposed in [7]), since the registers in the new model do not record the coordinate time corresponding to when the tick occurred.

8 Conclusions and Outlook

8.1 Conclusions

We started by discussing the ticking clock model of [7] which was one of the first theoretical models of a ticking clock. This revolutionary work inspired follow-up papers yet also some legitimate concerns from the community⁶ regarding the physicality of its foundations which we formalise and discuss. We then introduce an axiomatic definition of a new ticking clock based on physical principles and derive explicit solutions to its equations of motion. The aforementioned drawbacks do not apply to the new ticking clock model introduced here. In a nutshell, the main reasons are twofold: On the one hand the new formalism only requires finite running memory (see section 3.2), while on the other, the axioms imply the

⁶By “community” we refer to comments from anonymous referees and senior scientists during conference talks.

existence of a dynamical semi-group which describes the dynamics of the *entire* register and clockwork for all coordinate time. We furthermore show that the new equations of motion admit a fully autonomous realisation.

With every ticking clock, one can associate a set of delay functions which determine the accuracy measures of the ticking clock. We show that there is a one-to-one correspondence between the set of delay functions produced by the new ticking clock model introduced here and that of [7]. Consequently, bounds on the accuracy of the clocks in [7], derived in [1], apply also to the new ticking clock model presented here. Therefore, the main conclusion of [1], namely that quantum ticking clocks are more accurate than classical ones, applies also to the new ticking clock model presented here.

Another ticking clock model, based on thermodynamic principles was introduced in [2]. This model has many positive points, such as being fully autonomous and physically well motivated. It does however have some drawbacks, such as not being derived from first principles and having a reported accuracy which is substantially lower than that of the quantum ticking clock in [1]. Consequently, the results of this paper imply that both of the desirable properties of the ticking clock models [2, 7] are achievable in one physically transparent model: full autonomy and high accuracy. This is considered the most important conclusion of this paper. What's more we have seen that the ticking clock in [2] is in fact a special case of the ticking clock model introduced here.

Here by *autonomous* it is understood that for every ticking clock according to the new definition 1, there exists a large macroscopic environment, a time independent and local Hamiltonian over the clockwork, register, and environment, which represents said ticking clock. Note that this environment need not necessarily be thermal. Other possibilities such as a vacuum state, may turn out to be necessary in some cases. Whether indeed a (or several) thermal baths at various temperature(s) are sufficient for the most accurate clocks is an open question.

One would like to be able to continuously monitor the tick register of a ticking clock in real-time — analogously to how one can listen for the ticks of a wall clock in real-time. At first sight, this may seem impossible since the quantum Zeno effect dictates that continuous observation of a quantum system causes it to freeze its motion; which is clearly not a desirable property of a clock. We show that one can continuously observe the tick register without affecting its statistics and hence its accuracy.

8.2 Outlook

It is worth discussing some of the multiple future directions this work opens up. On the one hand there are questions such as what is the most accurate quantum clock, how much entropy is produced every time the ticking clock ticks, or how would one build such a ticking clock in practice. On the other hand, one can consider extensions to the formalism itself. One very practical such extension would be to include noise from the environment and the possibility of leveraging tools from quantum error correction to protect ticks against such adversarial noise. Such studies could also be applied to other types of extensions to the model. We provide 4 such examples:

1) Clocks in a network: One can readily extend the model introduced in this paper to take into account a source of external timing. One way to consider external timing in the case of a ticking clock was introduced in [10] and formulated in terms of the continuous time limit of the ticking clock of [7]. This extension could also be formulated for the ticking clocks introduced in this paper. We now discuss two other types of extensions which to date have not been considered in the literature thus far.

2) Relativistic ticking clock model: The ticking clock models in the literature (including this one), are not relativistic. Making them so would be an interesting endeavour. Finding a convincing axiomatic definition appears feasible, but since we do not have a fully credible theory of quantum gravity yet, solutions might turn out to be highly speculative. One approach is to attempt to make relativistic versions of the axioms for ticking clocks presented in section 4, by stating how the observable statistics and invariant quantities in these axioms transform relativistically. Another method would be to employ the same approach used in [9] to construct a relativistic quantum stopwatch. In such a semi-classical approach, one would include in the ticking clock set-up an additional kinematic degree of freedom associated with the ticking clock's momentum and position. One would then expand to leading order in relativistic corrections the general relativistic equations for time dilation, following a similar procedure to as in [25].

3) Relaxation of the axioms: One could consider variants of the ticking clock models presented here by changing or disregarding some of the conditions 1) to 5) in section 4. An obvious choice would be to consider “Time variant asymmetric ticking clocks”, namely those which do not satisfy condition 1) [eq. (9)]. An example of such a ticking clock channel with a classical register would be eqs. (21a) and (21b) in proposition 2 under the replacement $\tilde{J}_j^{(l)} \mapsto \sum_{k=0}^{N_T} J_{j,k} \otimes |k+1 \bmod N_T + 1\rangle\langle k|_{\text{RT}} \left(1 - \hat{\delta}_{k, N_T+l}\right)$, where $J_{j,k}$ are arbitrary operators on the clockwork. Note however that such generalisations clearly cannot

improve the accuracy of the clock.

4) Unitary ticking clock model: The model proposed in this paper takes on the form of a one-parameter dynamical semigroup over the clockwork and tick register. The clockwork provides the necessary timing while the register stores the tick information. The potential accuracy of the ticking clock depends on the properties of the clockwork, such as its dimensionality or energy. We have discussed how this can be dilated via the aid of an infinite-dimensional environment to unitary dynamics with a time independent Hamiltonian, using standard limiting procedures.

One could consider a ticking clock whose dynamics are unitarily driven by a time independent Hamiltonian with a *finite* environment instead. It would however have no classical counterpart (definition 5) nor allow for a classical register (definition 3). The lack of a classical register would mean that it could suffer from the Zeno effect, and thus there would be a critical observation frequency which if surpassed, the observations would start to change the accuracy of the clock significantly and if frequent enough, might even stop the clock ticking altogether. Understanding and quantifying the properties of such models could be an interesting future line of research.

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References

- [1] M. P. Woods, R. Silva, G. Pütz, S. Stupar, and R. Renner, “Quantum clocks are more accurate than classical ones,” (2018), arXiv:1806.00491v2 [quant-ph] .
- [2] P. Erker, M. T. Mitchison, R. Silva, M. P. Woods, N. Brunner, and M. Huber, Phys. Rev. X **7**, 031022 (2017).
- [3] H. Salecker and E. P. Wigner, Phys. Rev. **109**, 571 (1958).
- [4] A. Peres, Am. J. Phys **48**, 552 (1980).
- [5] V. Bužek, R. Derka, and S. Massar, Phys. Rev. Lett. **82**, 2207 (1999).
- [6] P. Erker, “The Quantum Hourglass,” (2014), ETH Zürich.
- [7] S. Ranković, Y.-C. Liang, and R. Renner, “Quantum clocks and their synchronisation — the Alternate Ticks Game,” (2015), arXiv:1506.01373v1 [quant-ph] .
- [8] M. P. Woods, R. Silva, and J. Oppenheim, Ann. Henri Poincaré (2018), 10.1007/s00023-018-0736-9.
- [9] S. Khandelwal, M. P. Lock, and M. P. Woods, Quantum **4**, 309 (2020).
- [10] Y. Yang, L. Baumgärtner, R. Silva, and R. Renner, “Accuracy enhancing protocols for quantum clocks,” (2019), arXiv:1905.09707 [quant-ph] .
- [11] N. Yunger Halpern and D. T. Limmer, Phys. Rev. A **101**, 042116 (2020).
- [12] P. A. Hoehn, A. R. H. Smith, and M. P. E. Lock, “The Trinity of Relational Quantum Dynamics,” (2019), arXiv:1912.00033 [quant-ph] .
- [13] W. Pauli, in *Handbuch der Physik*, Vol. 24 (Springer, 1933) pp. 83–272.
- [14] W. Pauli, *Encyclopedia of Physics*, Vol. 1 (Springer, Berlin, 1958) p. 60.
- [15] Á. Rivas, S. F. Huelga, and M. B. Plenio, Reports on Progress in Physics **77**, 094001 (2014).
- [16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Vol. 44 (Springer New York, New York, NY, 1983).
- [17] R. Gandy and C. Yates, *Mathematical Logic*, Vol. 4 (Elsevier, 2001) (see page 267).
- [18] A. Degasperis, L. Fonda, and G. C. Ghirardi, Il Nuovo Cimento A (1965-1970) **21**, 471 (1974).
- [19] B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).
- [20] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, 2007). See section 3.3 *Microscopic derivations*. In particular, 3.3.1 *Weak-coupling limit* and 3.3.3 *Singular-coupling limit*.
- [21] P. F. Palmer, J. Math. Phys. **18**, 527 (1977).
- [22] V. Gorini, A. Frigerio, M. Verri, A. Kossakowski, and E. Sudarshan, Rep. Math. Phys. **13**, 149 (1978).
- [23] S. Stupar, C. Klumpp, N. Gisin, and R. Renner, “Performance of Stochastic Clocks in the Alternate Ticks Game,” (2018), arXiv:1806.08812 [quant-ph] .
- [24] Y. Yang and R. Renner, “Ultimate limit on time signal generation,” (2020), arXiv:2004.07857 [quant-ph] .
- [25] I. Pikovski, M. Zych, F. Costa, and Č. Brukner, Nat. Phys. **11**, 668 (2015).
- [26] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
- [27] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
- [28] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th An-*

- niversary Edition*, 10th ed. (Cambridge University Press, USA, 2011).
- [29] K. Kraus, A. Böhm, J. D. Dollard, and W. H. Wootters, eds., *States, Effects, and Operations Fundamental Notions of Quantum Theory*, Lecture Notes in Physics, Vol. 190 (Springer Berlin Heidelberg, Berlin, Heidelberg, 1983).
- [30] M.-D. Choi, Linear Algebra and its Applications **10**, 285 (1975).

