Cellular automata in operational probabilistic theories

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The theory of cellular automata in operational probabilistic theories is developed. We start introducing the composition of infinitely many elementary systems, and then use this notion to define update rules for such infinite composite systems. The notion of causal influence is introduced, and its relation with the usual property of signalling is discussed. We then introduce homogeneity, namely the property of an update rule to evolve every system in the same way, and prove that systems evolving by a homogeneous rule always correspond to vertices of a Cayley graph. Next, we define the notion of locality for update rules. Cellular automata are then defined as homogeneous and local update rules. Finally, we prove a general version of the wrapping lemma, that connects CA on different Cayley graphs sharing some small-scale structure of neighbourhoods.

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In the last two decades a new approach to quantum foundations arose, grounded on the quantum information experience [1, 2, 3, 4]. This line of research on the fundamental aspects of quantum theory benefits from new ideas, concepts and

methods [5, 6, 7] that lead to a wealth of remarkable results [8, 9, 10, 11]. In particular, quantum theory can be now understood as a theory of information processing, that is selected among a universe of alternate theories [12, 13, 1, 14] by operational principles about the possibility or impossibility to perform specific information processing tasks [9, 15].

The scenario of alternate theories among which the principles select quantum theory is called the framework of Operational Probabilistic Theories (OPTs) [8, 15], and has connections with the less structured concept of Generalized Probabilistic Theory (GPT) [5, 16, 17], as well as the diagrammatic category theoretical approach often referred to as quantum pictorialism [18, 19]. Inspiration for the framework came also from quantum logic [20, 21].

Quantum theory, namely the theory of Hilbert spaces, density matrices, completely positive maps and POVMs (in particular we refer to the elegant exposition of Ref. [22]), can thus be reformulated as a special theory of information processing. Besides the many advantages of this result, one has to face a main issue: the theory as such is devoid of its physical content. Elementary systems are thought of as elementary information carriers—brutally speaking, memory cells—rather than elementary particles or fields in space-time. While this framework is satisfactory for an effective, empirical description of physical experiments, when it comes to provide a theoretical foundation for the physics of elementary systems, the informational approach at this stage calls for a way to re-embrace mechanical notions such as mass, energy, position, space-time, and complete the picture encompassing the dynamics of quantum systems.

A recent proposal for this endeavour is based on the idea that physical laws have to be ultimately understood as algorithms, that make systems evolve, changing their state, exactly as the memory cells of a computer are updated by the run of an algorithm [23, 24, 25]. Such a program already achieved successful results in the reconstruction of Weyl’s, Dirac’s and Maxwell’s equations (for a comprehensive review see Ref. [26]). The most natural candidate algorithm for describing a physical law in this context is a cellular automaton. The theory of cellular automata is a wide and established branch of computer sci-

ence. The notion of a cellular automaton for quantum systems was first devised as the quantum version of its classical counterpart, e.g. in Refs. [27, 28, 29, 30], but turned out to give rise to a rather independent theory, developed starting from Ref. [31]: the theory of Quantum Cellular Automata (QCAs). The latter counts presently various important results—see e.g. Refs. [32, 33], just to mention a few. We stress that, most commonly, QCAs are defined to be reversible algorithms, and most results in the literature are proved with this hypothesis. It is known, however, that many desirable preproperties fail to hold in the irreversible case (see e.g. Ref. [34]). In the present work we will not consider irreversible cellular automata, and leave this subject for further studies.

In order to use cellular automata as candidate physical laws in the foundational perspective based on OPTs, one has two choices at hand. The first one is to start treating automata within a definite theory, and this is the approach adopted so far, in particular within Fermionic theory [35, 36, 37]. In the linear case, Fermionic cellular automata reduce to Quantum Walks, and this brings in the picture all the tools from such widely studied topic [38, 39, 40, 41, 42, 43]. The non-linear case is far less studied [44, 45], and does not offer as many results for the analysis. Needless to say, this approach faces difficulties that are specific of the theory at hand, and prevents a comparison of different theories on a ground that is genuinely physical.