Appendices

A Implicit Ticking Clock Representation

In this appendix we will formulate a representation of the ticking clock channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t$ which holds if and only if the ticking clock satisfies the axiomatic definition 1, up to some stated equivalence. Unlike the representation of the ticking clock channel from section 5, this representation is technical in nature. Its presentation is followed by some technical implications such as how its delay functions are related to those of [1].

We start with the following lemma which asserts that the ticking clock from section 4 can equivalently be specified in terms of generators acting on the clockwork space $\mathcal{H}_C$.

Lemma 1 (Implicit ticking clock representation). *The pair $(\rho_{\text{CR}_T}^0, (\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)_{t \geq 0})$ form a ticking clock (definition 1) with a classical tick register (definition 3), up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator H as well as two finite sequences of operators $(L_j)_{j=1}^{N_L}$ and $(J_j)_{j=1}^{N_L}$ on $\mathcal{B}(\mathcal{H}_C)$; which are all k and t independent, such that for all $t \geq 0$ and $k = 0, 1, \dots, N_T$;*

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) = \lim_{\substack{N \rightarrow +\infty \\ N \in \mathbb{N}}} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ(N-1)} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,k}(\rho_C^0), \quad (30a)$$

where

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,k}(\cdot) = (\cdot) \otimes |k\rangle\langle k|_{\text{RT}} + \left(\frac{t}{N}\right) \mathcal{C}_{(1,k)}(\cdot) \otimes |k\rangle\langle k|_{\text{RT}} + \left(\frac{t}{N}\right) \mathcal{C}_{(2,k)}(\cdot) \otimes |k+1\rangle\langle k+1|_{\text{RT}} + F_{C \rightarrow \text{CR}_T}^{t/N,k}(\cdot), \quad (30b)$$

with

$$\mathcal{C}_{(1,k)}(\cdot) := -i[H, \cdot] - \sum_{j=1}^{N_L} \frac{1}{2} \{L_j^\dagger L_j + \theta(k) J_j^\dagger J_j, \cdot\} + L_j(\cdot) L_j^\dagger, \quad (30c)$$

$$\mathcal{C}_{(2,k)}(\cdot) := \theta(k) \sum_{j=1}^{N_L} J_j(\cdot) J_j^\dagger, \quad (30d)$$

and $F_{C \rightarrow \text{CR}_T}^{\delta,k}(\rho_C^0) = o(\delta)$ entry-wise. $\theta(k) = 1$ for all k in the periodic register case and $\theta(k) = 1 - \hat{\delta}_{k,N_T}$ in the cut-off register case, where $\hat{\delta}_{\cdot,\cdot}$ is the Kronecker delta.

The proof of this lemma, which uses some elements of the proof of Lindblad's representation theorem [26, 27], is provided in Appendix section B.2.

As regards to the dynamics on $\mathcal{H}_C \otimes \mathcal{H}_{\text{RT}}$, eq. (30a) is completely determined by eq. (30b) in terms of the initial state and operators H , $(L_j)_j$, $(J_j)_j$. To see this, 1st note that applying the channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N}$ to both sides of eq. (30b) one obtains the composition law

$$\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,l}(\cdot) = \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,l}(\cdot) + (t/N) \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,l}(\mathcal{C}_{(1,l)}(\cdot)) + (t/N) \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,l+1}(\mathcal{C}_{(2,l)}(\cdot)) + F_{C \rightarrow \text{CR}_T}^{\delta,k}(\cdot), \quad (31)$$

with $l = 0, 1, \dots, N_T$. This establishes eq. (30a) inductively. Every one of the N applications of the channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N}$ in eq. (30a), has a direct physical meaning. Up to order $o(t/N)$ contributions, the i^{th} application of $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N}$ takes the state from the previous time step ($i-1$ applications of $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N}$), and updates the state and probability so that if l ticks had occurred in the previous time step, then either no tick occurs or the ticking clock ticks once, in the i^{th} time step. Other processes, such as the clock “loosing a tick” or ticking more than once in the time step t/N can only occur with probability $o(t/N)$. However, when taking the $N \rightarrow +\infty$ limit in eq. (30a) the order $o(t/N)$ terms vanish. As such they are irrelevant and can be set to zero if one wishes. Furthermore, the requirement that $F_{C \rightarrow \text{CR}_T}^{\delta,k}(\rho_C) = o(\delta)$ entry-wise, holds if and only if $\|F_{C \rightarrow \text{CR}_T}^{\delta,k}(\rho_C)\|_p = o(\delta)$ for any $p > 0$ where $\|\cdot\|_p$ is the operator norm induced by the vector p -norm. This is shown in lemma 2 in the appendix.

Observe that we have not placed any restrictions on $N_L \in \mathbb{N}$. It turns out that without loss of generality, one can set $N_L = d^2 - 1$, where d is the Hilbert space dimension of the clockwork. This is because for any two finite sequences $(L_j)_{j=1}^{N_L}, (J_j)_{j=1}^{N_L}$ giving rise to eq. (30a), there exists a new set $(L'_j)_{j=1}^{d^2-1}, (J'_j)_{j=1}^{d^2-1}$ which gives rise to

the same dynamics in eq. (30a) upon their substitution. This follows from simple variants of well known proofs in quantum channel representation theory, as shown in lemma 3 in the appendix. Since the representations of the channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t$ in lemma 1 and proposition 2 are equivalent, the choice $N_L = d^2 - 1$ can also be made in proposition 2 w.l.o.g.

If in the definition of a ticking clock (definition 1), we remove the condition that the channel from $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_T})$ to $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_T})$ is Markovian [eq. (10)], then lemma 1 still holds if one traces out the register in both sides of eq. (30a) to produce the channel $\mathcal{M}_{C \rightarrow C}^t$. In such cases, it appears that the full channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t$ producing the dynamics on the register is undetermined other than for an infinitesimal time step. While such channels do not allow one to determine all properties of the clock, it does allow one to determine the probability of the ticking clock “ticking” during infinitesimal time step $[t, t + dt]$. Denoting this probability $P_{\text{tick}}(\rho_C(t)) dt$, one has that $P_{\text{tick}}(\rho_C(t)) = \sum_{k=0}^{N_T} p_k(t) P_{\text{tick}}^{(k)}(\rho_C(t))$ where $P_{\text{tick}}^{(k)}(\rho_C(t))$ is the probability density corresponding to ticking during coordinate time interval $[t, t + dt]$, given that the probability of the clockwork and register being in state $\rho_C(t) \otimes |k\rangle\langle k|_{\text{R}_T}$ at time t , was $p_k(t)$. For all probability distributions $(p_k(t))_k$ it takes the value

$$\begin{aligned} P_{\text{tick}}(\rho_C(t)) &= \sum_{k=0}^{N_T} p_k(t) \lim_{\delta \rightarrow 0^+} \frac{\text{tr} \left[|k+1\rangle\langle k+1|_{\text{R}_T} \left(\mathcal{M}_{C \rightarrow \text{CR}_T}^{\delta, k}(\rho_C(t)) - \rho_C(t) \otimes |k\rangle\langle k|_{\text{R}_T} \right) \right]}{\delta} \\ &= \left(\sum_{k=0}^{N_T} p_k(t) \theta(k) \right) \text{tr} \left[\left(\sum_{j=1}^{N_L} J_j^\dagger J_j \right) \rho_C(t) \right]. \end{aligned} \quad (32)$$

where $\rho_C(t) = \mathcal{M}_{C \rightarrow C}^t(\rho_C)$ is the clockwork state at coordinate time t . Since $\sum_{k=0}^{N_T} p_k(t) = 1$, in the case of the periodic register, the factor $\sum_{k=0}^{N_T} p_k(t) \theta(k)$ vanishes from eq. (32) and $P_{\text{tick}}(\rho_C(t))$ only depends on the local clockwork dynamics. However, such probabilities are not so useful for determining measures of ticking clock accuracy, since they do not provide information about individual ticks.