The second approach is initiated in the present paper, and consists in defining cellular automata in the general context of OPTs. This perspective offers the possibility of extracting the essential features of the theory of quantum or Fermionic cellular automata, those that are not specific of the theory but are well suited in any theory of information processing. As a consequence, one can generalise some results, and figure out why and how others fail to extend to the broader scenario. Moving a much less structured mathematical context, this approach can use only few tools, but provides results that have the widest applicability range.

2 Detailed outlook

In this section we provide a short, non technical discussion of the main results. The purpose of this work is to define cellular automata in OPTs, and prove some general results that will help applying the theory to special cases of interest.

A cellular automaton is an algorithm that updates the information stored in an array of memory cells in discrete steps, in such a way that one needs to read the content of a few neighbouring cells to determine the state of a given cell at the next step. Cellular automata in the literature are often, but not always, defined to be homogeneous: in this case the local rule for the update of the cell is the same for every cell. Here we will adopt homogeneity, but the subject is presented so that the generalisation of definitions to inhomogeneous CA is straightforward.

Typically, the interesting case of a cellular automaton is the one involving an infinite memory array. The first challenge we have to face is then to extend the theory of OPTs from finite, arbitrarily large composite systems to actually infinite ones. This piece of theory has an interest per se, for many reasons ranging from the possibility to introduce thermodynamic limits to the extension of the theory of C^* and von Neumann algebras.

We then start with a review of the framework of OPTs, and build the necessary tools to define infinite composite systems. The starting point is the construction of mathematical objects that describe measurements on finitely many systems within an infinite array—the OPT counterpart of the space of local effects of quantum theory. Effects for the infinite system are then defined as limits of Cauchy sequences of local effects. Since the introduction of a suitable topology for the definition of limits is needed, we open the paper with section 3, where a review of OPTs is provided, along with a few new results that will be useful, and a rather consistent part of the section will be dedicated to the introduction and discussion of norms that will provide the necessary topological framework for a consistent definition of limits.

The subject of the subsequent Section 4 is then the construction of the Banach space of effects for the infinite composite system, and consequently the construction of the space of states as suitable linear functionals on effects. An important subsection will be dedicated to the construction of

the algebra of quasi-local transformations, that allows for the description of transformations on an infinite system that can be arbitrarily well approximated by local operations on finitely many subsystems. Indeed, an important part of the theory of CAs in OPTs is built by ruling the way in which quasi-local transformations are transformed by the CA. In particular, the way in which the CA propagates the effects of a local transformation on surrounding subsystems will be the key to the definition of the neighbourhood of the subsystem, a concept that is central to the theory of CAs.

Once this is done, the next step consists in defining update rules, and their admissibility conditions, which are the subject of Section 5, along with causal influence and a block-decomposition theorem. Update rules are defined in the first place as automorphisms $\mathcal{V}$ of the space of quasi-local effects, but an important request they must abide is that when they act on a quasi-local transformation $\mathcal{A}$ by conjugation as $\mathcal{V}\mathcal{A}\mathcal{V}^{-1}$, the obtained transformation is again quasi-local.

The next step is taken in section 6, and consists in defining the property of homogeneity. The latter presents with some difficulties, stemming from the fact that, as we mentioned above, we are defining update rules *prior* to any mechanical notion. This implies that even space-time is not available at this fundamental stage, and without geometry we cannot define homogeneity as translational invariance under the group corresponding to a given space-time. On the contrary, we will define homogeneity by formalising the idea that every single cell has to be treated equally by the update rule, and this notion will be defined operationally, requiring that no experiment made of local operations will allow establishing any difference between cells. As a consequence of homogeneity, one can prove that every homogeneous update rule is underpinned by the Cayley graph of some group, generalising a result of Refs. [25, 46]. The mathematical theory known as *geometric group theory* then tells us that the Cayley graph representing the causal connections of cells in the memory array, being a metric space, uniquely identifies an equivalence class of metric spaces, that captures both the algebraic and geometric essential features of the group. Astonishingly, for Cayley graphs of finitely presented groups—which is the case for cellular automata

as we define them here—the equivalence class always contains a smooth manifold of dimension at most four (see Ref. [47], pag. 90). This result means that we can always think of a cellular automaton as if it was embedded in a Riemannian manifold, in such a way that the Riemannian distance between nodes is almost the same as the distance between nodes given in terms of steps along the graph edges.