For example, a more useful quantity is the probability of producing the k^{th} tick during coordinate time interval $[t, t + dt]$. Or in other words, the probability that during time interval $[0, t]$, the ticking clock ticked $k - 1$ times and then produced a tick during time interval $[t, t + dt]$. We denote this probability measure $P_{\text{ticks}}^{(k)}(t) dt$, where $P_{\text{ticks}}^{(k)}(t)$ is called the k^{th} tick delay function. One finds

$$P_{\text{ticks}}^{(k)}(t) = \lim_{\delta \rightarrow 0^+} \frac{\text{tr} \left[|k\rangle\langle k|_{\text{R}_T} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{\delta}(\rho_{\text{CR}_T}^{(k-1)}(t)) - \rho_{\text{CR}_T}^{(k-1)}(t) \right) \right]}{\delta}, \quad (33)$$

where $\rho_{\text{CR}_T}^{(k-1)}(t)$ is the un-normalised outcome of a measurement when the register is found to be in the $|k-1\rangle\langle k-1|_{\text{R}_T}$ state,

$$\rho_{\text{CR}_T}^{(k-1)}(t) = |k-1\rangle\langle k-1|_{\text{R}_T} \left(\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, 0}(\rho_C^0) \right) |k-1\rangle\langle k-1|_{\text{R}_T}, \quad (34)$$

and we have assumed that the clockwork and register are initialised to $\rho_C^0 \otimes |0\rangle\langle 0|_{\text{R}_T}$ at coordinate time $t = 0$. In the case of a cut-off register, from eq. (30b), we observe that in every infinitesimal time step, the dynamics incurred on the clockwork for the first $N_T - 1$ ticks is identical to that derived in [1] for the model [7] in the continuous time limit. Therefore, the k^{th} tick delay function, $P_{\text{ticks}}^{(k)}(t)$, is the same for the ticking clocks in proposition 2 and those in [1] for $k = 1, 2, 3, \dots, N_T$ in the cut-off register case. This has important consequences as discussed in section 7.

For example, consider the case of the 1st tick in the case of the cut-off register model. A simple calculation finds that $\rho_C^{(0)}(t)$ is generated via the clockwork’s dynamics with the tick generating channel removed:

$$\rho_C^{(0)}(t) = \text{tr}_{\text{R}_T} \left[|0\rangle\langle 0|_{\text{R}_T} \left(\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, 0}(\rho_C) \right) \right] \quad (35)$$

$$= \lim_{N \rightarrow +\infty} \text{tr}_{\text{R}_T} \left[\left(|0\rangle\langle 0|_{\text{R}_T} \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N, 0}(\rho_C) |0\rangle\langle 0|_{\text{R}_T} \right)^{\circ N} \right] \quad (36)$$

$$= e^{t\mathcal{L}_C^{(0)}}(\rho_C), \quad (37)$$

$$\mathcal{L}_C^{(0)}(\cdot) = \mathcal{C}_{(1,0)}(\cdot). \quad (38)$$

In the case of the periodic register model, the above expression for $\rho_C^{(0)}(t)$ does not hold, since the equality in line (36) is false. Physically speaking, this is because in the periodic register case, when the register runs out

of memory, a tick is produced in the initial memory state $|0\rangle\langle 0|_{\text{R}_T}$ and thus the dynamics of the clock at times after the 1st tick has occurred are still relevant for the 1st tick's statistics. This is not the case in the infinite register limit $N_T \rightarrow +\infty$ nor for the cut-off register case.

B Proofs

B.1 Proof of proposition 1

Proposition 1 (Measured register equivalence). *For all coordinate times $t_n \geq 0$ s.t. $\sum_{n=1}^N t_n = t$ and for all $N \in \mathbb{N}_{>0}$, the dynamics of any ticking clock with a classical register is equal to that of the ensemble of classical register measured channels, where the ensemble is weighted by the probability of the classical register measured channel occurring:*

$$\begin{aligned} \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t(\rho_C \otimes |k\rangle\langle k|_{\text{R}_T}) \\ = \sum_{\mathbf{s}_N \in \mathcal{S}_N(t)} \text{Prob}[\mathbf{s}_N] \mathcal{C}\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t[\mathbf{s}_N](\rho_C \otimes |k\rangle\langle k|_{\text{R}_T}) \end{aligned} \quad (19)$$

for all $(t_n)_{n=1}^N$, $N \in \mathbb{N}_{>0}$, $t \geq 0$, $k = 0, 1, \dots, N_T$, and $\rho_C \in \mathcal{S}(\mathcal{H}_C)$.

Proof. To start with, observe that for a ticking clock $(\rho_C^0, \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)$ with a classical register, one has for all $t \geq 0$; $k = 0, 1, \dots, N_T$; $\rho_C^0 \in \mathcal{S}(\mathcal{H}_C)$,

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) = \sum_{l=0}^{N_T} \text{tr}_{\text{R}_T} \left[|l\rangle\langle l|_{\text{R}_T} \mathcal{M}_{C \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) \right] \otimes |l\rangle\langle l|_{\text{R}_T} \quad (39)$$

$$= \sum_{l=0}^{N_T} \text{Prob}[l, t] \mathcal{P}_l \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{t,k}(\rho_C^0), \quad (40)$$

where $\text{Prob}[l, t] := \text{tr} \left[|l\rangle\langle l|_{\text{R}_T} \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) \right]$. Thus, using the Markovian property of channel $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t$ [condition 2)], followed by iteratively substituting the above equation,

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) \quad (41)$$

$$= \bigcirc_{n=1}^N \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_n}(\rho_C^0 \otimes |k\rangle\langle k|_{\text{R}_T}) \quad (42)$$

$$= \bigcirc_{n=2}^N \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_n} \circ \sum_{l_1=0}^{N_T} \text{Prob}[l_1, t_1] \mathcal{P}_{l_1} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1}(\rho_C^0 \otimes |k\rangle\langle k|_{\text{R}_T}) \quad (43)$$

$$= \bigcirc_{n=3}^N \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_n} \circ \sum_{l_2=0}^{N_T} \text{Prob}[l_2, t_2] \mathcal{P}_{l_2} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2} \circ \sum_{l_1=0}^{N_T} \text{Prob}[l_1, t_1] \mathcal{P}_{l_1} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1}(\rho_C^0 \otimes |k\rangle\langle k|_{\text{R}_T}) \quad (44)$$

$$= \sum_{l_1=0}^{N_T} \dots \sum_{l_N=0}^{N_T} \text{Prob}[l_1, t_1] \dots \text{Prob}[l_N, t_N] \bigcirc_{n=1}^N \mathcal{P}_{l_n} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_n}(\rho_C^0 \otimes |k\rangle\langle k|_{\text{R}_T}). \quad (45)$$

Observe that $\text{Prob}[l_1, t_1] \dots \text{Prob}[l_N, t_N] = \text{Prob}[\mathbf{s}_N]$. Taking into account definition 4 we complete the proof. $\blacksquare$

B.2 Proof of lemma 1

Lemma 1 (Implicit ticking clock representation). *The pair $(\rho_{\text{CR}_T}^0, (\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)_{t \geq 0})$ form a ticking clock (definition 1) with a classical tick register (definition 3), up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator H as well as two finite sequences of operators $(L_j)_{j=1}^{N_L}$ and $(J_j)_{j=1}^{N_L}$ on $\mathcal{B}(\mathcal{H}_C)$; which are all k and t independent, such that for all $t \geq 0$ and $k = 0, 1, \dots, N_T$;*

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) = \lim_{\substack{N \rightarrow +\infty \\ N \in \mathbb{N}}} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ(N-1)} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N,k}(\rho_C^0), \quad (30a)$$

where

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N, k}(\cdot) = (\cdot) \otimes |k\rangle\langle k|_{\text{RT}} + \left(\frac{t}{N}\right) \mathcal{C}_{(1,k)}(\cdot) \otimes |k\rangle\langle k|_{\text{RT}} + \left(\frac{t}{N}\right) \mathcal{C}_{(2,k)}(\cdot) \otimes |k+1\rangle\langle k+1|_{\text{RT}} + F_{C \rightarrow \text{CR}_T}^{t/N, k}(\cdot), \quad (30b)$$

with

$$\mathcal{C}_{(1,k)}(\cdot) := -i[H, \cdot] - \sum_{j=1}^{N_L} \frac{1}{2} \{L_j^\dagger L_j + \theta(k) J_j^\dagger J_j, \cdot\} + L_j(\cdot) L_j^\dagger, \quad (30c)$$

$$\mathcal{C}_{(2,k)}(\cdot) := \theta(k) \sum_{j=1}^{N_L} J_j(\cdot) J_j^\dagger, \quad (30d)$$

and $F_{C \rightarrow \text{CR}_T}^{\delta, k}(\rho_C^0) = o(\delta)$ entry-wise. $\theta(k) = 1$ for all k in the periodic register case and $\theta(k) = 1 - \hat{\delta}_{k, N_T}$ in the cut-off register case, where $\hat{\delta}_{\cdot, \cdot}$ is the Kronecker delta.