Also the notion of locality, presented in section 7, comes with its own difficulties, that can be overcome by proving a generalisation of the result known as “unitarity plus causality implies localizability” [32]. This is where the theory, which is inspired by that of QCAs, deeply differs from the theory of classical cellular automata. In particular, considering the collection of transformations $\mathcal{V}\mathcal{A}\mathcal{V}^{-1}$ for all $\mathcal{A}$ acting on a given system g , we will define the neighbourhood of g as the set of systems on which transformations $\mathcal{V}\mathcal{A}\mathcal{V}^{-1}$ act non trivially.

Once the theory of cellular automata is fully developed, some results are proved in Section 8. In particular, a very useful theorem is the wrapping lemma, which under very wide hypotheses—though not universal—allows for the classification of automata on a given infinite graph by classifying automata on any suitably “wrapped” finite version of the same graph.

In Section 9, a few examples are reviewed. In particular, using the general notion of locality, we apply the definition of causal influence and the neighbourhood scheme to the case of classical cellular automata. With the above definitions at hand, we show that allegedly local automata are actually non-local. This solves a long standing puzzle about the connection between classical and quantum automata—the so-called quantisation of classical automata [31, 48].

The paper is concluded by Section 10 with a summary of the results and some closing remarks.

3 Operational Probabilistic Theories

In this section we will provide a thorough introduction to OPTs, which is partly a review of the literature, and partly presentation of new results that will be used in the remainder. We provide here a sketchy introduction before going through formal definitions, and use Quantum Theory as an illustrative example for the main notions in

the framework.

Quantum theory is about system types A ¹—namely complex Hilbert spaces H_A that are classified by their dimension $d_A = \dim(H_A)$ —and transformations that can occur on systems as a consequence of undergoing a *test*. Tests are represented by quantum instruments $E_X = \{\mathcal{E}_i\}_{i \in X}$, i.e. a collection of Completely Positive (CP) maps $\mathcal{E}_i$ that sum to a trace-preserving one (a channel): $\mathcal{E} = \sum_{i \in X} \mathcal{E}_i$. The quantum operation $\mathcal{E}_i$ —a CP trace non-increasing map—represents the change in the system occurring upon the event of a specific outcome $i \in X$ in the test. In the diagrammatic language of OPTs systems are represented as labelled wires, and instruments or quantum operations as boxes with an input and output wire, e.g.

$$\text{---}^A \boxed{E_X} \text{---}^B, \quad \text{---}^A \boxed{\mathcal{E}_i} \text{---}^B,$$

respectively. States can be considered as special transformations where the input system is I , having $H_I = \mathbb{C}$, i.e. $d_I = 1$. Notice that the *preparation test*, i.e. the most general test from I to A , represents a *probabilistic* preparation procedure where some state in the collection $P_X = \{\rho_i\}_{i \in X}$ is prepared, with the sub-normalised density matrix ρ_i occurring upon reading the outcome $i \in X$. The probability of occurrence of ρ_i is $\text{Tr}[\rho_i]$, and $\rho = \sum_{i \in X} \rho_i$ has unit trace. Preparation tests or states for system A are represented by special diagrams

$$\boxed{P_X} \text{---}^A, \quad \boxed{\rho_i} \text{---}^A,$$

respectively. The space $\mathcal{H}_A$ of Hermitian operators on H_A , whose dimension is $D_A = d_A^2$, is the real span of states of system A .

A POVM $Q_Y = \{Q_j\}_{j \in Y}$ is a collection of *effects* $0 \leq Q_j \leq I_A$ that sum to the identity operator: $\sum_{j \in Y} Q_j = I_A$. This kind of test can be seen as a quantum instrument from A to I , namely a collection of linear functionals on the real space spanned by density matrices, where the functionals $a_j(\rho)$ are defined as $a_j(\rho) := \text{Tr}[\rho Q_j]$. The

¹We remark that for the purpose of information processing the type of a system is captured by the dimension of the corresponding Hilbert space, e.g. the electron spin is of the same type as the photon polarisation. This is slightly different than the usual notion of system type in physics, where the two types in the above example are different.

diagrammatic representation of POVMs and effects is the following

$$\text{---} \overset{A}{\text{---}} \boxed{Q_Y} \text{---}, \quad \text{---} \overset{A}{\text{---}} \boxed{Q_j} \text{---},$$

respectively.

Composite systems, such as AB , correspond to the Hilbert space $H_A \otimes H_B$ of dimension $d_{AB} = d_A d_B$. Tests $\{\mathcal{E}_i\}_{i \in X}$ from A to B and $\{\mathcal{F}_j\}_{j \in Y}$ from B to C can be run in sequence, obtaining a new test: $\{\mathcal{D}_{ij}\}_{(i,j) \in X \times Y}$ where $\mathcal{D}_{ij} := \mathcal{F}_j \mathcal{E}_i$. On the other hand, tests $\{\mathcal{E}_i\}_{i \in X}$ from A to B and $\{\mathcal{F}_j\}_{j \in Y}$ from B to B' on subsystems of a composite system AA' can be run in parallel, obtaining the test $\{\mathcal{D}_{ij}\}_{(i,j) \in X \times Y}$ with $\mathcal{D}_{ij} := \mathcal{E}_i \otimes \mathcal{F}_j$. In diagrams, we draw for quantum operations

$$\begin{aligned} \text{---} \overset{A}{\text{---}} \boxed{\mathcal{F}_j \mathcal{E}_i} \text{---} \overset{C}{\text{---}} &= \text{---} \overset{A}{\text{---}} \boxed{\mathcal{E}_i} \text{---} \overset{B}{\text{---}} \boxed{\mathcal{F}_j} \text{---} \overset{C}{\text{---}}, \\ \text{---} \overset{AA'}{\text{---}} \boxed{\mathcal{E}_i \otimes \mathcal{F}_j} \text{---} \overset{BB'}{\text{---}} &= \begin{array}{c} \text{---} \overset{A}{\text{---}} \boxed{\mathcal{E}_i} \text{---} \overset{B}{\text{---}} \\ \text{---} \overset{A'}{\text{---}} \boxed{\mathcal{F}_j} \text{---} \overset{B'}{\text{---}} \end{array}, \end{aligned}$$

and analogously for instruments.

Very relevant structures in the theory are the real space $\mathcal{H}_A$ of Hermitian operators on H_A , spanned by quantum states, the cone of positive operators in $\mathcal{P}_A \subseteq \mathcal{H}_A$, the convex set of states $[\![A]\!]$, obtained by intersecting the positive cone with the half-space $\text{Tr}[X] \leq 1$, and the convex set of deterministic states $[\![A]\!]_1$, obtained by intersecting the positive cone with the affine hyperplane $\text{Tr}[X] = 1$. Similar structures can be generalised to the space spanned by quantum operations. In the general framework of OPTs, we will systematically refer to the generalisation of the above concepts.

3.1 Formal framework

The framework of OPTs is meant to capture the main traits of Quantum Theory (shared e.g. by Classical Theory, or Fermionic Theory, etc.) summarised above, and use them as defining properties of a family of abstract theories that might be candidates for an alternative representation of elementary physical systems and their transformations. In the remainder of this section we provide a brief review of the framework of OPTs (for reference see e.g. [15, 8, 9]). Some of the most relevant differences with respect to the quantum case stem from the fact that systems might not compose with the tensor product rule.

We warn the reader that this presentation has some elements that are slightly different from other reviews in the literature, and is tailored to ease the presentation of subsequent material. Most of the results in the present sections are new, and their proof will be given. Those theorems that are not original are not proved, and due reference is provided.

An operational theory Θ consists in i) a collection $\text{Test}(\Theta)$ of tests $T_X^{A \rightarrow B}$, each labelled by input and output letters from a collection $\text{Sys}(\Theta)$ denoting system types, e.g. $A \rightarrow B$ —that will be systematically omitted—and by a finite set of outcomes X ; for every pair of types $A, B \in \text{Sys}(\Theta)$ the set of tests of type $A \rightarrow B$ is denoted $\langle\langle A \rightarrow B \rangle\rangle$; ii) a very basic associative rule for sequential composition: the test $T_X \in \langle\langle A \rightarrow B \rangle\rangle$ can be followed by the test $S_Y \in \langle\langle A' \rightarrow B' \rangle\rangle$ if $A' \equiv B$, thus obtaining the sequential composition $ST_{X \times Y} \in \langle\langle A \rightarrow B' \rangle\rangle$; iii) a rule $\otimes : (A, B) \mapsto AB$ for composing labels in parallel, and a corresponding rule for tests $\otimes : (S_X, T_Y) \mapsto (S \otimes T)_{X \times Y}$, with the following properties