Proof. First we will prove the proposition for the case of a ticking clock with a periodic register. The case of a cut-off register will then be straightforward. To start with, we consider the most general representation of a channel in the Kraus form, namely

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, k}(\rho_C) = \sum_{j=0}^{N_L} Q_j^{(k)}(t) \rho_C Q_j^{(k)\dagger}(t), \quad (46)$$

where

$$Q_j^{(k)}(t) : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\text{RT}}) \quad (47)$$

and $\sum_j Q_j^{(k)}(t)^\dagger Q_j^{(k)}(t) = \mathbb{1}_C$. We now expand the $Q_j^{(k)}(t)$ operators in the register basis. By making the identification $N_j^{(k)}(l, t) := \langle l|_{\text{RT}} Q_j^{(k)}(t) : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C)$ this yields

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, k}(\rho_C) = \sum_{l, l'=0}^{N_T} \sum_{j=0}^{N_L} N_j^{(k)}(l, t) \rho_C N_j^{(k)\dagger}(l', t) \otimes |l\rangle\langle l'|_{\text{RT}}, \quad (48)$$

since the register is classical, the off diagonal terms must vanish due to compatibility with eq. (16). We therefore have

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, k}(\rho_C) = \sum_{l=0}^{N_T} \sum_{j=0}^{N_L} N_j^{(k)}(l, t) \rho_C N_j^{(k)\dagger}(l, t) \otimes |l\rangle\langle l|_{\text{RT}}, \quad (49)$$

for all $k = 0, 1, \dots, N_T$; $t \geq 0$. Observe that

$$\text{tr}_{\text{RT}} \left[\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, k}(\rho_C) |l\rangle\langle l|_{\text{RT}} \right] = \sum_{j=0}^{N_L} N_j^{(k)}(l, t) \rho_C N_j^{(k)\dagger}(l, t). \quad (50)$$

Moreover, in the periodic register case, $|l\rangle_{\text{RT}} = |l \bmod. N_T + 1\rangle_{\text{RT}}$. For convenience, we therefore extend the definition of the operators $N_j^{(k)}(l, t)$ in the periodic case from $l = 0, 1, \dots, N_T$ to $l \in \mathbb{Z}$ by defining

$$N_j^{(k)}(l, t) = N_j^{(k)}(l \bmod. N_T + 1, t) \quad (51)$$

Consider an expansion of the form $N_j^{(k)}(l, t) = \sum_{n, m} a_{n, m}^{(j, k)}(l, t) |n\rangle\langle m|_C$ and states $\rho_C(p)$ which are pure and diagonal in this basis, $\rho_C(p) = |p\rangle\langle p|_C$. Therefore

$$\frac{\text{tr}_{\text{RT}} \left[\mathcal{M}_{C \rightarrow \text{CR}_T}^{t, k}(\rho_C(p)) |l\rangle\langle l|_{\text{RT}} \right]}{\tilde{p}_l^{(k)}(t)} = \sum_{j=0}^{N_L} \frac{\sum_{n \neq m} a_{n, p}^{(j, k)}(l, t) a_{m, p}^{(j, k)*}(l, t) |n\rangle\langle m|}{\tilde{p}_l^{(k)}(t)} + \frac{\sum_n a_{n, p}^{(j, k)}(l, t) a_{n, p}^{(j, k)*}(l, t) |n\rangle\langle n|}{\tilde{p}_l^{(k)}(t)}. \quad (52)$$

Now noting the definition of $\tilde{p}_l^{(k)}(t)$ (eq. (12)) and taking the trace on both sides, we find

$$1 = \frac{\sum_{n,j} |a_{n,p}^{(j,k)}(l,t)|^2}{\tilde{p}_l^{(k)}(t)}. \quad (53)$$

Therefore,

$$|a_{n,p}^{(j,k)}(l,t)| \leq \sqrt{\tilde{p}_l^{(k)}(t)} \quad \forall j = 0, 1, \dots, N_L; n, p = 0, 1, \dots, d-1; l, k = 0, 1, \dots, N_T; t \geq 0, \quad (54)$$

where $d \in \mathbb{N}_{>0}$ is the dimension of the Hilbert space of the clockwork. We will now use inequality eq. (54) together with condition 4) [eq. (13)] to show an important limit. To start with, denote the entries of a matrix M by $[M]_{[ab]}$ and observe that eq. (54) implies

$$\begin{aligned} \lim_{t \rightarrow 0^+} \left| \frac{\sum_{l=0}^{N_T} \sum_{l \notin \{k, f(k)\}} \sum_{j=0}^{N_L} \left[N_j^{(k)}(l,t) \rho_C N_j^{(k)\dagger}(l,t) \right]_{[ab]}}{\tilde{p}_{f(k)}^{(k)}(t)} \right| &= \lim_{t \rightarrow 0^+} \left| \frac{\sum_{l=0}^{N_T} \sum_{l \notin \{k, f(k)\}} \sum_{j=0}^{N_L} \sum_{m,n} a_{a,m}^{(j,k)}(l,t) [\rho_C]_{[m,n]} a_{b,n}^{(j,k)*}(l,t)}{\tilde{p}_{f(k)}^{(k)}(t)} \right| \\ &\leq (N_L + 1) \left(\sum_{m,n} |[\rho_C]_{[m,n]}| \right) \lim_{t \rightarrow 0^+} \frac{\sum_{l=0}^{N_T} \sum_{l \notin \{k, f(k)\}} \tilde{p}_l^{(k)}(t)}{\tilde{p}_{f(k)}^{(k)}(t)} = 0, \end{aligned} \quad (55)$$

for all $k = 0, 1, \dots, N_T$; $a, b = 0, 1, \dots, d-1$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. In the last line in eq. (55) we have used eq. (13). Therefore,

$$\sum_{l=0}^{N_T} \sum_{l \notin \{k, f(k)\}} \sum_{j=0}^{N_L} N_j^{(k)}(l,t) \rho_C N_j^{(k)\dagger}(l,t) = o\left(\tilde{p}_{f(k)}^{(k)}(t)\right) \quad (56)$$

entry-wise in the $t \rightarrow 0^+$ limit for all $k = 0, 1, \dots, N_T$. Since every term $N_j^{(k)}(l,t) \rho_C N_j^{(k)\dagger}(l,t)$ in the above summation is positive semi-definite, we thus have

$$\mathcal{M}_{C \rightarrow CR_T}^{\delta t, k}(\rho_C) = \sum_{l \in \{k, f(k)\}} \sum_{j=0}^{N_L} N_j^{(k)}(l, \delta t) \rho_C N_j^{(k)\dagger}(l, \delta t) \otimes |l\rangle\langle l|_{R_T} + o\left(\tilde{p}_{f(k)}^{(k)}(\delta t)\right), \quad (57)$$

$$= \sum_{l \in \{0, f(k)-k\}} \sum_{j=0}^{N_L} N_j^{(k)}(l+k, \delta t) \rho_C N_j^{(k)\dagger}(l+k, \delta t) \otimes |l+k\rangle\langle l+k|_{R_T} + o\left(\tilde{p}_{f(k)}^{(k)}(\delta t)\right), \quad (58)$$

for all $k = 0, 1, \dots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. On the other hand, from eq. (49) it follows that the state of the clockwork, given the register is measured to be in the state $|k+l\rangle\langle k+l|_{R_T}$ for $l \in \mathbb{Z}$ is

$$\text{tr}_{R_T} \left[\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\rho_C) |k+l\rangle\langle k+l|_{R_T} \right] = \sum_{j=0}^{N_L} N_j^{(k)}(l+k, t) \rho_C N_j^{(k)\dagger}(l+k, t). \quad (59)$$

By virtue of condition 1) (eq. (9)), we have that every matrix component of the r.h.s. of eq. (59) is k independent for all $t \geq 0$, $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ and $l \in \mathbb{Z}$; $k = 0, 1, \dots, N_T$ s.t. $k+l = 0, 1, \dots, N_T$ in the periodic register case. Therefore, in particular, we have

$$\sum_{j=0}^{N_L} N_j^{(k)}(l+k, t) \rho_C N_j^{(k)\dagger}(l+k, t) = \sum_{j=0}^{N_L} N_j^{(0)}(l, t) \rho_C N_j^{(0)\dagger}(l, t) \quad (60)$$

for all $t \geq 0$, $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ and $l \in \mathbb{Z}$; $k = 0, 1, \dots, N_T$ s.t. $k+l = 0, 1, \dots, N_T$ in the periodic case. Plugging eq. (60) into eq. (58), yields

$$\mathcal{M}_{C \rightarrow CR_T}^{\delta t, k}(\rho_C) = \sum_{l \in \{0, f(k)-k\}} \sum_{j=0}^{N_L} N_j^{(0)}(l, \delta t) \rho_C N_j^{(0)\dagger}(l, \delta t) \otimes |l+k\rangle\langle l+k|_{R_T} + o\left(\tilde{p}_{f(k)}^{(k)}(\delta t)\right), \quad (61)$$

for all $k = 0, 1, \dots, N_T - 1$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ in the periodic register case. For the case $k = N_T$ in the above equation, recall that due to eq. (51) we have that

$$\sum_{j=0}^{N_L} N_j^{(0)}(-N_T, t) \rho_C N_j^{(0)\dagger}(-N_T, t) = \sum_{j=0}^{N_L} N_j^{(0)}(1, t) \rho_C N_j^{(0)\dagger}(1, t). \quad (62)$$

For the periodic register case, taking into account eqs. (60) and (62) we have

$$\tilde{p}_{f(k)}^{(k)}(t) = \text{tr} \left[\sum_{j=0}^{N_L} N_j^{(k)}(k+1, t) \rho_C N_j^{(k)\dagger}(k+1, t) \right] = \text{tr} \left[\sum_{j=0}^{N_L} N_j^{(0)}(1, t) \rho_C N_j^{(0)\dagger}(1, t) \right] = \tilde{p}_1^{(0)}(t), \quad (63)$$

for all $k = 0, 1, \dots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. Therefore, for all $k = 0, 1, \dots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; eq. (61) reduces to

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{\delta t, k}(\rho_C) = \sum_{l \in \{0, 1\}} \sum_{j=0}^{N_L} N_j^{(0)}(l, \delta t) \rho_C N_j^{(0)\dagger}(l, \delta t) \otimes |l+k\rangle\langle l+k|_{\text{R}_T} + o\left(\tilde{p}_1^{(0)}(\delta t)\right). \quad (64)$$

It follows from eqs. (10), (11a) and (11b) of condition 3), that the clockwork channel admits a power-law expansion in t (see uniformly continuous semigroup in [16]). Specifically, there exists an operator A_{CR_T} on $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{\text{R}_T})$ such that

$$\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t = e^{t A_{\text{CR}_T}} := \sum_{n=0}^{\infty} t^n \frac{A_{\text{CR}_T}^n}{n!}. \quad (65)$$

Therefore, w.l.o.g. we can use the following ansatz:

Let $n_1 \leq N_L$ of the elements of the sequence $(N_j^{(0)}(0, \delta t))_j$ be of linear order in δt while the others are of order $\sqrt{\delta t}$. Specifically, let

$$\left(N_j^{(0)}(0, \delta t) = I_j + (-iH_j + K_j)\delta t \right)_{j=0}^{n_1}, \quad \left(N_j^{(0)}(0, \delta t) = L_j \sqrt{\delta t} \right)_{j=n_1+1}^{N_L}, \quad (66)$$

where H_j, K_j are Hermitian and the operators L_j, H_j, K_j, I_j , are all t independent. Similarly, we can employ the same form of the expansion for the sequence $(N_j^{(0)}(1, \delta t))_j$ associated with the register state $|k+1\rangle\langle k+1|_{\text{R}_T}$:

$$\left(N_j^{(0)}(1, \delta t) = I'_j + (-i\bar{H}_j + \bar{K}_j)\delta t \right)_{j=0}^{n_1}, \quad \left(N_j^{(0)}(1, \delta t) = J_j \sqrt{\delta t} \right)_{j=n_1+1}^{N_L}, \quad (67)$$

where $\bar{H}_j, \bar{K}_j$ are Hermitian and the operators $J_j, \bar{H}_j, \bar{K}_j, I'_j$, are all t independent operators.

We first fix the zeroth order terms by noting that $\mathcal{M}_{C \rightarrow C}^0 := \text{tr}_{\text{R}_T} [\mathcal{M}_{C \rightarrow \text{CR}_T}^{0, k}]$ has to be the identity channel due to condition 3) [eq. (11a)] and eq. (64). It hence follows:

$$\mathcal{M}_{C \rightarrow C}^0(\rho_C) = \sum_{j=0}^{n_1} I_j \rho_C I_j^\dagger + I'_j \rho_C I'_j{}^\dagger = \rho_C \quad (68)$$

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. Two sets of Kraus operators $(\tilde{K}_l)_{l=0}^{n_1}, (\tilde{K}'_l)_{l=0}^{n_1}$ give rise to the same quantum channel (i.e. $\sum_l \tilde{K}'_l(\cdot) \tilde{K}'_l{}^\dagger = \sum_l \tilde{K}_l(\cdot) \tilde{K}_l{}^\dagger$) iff there exists an n_1 by n_1 unitary V with entries $V_{[lm]} \in \mathbb{C}$ such that $\tilde{K}'_l = \sum_m V_{[lm]} \tilde{K}_m$ for all $l = 0, 1, \dots, n_1$. See e.g. [28, 29] for a proof. Note that this even covers the case in which one or both of the channels have less than $n_1 + 1$ Kraus operator elements, since we can always choose to define additional Kraus operators which are equal to zero. Therefore, since $(I_j = c_j \mathbb{1}, I'_j = c'_j \mathbb{1})_{j=0}^{n_1}$ with $\sum_{j=0}^{n_1} |c_j|^2 + |c'_j|^2 = 1$ are solutions to eq. (68), this unitary equivalence theorem implies that it is the only family of solutions. On the other hand, one also finds

$$\lim_{t \rightarrow 0^+} \tilde{p}_0^{(0)}(t) = \sum_{j=0}^{n_1} \text{tr} [I_j \rho_C I_j^\dagger] = 1, \quad (69)$$

$$\lim_{t \rightarrow 0^+} \tilde{p}_1^{(0)}(t) = \sum_{j=0}^{n_1} \text{tr} [I'_j \rho_C I'_j{}^\dagger] = 0, \quad (70)$$

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. The last equality in eq. (69) follows from invoking condition 3) [eq. (11b)] and the definition of $\tilde{p}_0^{(0)}$, while the last equality in eq. (70) follows from conservation of probability. Since $I_j' \rho_C I_j'^\dagger$ is positive semi-definite, it follows that $I_j' = 0$ for all $j = 0, 1, \dots, n_1$ and thus we find $I_j = c_j \mathbb{1}$ with $\sum_{j=0}^{n_1} |c_j|^2 = 1$ for all $j = 0, 1, \dots, n_1$. It thus follows from plugging in the above ansatz for $N_j^{(0)}(0, t)$ and $N_j^{(0)}(1, t)$ into the definition of $\tilde{p}_1^{(0)}$ that

$$o(\tilde{p}_1^{(0)}(\delta t)) = o(\delta t). \quad (71)$$

Expanding eq. (64) we thus find

$$\begin{aligned} \mathcal{M}_{C \rightarrow C}^{\delta t, k}(\rho_C) &= \rho_C \otimes |k\rangle\langle k|_{\text{RT}} + \left[\rho_C, \sum_{j=0}^{n_1} (c_j^R H_j + c_j^I K_j) \right] \otimes |k\rangle\langle k|_{\text{RT}} \delta t + \left\{ \rho_C, \sum_{j=0}^{n_1} (c_j^I H_j + c_j^R K_j) \right\} \otimes |k\rangle\langle k|_{\text{RT}} \delta t \\ &+ \sum_{j=n_1+1}^{N_L} \left(L_j \rho_C L_j^\dagger \otimes |k\rangle\langle k|_{\text{RT}} + J_j \rho_C J_j^\dagger \otimes |k+1\rangle\langle k+1|_{\text{RT}} \right) \delta t + o(\delta t), \end{aligned} \quad (72)$$

where c_j^R, c_j^I are the real and imaginary parts of c_j respectively. Now observe that by defining Kraus operators $H := \sum_{j=0}^{n_1} (c_j^R H_j + c_j^I K_j)$, $K := \sum_{j=0}^{n_1} (c_j^I H_j + c_j^R K_j)$ one can exchange eqs. (66) and (67) with

$$\begin{aligned} N_0^{(0)}(0, \delta t) &= \mathbb{1} + (-iH + K)\delta t, & \left(N_j^{(0)}(0, \delta t) = L_j \sqrt{\delta t} \right)_{j=1}^{N_L}, \\ N_0^{(0)}(1, \delta t) &= 0, & \left(N_j^{(0)}(1, \delta t) = J_j \sqrt{\delta t} \right)_{j=1}^{N_L}, \end{aligned} \quad (73)$$

and obtain the same solution as eq. (72) to order $o(\delta t)$ up to a relabelling of the summation indices obtaining

$$\begin{aligned} \mathcal{M}_{C \rightarrow \text{CR}_T}^{\delta t, k}(\rho_C) &= \rho_C \otimes |k\rangle\langle k|_{\text{RT}} - \delta t \left(i[H, \rho_C] - \{K, \rho_C\} - \sum_{j=1}^{N_L} L_j \rho_C L_j^\dagger \right) \otimes |k\rangle\langle k|_{\text{RT}} \\ &+ \delta t \sum_{j=1}^{N_L} J_j \rho_C J_j^\dagger \otimes |k+1\rangle\langle k+1|_{\text{RT}} + o(\delta t) \end{aligned} \quad (74)$$

for all $k = 0, 1, \dots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. Furthermore, taking into account the normalisation of the Kraus operators [eq. (47)], from eq. (73) we obtain a solution for K , namely

$$K = -\frac{1}{2} \sum_{j=1}^{N_L} \left(L_j^\dagger L_j + J_j^\dagger J_j \right). \quad (75)$$

In the case of a periodic register, eq. (30b) in the lemma statement follows by pugging in eq. (75) into eq. (74). Equation (31) in the lemma statement follows by recalling eq. (8) and using eq. (74). By applying the divisibility of the channel [condition 2), eq. (10)] recursively $N \in \mathbb{N}_{>0}$ times we find

$$\left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ N} = \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t, \quad (76)$$

for all $t \geq 0$. Equation (30a) then follows by recalling the notation eq. (8) and taking the limit $N \rightarrow +\infty$. This concludes the “only if” part of the lemma for a periodic register.