1. Associativity: $(AB)C = A(BC)$.
2. For every $S_X \in \langle\langle A \rightarrow B \rangle\rangle$ and $T_Y \in \langle\langle C \rightarrow D \rangle\rangle$, one has $S_X \otimes T_Y \in \langle\langle AC \rightarrow BD \rangle\rangle$. Associativity of $\otimes$ holds:

$$(S_X \otimes T_Y) \otimes W_Z = S_X \otimes (T_Y \otimes W_Z).$$

3. Unit: there is a label I such that $IA = AI = A$ for every $A \in \text{Sys}(\Theta)$.
4. Identity: for every $A \in \text{Sys}(\Theta)$, a test $I_A \in \langle\langle A \rightarrow A \rangle\rangle$ such that $I_B S_X = S_X I_A = S_X$, for every $S_X \in \langle\langle A \rightarrow B \rangle\rangle$.
5. For every $A_X \in \langle\langle A \rightarrow B \rangle\rangle$, $B_Y \in \langle\langle B \rightarrow C \rangle\rangle$, $D_Z \in \langle\langle D \rightarrow E \rangle\rangle$, $E_W \in \langle\langle E \rightarrow F \rangle\rangle$, one has

$$(B_Y \otimes E_W)(A_X \otimes D_Z) = (B_Y A_X) \otimes (E_W D_Z). \quad (1)$$

6. Braiding: for every pair of system types A, B , there exist tests $S_{AB}, S_{AB}^* \in \langle\langle AB \rightarrow BA \rangle\rangle$ such that $S_{BA}^* S_{AB} = S_{BA} S_{AB}^* = I_{AB}$, and $S_{AB}(A_X \otimes B_Y) = (B_Y \otimes A_X)S_{AB}$. Moreover,

$$\begin{aligned} S_{(AB)C} &= (I_A \otimes S_{BC})(S_{AC} \otimes I_B), \\ S_{A(BC)} &= (S_{AB} \otimes I_C)(I_B \otimes S_{AC}). \end{aligned}$$

When $S_{AB}^* \equiv S_{AB}$, the theory is *symmetric*.

All the theories developed so far are symmetric.

All tests of an operational theory are (finite) collections of *events*: $\langle\langle A \rightarrow B \rangle\rangle \ni R_X = \{\mathcal{R}_i\}_{i \in X}$. If $\langle\langle A \rightarrow B \rangle\rangle \ni R_X = \{\mathcal{R}_i\}_{i \in X}$ and $\langle\langle B \rightarrow C \rangle\rangle \ni T_Y = \{\mathcal{T}_j\}_{j \in Y}$, then

$$\langle\langle A \rightarrow C \rangle\rangle \ni (TR)_{X \times Y} := \{\mathcal{T}_j \mathcal{R}_i\}_{(i,j) \in X \times Y}.$$

Similarly, for $R_X \in \langle\langle A \rightarrow B \rangle\rangle$ and $T_Y \in \langle\langle C \rightarrow D \rangle\rangle$,

$$(R \otimes T)_{X \times Y} := \{\mathcal{R}_i \otimes \mathcal{T}_j\}_{(i,j) \in X \times Y}.$$

The set of events of tests in $\langle\langle A \rightarrow B \rangle\rangle$ is denoted by $\llbracket A \rightarrow B \rrbracket$. By the properties of sequential and parallel composition of tests, one can easily derive associativity of sequential and parallel composition of events, as well as the analogue of Eq. (1). For every test $T_X \in \langle\langle A \rightarrow B \rangle\rangle$ with $T_X = \{\mathcal{T}_i\}_{i \in X}$, and every disjoint partition $\{X_j\}_{j \in Y}$ of $X = \bigcup_{j \in Y} X_j$, one has a *coarse graining* operation that maps T_X to $T'_Y \in \langle\langle A \rightarrow B \rangle\rangle$, with $T'_Y = \{\mathcal{T}'_j\}_{j \in Y}$. We define $\mathcal{T}_{X_j} := \mathcal{T}'_j$. The parallel and sequential compositions distribute over coarse graining:

$$\begin{aligned} \mathcal{T}_{X_j} \otimes \mathcal{R}_k &= (\mathcal{T} \otimes \mathcal{R})_{X_j \times \{k\}}, \\ \mathcal{A} \mathcal{T}_{X_j} \mathcal{B}_k &= (\mathcal{A} \mathcal{T} \mathcal{B})_{\{l\} \times X_j \times \{k\}}. \end{aligned}$$