Finally, to verify the converse part of the lemma in the case of a periodic register, one simply has to check that for all Hermitian operators H and families of operators $(L_j)_{j=1}^m, (J_j)_{j=1}^m$ acting on $\mathcal{B}(\mathcal{H}_C)$, the equations in the lemma statement satisfy the conditions 1) to 4) in section 4. We do this in the following. For conciseness, we will refer to the sequence of such operators on $\mathcal{B}(\mathcal{H}_C)$ by $\mathcal{D} = (H, (L_j)_j, (J_j)_j)$.

To verify that 1) [eq. (9)] holds for all $\mathcal{D}$, first note that $\text{tr}_{\text{RT}} \left[\left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^\tau \right)^{\circ l} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{\tau, k}(\rho_C) |k+m\rangle\langle k+m| \right]$ is k independent (up to an order $o(\tau)$ term) for all $l \in \mathbb{N}_{>0}$, $m \in \mathbb{Z}$, $k = 0, 1, \dots, N_T$ s.t. $k+m = 0, 1, \dots, N_T$, for all $\tau \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; and $\mathcal{D}$. Hence condition 1) [eq. (9)] follows by choosing $\tau = t/N$, $l = N-1$ and taking the $N \rightarrow \infty$ limit such that the $o(\tau)$ terms vanish.

To verify that eq. (10) in condition 2) holds for all $\mathcal{D}$, one needs to verify that

$$\lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{(t_1+t_2)/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \quad (77)$$

and

$$\lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N_1} \right)^{\circ N_1} \circ \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N_2} \right)^{\circ N_2} (\rho_C \otimes |k\rangle\langle k|) \quad (78)$$

are equal for all $t_1, t_2 \geq 0$; $k = 0, 1, \dots, N_T$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; and $\mathcal{D}$. To do so, we first note by explicit calculation using eq. (30b) that $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} (\rho_C \otimes |k\rangle\langle k|) = \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{(t_1+t_2)/N} (\rho_C \otimes |k\rangle\langle k|) + o(1/N)$. Therefore

$$\lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{(t_1+t_2)/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \quad (79)$$

$$= \lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} + o(1/N) \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \quad (80)$$

$$= \lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \quad (81)$$

$$+ No(t/N) + (N-1)o(t/N)^2 + (N-2)o(t/N)^3 + \dots + (N-(N-1))o(t/N)^N \quad (82)$$

$$= \lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} (\rho_C \otimes |k\rangle\langle k|) \right)^{\circ N} \quad (83)$$

$$+ No(t/N) + (N-1)^2 o(t/N)^2 \quad (84)$$

$$= \lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \right)^{\circ N} \circ \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \quad (85)$$

which is equal to eq. (78) due to continuity. The confirmation that eqs. (11a) and (11b) in condition 3) hold for all $\mathcal{D}$, follows straightforwardly from eq. (30b):

$$\lim_{t \rightarrow 0^+} \lim_{N \rightarrow \infty} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \quad (86)$$

$$= \lim_{t \rightarrow 0^+} \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t (\rho_C \otimes |k\rangle\langle k|) = \rho_C \otimes |k\rangle\langle k|, \quad (87)$$

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$, $\mathcal{D}$ and where in the second equality we used the Markovianity of the channel (which we have just proven) and the penultimate line uses eq. (30b).

Finally, the verification that condition 4) holds for all $\mathcal{D}$ is straightforward. Plugging in eq. (30b) into definition eq. (12) and proceeding similarly to as in the above equation, one finds

$$\lim_{t \rightarrow 0^+} \frac{\sum_{l=0}^{N_T} \tilde{p}_l^{(k)}(t)}{\tilde{p}_{f(k)}^{(k)}(t)} = \lim_{t \rightarrow 0^+} \frac{o(t)}{ct} = 0, \quad (88)$$

for all $k = 0, 1, \dots, N_T$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; and $\mathcal{D}$, where $c > 0$ is a constant. This concludes the proof of the converse part of the lemma for the case of a periodic register. We now proceed to the case of a cut-off register.

We have just proven that $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}$ is the channel of a ticking clock with a classical register of the periodic type, iff it has the form stated in the lemma. Therefore a ticking clock with a classical register of the cut-off type can only satisfy the 1st part of condition 5) (eq. (14a)) iff the ticking clock $\tilde{\mathcal{M}}_{\text{CR}_T \rightarrow \text{CR}_T}$ in eq. (14a) is of the form of that in the lemma statement for $k = 0, 1, \dots, N_T - 1$. Since eq. (30b) is the same for both cut-off and periodic register types, for $k = 0, 1, \dots, N_T - 1$, this holds true. Furthermore, by direct calculation of eq. (30b) in the case of a cut-off register and $k = N_T$, we see that it satisfies eq. (14b) for $k = N_T$. While eq. (30b) in the case of a cut-off register and $k = N_T$ is clearly not necessary for it to satisfy eq. (14b) for $k = N_T$, it is necessary to satisfy eq. (14b) for $k = N_T$ up to clockwork equivalence (definition 2). This can be verified by noting that eq. (14b) for $k = N_T$ implies that the state of the register and clockwork must be a product state up to order $o(\delta t)$.

We have thus far verified that condition 5) holds, up to clockwork equivalence, for a ticking clock with a classical register of the cut-off type iff eq. (30b) in the lemma statement holds. By definition of a ticking clock (definition 1), we only need to verify that condition 2) [eq. (10)] holds for a ticking clock with a cut-off register. The case eq. (10) is verified analogously to the periodic register case above. ■

B.3 Proof of proposition 2

Proposition 2 (Explicit ticking clock representation). *The pair $(\rho_C^0, (\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)_{t \geq 0})$ form a ticking clock (definition 1) with a classical tick register (definition 3), up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator H as well as two finite sequences of operators $(L_j)_{j=1}^{N_L}$ and $(J_j)_{j=1}^{N_L}$ on $\mathcal{B}(\mathcal{H}_C)$; which are all t independent, such that for all $t \geq 0$ and $N_T \in \mathbb{N}_{>0}$,*

$$\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t(\cdot) = e^{t\mathcal{L}_{\text{CR}_T}}(\cdot), \quad (21a)$$

$$\begin{aligned} \mathcal{L}_{\text{CR}_T}(\cdot) = & -i[\tilde{H}, (\cdot)] + \sum_{j=1}^{N_L} \tilde{L}_j(\cdot) \tilde{L}_j^\dagger - \frac{1}{2} \{ \tilde{L}_j^\dagger \tilde{L}_j, (\cdot) \} \\ & + \sum_{j=1}^{N_L} \tilde{J}_j^{(l)}(\cdot) \tilde{J}_j^{(l)\dagger} - \frac{1}{2} \{ \tilde{J}_j^{(l)\dagger} \tilde{J}_j^{(l)}, (\cdot) \}, \end{aligned} \quad (21b)$$

where the operators are $\tilde{H} = H \otimes \mathbb{1}_{\text{R}_T}$, $\tilde{L}_j = L_j \otimes \mathbb{1}_{\text{R}_T}$, $\tilde{J}_j^{(l)} = J_j \otimes O_{\text{R}_T}^{(l)}$, with

$$\begin{aligned} O_{\text{R}_T}^{(l)} := & |1\rangle\langle 0|_{\text{R}_T} + |2\rangle\langle 1|_{\text{R}_T} + |3\rangle\langle 2|_{\text{R}_T} + \dots + \\ & |N_T\rangle\langle N_T - 1|_{\text{R}_T} + l|0\rangle\langle N_T|. \end{aligned} \quad (21c)$$

In the cut-off register case $l = 0$, while $l = 1$ for the periodic register case.

Proof. The proposition follows straightforwardly from a more technical representation (lemma 1) discussed in appendix section A. Specifically, if one expands to leading order in t the channel $\mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t,k}$ for $k = 0, 1, \dots, N_T$ using eq. (21a), one finds eq. (30b). Furthermore, since eq. (21a) is manifestly Markovian, the expression eq. (30a) also holds. Since it was established in lemma 1, that this form of the channel is both necessary and sufficient for the channel to be a ticking clock (definition 1), we conclude the proof of the proposition. ■

B.4 Proof of proposition 3

Proposition 3 (Clockwork representation). *Consider a ticking clock with a classical periodic register (definitions 1 and 3) written in the representation of proposition 2. Its clockwork channel, defined via*

$$\mathcal{M}_{\text{C} \rightarrow \text{C}}^t(\cdot) := \text{tr}_{\text{R}_T}[\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t((\cdot) \otimes |k\rangle\langle k|_{\text{R}_T})] \quad (22)$$

is k -independent for all $t \geq 0$ and of the form

$$\mathcal{M}_{\text{C} \rightarrow \text{C}}^t(\cdot) = e^{t\mathcal{L}_C}(\cdot), \quad (23)$$

with $\mathcal{L}_C$ equal to the r.h.s. of eq. (21b) under the replacements $\tilde{H} \mapsto H$, $\tilde{L}_j \mapsto L_j$ and $\tilde{J}_j^{(l)} \mapsto J_j$. What is more, for every ticking clock with a classical cut-off register written in the representation of proposition 2, there exists a ticking clock which is clockwork equivalent (definition 2), such that its clockwork is k -independent and given by eq. (23).

Proof. Consider a ticking clock $(\rho_{\text{CR}_T}^0, (\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^t)_{t \geq 0})$ with a classical register given by the following expression for all $t \geq 0$ and $k = 0, 1, \dots, N_T$:

$$\mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t,k}(\rho_C^0) = \lim_{\substack{N \rightarrow +\infty \\ N \in \mathbb{N}}} \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ(N-1)} \circ \mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t/N,k}(\rho_C^0), \quad (89a)$$

where

$$\mathcal{M}_{\text{C} \rightarrow \text{CR}_T}^{t/N,k}(\cdot) = (\cdot) \otimes |k\rangle\langle k|_{\text{R}_T} + \left(\frac{t}{N}\right) \mathcal{C}_{(1)}(\cdot) \otimes |k\rangle\langle k|_{\text{R}_T} + \left(\frac{t}{N}\right) \mathcal{C}_{(2)}(\cdot) \otimes |k+1\rangle\langle k+1|_{\text{R}_T}, \quad (89b)$$

with

$$\mathcal{C}_{(1)}(\cdot) := -i[H, \cdot] - \sum_{j=1}^{N_L} \frac{1}{2} \{ L_j^\dagger L_j + J_j^\dagger J_j, \cdot \} + L_j(\cdot) L_j^\dagger, \quad (89c)$$

$$\mathcal{C}_{(2)}(\cdot) := \sum_{j=1}^{N_L} J_j(\cdot) J_j^\dagger, \quad (89d)$$

and where $H \in \mathcal{B}(\mathcal{H}_C)$ is Hermitian and $(L_j)_j, (J_j)_j$ are arbitrary operators in $\mathcal{B}(\mathcal{H}_C)$ and

$$|l\rangle_{\text{RT}} := \begin{cases} |l \bmod N_T + 1\rangle_{\text{RT}} & \text{for } l \in \mathbb{N} \text{ in the periodic register case.} \\ |N_T\rangle_{\text{RT}} & \text{for } l = N_T, N_T + 1, N_T + 2, \dots \text{ in the cut-off register case.} \end{cases} \quad (90)$$

To start with, we have to justify that the above channel is a ticking clock. In the case of a periodic register, this is obvious since it is identical to the channel in lemma 1. In the case of the cut-off register, it clearly satisfies condition 5) [eq. (14)]. Thus the only condition remaining to conclude that it is indeed a representation of a ticking clock with a classical cut-off register, is condition 2) [eq. (10)]. There are numerous ways to show this, the most direct is to note by direct substitution that the following Lindbladian is a generator for the channel:

$$\mathcal{L}'_{\text{CR}_T} := -i[\tilde{H}, (\cdot)] + \sum_{j=1}^{N_L} \tilde{L}_j(\cdot) \tilde{L}_j^\dagger - \sum_{j=1}^{N_L} \frac{1}{2} \{ \tilde{L}_j^\dagger \tilde{L}_j, (\cdot) \} + \sum_{l=0}^{N_T} \sum_{j=1}^{N_L} \bar{J}_j^{(l)}(\cdot) \bar{J}_j^{(l)\dagger} - \sum_{l=0}^{N_T} \sum_{j=1}^{N_L} \frac{1}{2} \{ \bar{J}_j^{(l)\dagger} \bar{J}_j^{(l)}, (\cdot) \}, \quad (91)$$

where, as before $\tilde{H} := H \otimes \mathbb{1}_{\text{RT}}$, $\tilde{L}_j := L_j \otimes \mathbb{1}_{\text{RT}}$, and the new tick generators are

$$\bar{J}_j^{(l)} := \begin{cases} J_j \otimes |l+1\rangle\langle l|_{\text{RT}} & \text{for } l = 0, 1, 2, \dots, N_T - 1 \\ J_j \otimes |N_T\rangle\langle N_T|_{\text{RT}} & \text{for } l = N_T. \end{cases} \quad (92)$$

What is more, even in the cut-off register case, eq. (89) is the same as that in lemma 1 up to clockwork equivalence.

Now that we have justified the form of the channel eq. (89), we proceed to calculate its clockwork channel as per the definition eq. (22):

$$\mathcal{M}_{C \rightarrow C}^t(\cdot) = \lim_{\substack{N \rightarrow +\infty \\ N \in \mathbb{N}}} \text{tr}_{\text{RT}} \left[\left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ(N-1)} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{t/N, k}(\cdot) \right]. \quad (93)$$

Furthermore, observe that one has the following expansion

$$\left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{\delta t} \right)^{\circ(m-1)} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{\delta t, k}(\cdot) = \sum_{l=0}^m M_m^{(l, \delta t)}(\cdot) \otimes |l+k\rangle\langle l+k|_{\text{RT}}, \quad (94)$$

for some $M_m^{(l, \delta t)} \in \mathcal{B}(\mathcal{H}_C)$ which may be k -dependent. To see that a solution of the form eq. (94) exists, note that every application of the channel eq. (89b) only contains terms which either keep the support of the register the same, i.e. has support on $|k\rangle\langle k|_{\text{RT}}$, or increases by one, i.e. has support on $|k+1\rangle\langle k+1|_{\text{RT}}$. Furthermore, the summation ranges from 0 to m after m applications of the channel, which follows easily inductively.

Hence by comparing eqs. (93) and (94) one sees that to prove that $\mathcal{M}_{C \rightarrow C}^t$ is k -independent, it suffices to show that the channels $\left(M_m^{(l, \delta t)}(\cdot) \right)_{l=0}^m$ are k -independent for all $m \in \mathbb{N}_{>0}$, $\delta t \geq 0$.

This is most easily shown by induction. We start by showing that $\left(M_1^{(l, \delta t)}(\cdot) \right)_{l=0}^1$ are k -independent by equating

$$\mathcal{M}_{C \rightarrow \text{CR}_T}^{\delta t, k}(\cdot) = \sum_{l=0}^1 M_1^{(l, \delta t)}(\cdot) \otimes |l+k\rangle\langle l+k|_{\text{RT}} \quad (95)$$

with eqs. (89c) and (89d) to find $M_1^{(0, \delta t)} = \mathcal{I}_C + \delta t \mathcal{C}_{(1)}(\cdot)$, $M_1^{(1, \delta t)} = \delta t \mathcal{C}_{(2)}(\cdot)$. Therefore, $\left(M_1^{(l, \delta t)}(\cdot) \right)_{l=0}^1$ are k -independent since $\mathcal{C}_{(1)}$ and $\mathcal{C}_{(2)}$ are. Now assume $\left(M_m^{(l, \delta t)}(\cdot) \right)_{l=0}^m$ are k -independent. We show that it follows that $\left(M_{m+1}^{(l, \delta t)}(\cdot) \right)_{l=0}^{m+1}$ are k -independent:

$$\sum_{l=0}^{m+1} M_{m+1}^{(l, \delta t)}(\cdot) \otimes |l+k\rangle\langle l+k|_{\text{RT}} \quad (96)$$

$$= \left(\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{\delta t} \right)^{\circ m} \circ \mathcal{M}_{C \rightarrow \text{CR}_T}^{\delta t, k}(\cdot) \quad (97)$$

$$= \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{\delta t} \circ \left(\sum_{l=0}^m M_m^{(l, \delta t)}(\cdot) \otimes |l+k\rangle\langle l+k|_{\text{RT}} \right) \quad (98)$$

$$= \sum_{l=0}^m M_1^{(0, \delta t)} \circ M_m^{(l, \delta t)}(\cdot) \otimes |l+k\rangle\langle l+k|_{\text{RT}} + M_1^{(1, \delta t)} \circ M_m^{(l, \delta t)}(\cdot) \otimes |l+k+1\rangle\langle l+k+1|_{\text{RT}}. \quad (99)$$

Therefore, since $M_1^{(0,\delta t)}$, $M_1^{(1,\delta t)}$ are manifestly k -independent and $M_m^{(l,\delta t)}$ are k -independent by assumption, it follows by equating terms in lines (96) and (99), that $\left(M_{m+1}^{(l,\delta t)}(\cdot)\right)_{l=0}^{m+1}$ are k -independent. Hence by induction, we conclude that $\left(M_m^{(l,\delta t)}(\cdot)\right)_{l=0}^m$ are k -independent for all $m \in \mathbb{N}_{>0}$ and thus that eq. (93) is k -independent also.

Observe that the only difference between the periodic register case and the cut-off register case, are the kets $|l\rangle_{\text{RT}}$, which are defined in eq. (90). Since $\text{tr}[|l\rangle\langle l|_{\text{RT}}] = 1$ for all $l \in \mathbb{N}_{\geq 0}$ in both cases, we conclude that eq. (93) is the same in both cases.

Finally, proceeding similarly to the above inductive proof, we observe that

$$\mathcal{M}_{\text{C} \rightarrow \text{C}}^t(\cdot) = \lim_{\substack{N \rightarrow +\infty \\ N \in \mathbb{N}}} \left(\mathcal{M}_{\text{C} \rightarrow \text{C}}^{t/N}\right)^{\circ N}, \quad (100)$$

where

$$\left(\mathcal{M}_{\text{C} \rightarrow \text{C}}^{t/N}\right)^{\circ N}(\cdot) = \sum_{l=0}^N M_N^{(l, t/N)}(\cdot). \quad (101)$$

It is now straightforward to verify that the above channel constitutes a dynamical semigroup thus admitting a generator representation of the form $\mathcal{M}_{\text{C} \rightarrow \text{C}}^t(\cdot) = e^{t\mathcal{L}_{\text{C}}}(\cdot)$ with

$$\mathcal{L}_{\text{C}}(\cdot) = \lim_{t \rightarrow 0^+} \frac{\mathcal{M}_{\text{C} \rightarrow \text{C}}^t - \mathcal{I}_{\text{CRT}}}{t} = \lim_{t \rightarrow 0^+} \frac{M_1^{(0,t)} + M_1^{(1,t)} - \mathcal{I}_{\text{CRT}}}{t} = \mathcal{C}_{(1)}(\cdot) + \mathcal{C}_{(2)}(\cdot). \quad (102)$$

Thus $\mathcal{L}_{\text{C}}(\cdot)$ is equal to the r.h.s. of eq. (21b) under the replacements $\tilde{H} \mapsto H$, $\tilde{L}_j \mapsto L_j$ and $\tilde{J}_j^{(l)} \mapsto J_j$. $\blacksquare$

B.5 Proofs of lemmas 2 and 3

Lemma 2 (Entry-wise and p -norm equivalence). *Let the complex finite dimensional matrix $A \in \mathbb{C}^l \times \mathbb{C}^m$ have entries denoted by $A_{qr} \in \mathbb{C}$. Let $\|\cdot\|_p$ denote the operator norm on $\mathbb{C}^l \times \mathbb{C}^m$ induced by the vector p -norm on vector in $\mathbb{C}^m$. Let $o(\delta)$ denote “little o ” notation for some limit $\delta \rightarrow a$. It follows that*

$$A_{qr} = o(\delta) \quad (103)$$

for all $q = 1, 2, 3, \dots, l; r = 1, 2, 3, \dots, m$ if and only if

$$\|A\|_p = o(\delta). \quad (104)$$

The statement holds for any $p > 0$.

Proof. Given the expression for the operator norm, namely

$$\|A\|_p = \sup_{v \in \mathbb{R}^m; \|v\|_p \leq 1} \left(\sum_{q=1}^l \left| \sum_{r=1}^m A_{qr} v_r \right|^p \right)^{1/p}, \quad (105)$$

the direction $A_{qr} = o(\delta) \forall q, r \implies \|A\|_p = o(\delta)$ follows easily. To prove the converse, we will use proof by contradiction. Suppose $\|A\|_p = o(\delta)$, and by contradiction, assume that there exists matrix entry A_{st} s.t. $A_{st} \neq o(\delta)$. Therefore,

$$\lim_{\delta \rightarrow a} \frac{|A_{st}|}{\delta} > 0. \quad (106)$$

We can now use the definition of the operator norm to achieve the lower bound

$$\|A\|_p \geq \left(\sum_{q=1}^l \left| \sum_{r=1}^m A_{qr} \delta_{r,t} \right|^p \right)^{1/p} = \left(\sum_{q=1}^l |A_{qt}|^p \right)^{1/p} \geq |A_{st}|. \quad (107)$$

Therefore, dividing both sides by δ followed by taking the limit $\delta \rightarrow a$ we achieve using eq. (106) that $\lim_{\delta \rightarrow a} \frac{\|A\|_p}{\delta} > 0$. This contradicts the assertion that $\|A\|_p = o(\delta)$. $\blacksquare$

Lemma 3 (Maximum number of Lindblad operators needed). *Consider a clockwork of Hilbert space dimension $d \in \mathbb{N}_{>0}$. For every Hermitian operator H and two finite sequences of operators $(L_j)_{j=1}^{N_L}, (J_j)_{j=1}^{N_L}$ on $\mathcal{B}(\mathcal{H}_C)$ giving rise to the channel $\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\cdot)$ via eq. (30a); there exists $2(d^2 - 1)$ new operators $(L'_j)_{j=1}^{d^2-1}, (J'_j)_{j=1}^{d^2-1}$ on $\mathcal{B}(\mathcal{H}_C)$ such that the channel $\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\cdot)$ is invariant under the mappings*

$$\sum_{j=1}^{N_L} -\frac{1}{2} \{L_j^\dagger L_j + \theta(k) J_j^\dagger J_j, \cdot\} + L_j(\cdot) L_j^\dagger \mapsto \sum_{j=1}^{d^2-1} -\frac{1}{2} \{L_j'^\dagger L'_j + \theta(k) J_j'^\dagger J'_j, \cdot\} + L'_j(\cdot) L_j'^\dagger \quad (108)$$

$$\sum_{j=1}^{N_L} J_j(\cdot) J_j^\dagger \mapsto \sum_{j=1}^{d^2-1} J'_j(\cdot) J_j'^\dagger \quad (109)$$

in eqs. (30b) and (30d) respectively.

Proof. In the proof of lemma 1, N_L is simply a non negative integer arising from writing an arbitrary implementation of the channel $\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\cdot)$ is Kraus form. To prove lemma 3, it will suffice to prove that without loss of generality, N_L in the proof of lemma 1 can be chosen to be equal to $d^2 - 1$. To do so, we start by recalling eq. (49) in the proof of lemma 1:

$$\mathcal{M}_{C \rightarrow CR_T}^{t,k}(\rho_C) = \sum_{l=0}^{N_T} \sum_{j=0}^{N_L} N_j^{(k)}(l, t) \rho_C N_j^{(k)\dagger}(l, t) \otimes |l\rangle\langle l|_{R_T}, \quad (110)$$

for all $k = 0, 1, \dots, N_T; t \geq 0$. Recall also that $N_j^{(k)}(l, t) := \langle l|_{R_T} Q_j^{(k)}(t) : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C)$ where $Q_j^{(k)}(t) : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{R_T})$ and

$$\sum_{j=0}^{N_L} Q_j^{(k)}(t)^\dagger Q_j^{(k)}(t) = \mathbb{1}_C. \quad (111)$$

Using the resolution of the identity, eq. (111) implies

$$\sum_{l=0}^{N_T} \sum_{j=0}^{N_L} N_j^{(k)}(l, t)^\dagger N_j^{(k)}(l, t) = \mathbb{1}_C. \quad (112)$$

Thus since the basis $\{|l\rangle_{R_T}\}_{l=0}^{N_T}$ is orthogonal, the operators $\{N_j^{(k)}(l, t)\}_{j=0}^{N_L}$ are completely arbitrary and independent from $\{N_j^{(k)}(l', t)\}_{j=0}^{N_L}$ for all $l \neq l', l = 0, 1, 2, \dots, N_T$; up to the normalisation imposed by eq. (112). The lemma now follows directly from Choi's theorem [30]. To see this, note that the channel

$$\sum_{j=0}^{N_L} N_j^{(k)}(l, t) (\cdot) N_j^{(k)\dagger}(l, t) : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_C), \quad (113)$$

is completely positive for all $N_L \in \mathbb{N}_{>0}$. Therefore, via Choi's theorem there exists operators $(N_j^{(k)'})_{j=1}^{d^2-1}$ such that

$$\sum_{j=0}^{N_L} N_j^{(k)}(l, t) (\cdot) N_j^{(k)\dagger}(l, t) = \sum_{j=0}^{d^2-1} N_j^{(k)'}(l, t) (\cdot) N_j^{(k)'\dagger}(l, t), \quad (114)$$

where

$$\sum_{l=0}^{N_T} \sum_{j=0}^{d^2-1} N_j^{(k)'}(l, t)^\dagger N_j^{(k)'}(l, t) = \mathbb{1}_C. \quad (115)$$

However, since the operators $(N_j^{(k)})_{j=1}^{N_L}$ were arbitrary to begin with (up to the aforementioned normalisation which the operators $(N_j^{(k)'}(l, t))_{j=1}^{d^2-1}$ also satisfy), we can always choose $N_L = d^2 - 1$ from the outset. ■