Notice that for every test $T_X \in \langle\langle A \rightarrow B \rangle\rangle$ there exists the singleton test $T'_* := \{\mathcal{T}_X\}$. One can easily prove that the identity test I_A is a singleton: $I_A = \{\mathcal{I}_A\}$, and $\mathcal{I}_B \mathcal{T} = \mathcal{T} \mathcal{I}_A$ for every event $\mathcal{T} \in \llbracket A \rightarrow B \rrbracket$. Similarly, for $S_{AB} = \{\mathcal{S}_{AB}\}$ and $S_{AB}^* = \{\mathcal{S}_{AB}^*\}$ we have $\mathcal{S}_{BA}^* \mathcal{S}_{AB} = \mathcal{S}_{BA} \mathcal{S}_{AB}^* = \mathcal{I}_{AB}$. The collection of events of an operational theory Θ will be denoted by $\text{Ev}(\Theta)$. The above requirements make the collections $\text{Test}(\Theta)$ and $\text{Ev}(\Theta)$ the families of morphisms of two braided monoidal categories with the same objects—system types $\text{Sys}(\Theta)$.

An operational theory is an OPT if the tests $\langle\langle I \rightarrow I \rangle\rangle$ are probability distributions: $1 \geq \mathcal{T}_i = p_i \geq 0$, so that $\sum_{i \in X} p_i = 1$, and given two tests $S_X, T_Y \in \langle\langle I \rightarrow I \rangle\rangle$ with $\mathcal{S}_i = p_i$ and $\mathcal{T}_i = q_i$, the following identities hold

$$\begin{aligned} \mathcal{S}_i \otimes \mathcal{T}_j &= \mathcal{S}_i \mathcal{T}_j := p_i q_j, \\ \mathcal{T}_{X_j} &:= \sum_{i \in X_j} p_i, \end{aligned}$$

meaning that events in the same test are mutually exclusive and events in different tests of system I

are independent. While it is immediate that $1 \in \llbracket I \rightarrow I \rrbracket$ —since $\{1\}_* = \{\mathcal{I}_I\}$ is the only singleton test—we will assume that $0 \in \llbracket I \rightarrow I \rrbracket$. This means that we can consider e.g. tests of the form $\{1, 0, 0\}$.

Events in $\llbracket A \rightarrow B \rrbracket$ are called *transformations*. As a consequence of the above definitions, every set $\llbracket A \rrbracket := \llbracket I \rightarrow A \rrbracket$ can be viewed as a set of functionals on $\llbracket \bar{A} \rrbracket$. As such, it can be viewed as a spanning subset of the real vector space $\llbracket A \rrbracket_{\mathbb{R}}$ of linear functionals on $\llbracket \bar{A} \rrbracket$. On the other hand $\llbracket \bar{A} \rrbracket := \llbracket A \rightarrow I \rrbracket$ is a separating set of positive linear functionals on $\llbracket A \rrbracket$, which then spans the dual space $\llbracket A \rrbracket_{\mathbb{R}}^* =: \llbracket \bar{A} \rrbracket_{\mathbb{R}}$. The dimension D_A of $\llbracket A \rrbracket_{\mathbb{R}}$ (which is the same as that of $\llbracket \bar{A} \rrbracket_{\mathbb{R}}$) is called *size* of system A. One can easily prove that in any OPT Θ , I is the unique system with unit size $D_I = 1$. Using the properties of parallel composition, one can also prove that $D_{AB} \geq D_A D_B$. Events in $\llbracket A \rrbracket$ are called *states*, and denoted by lower-case greek letters, e.g. ρ , while events in $\llbracket \bar{A} \rrbracket$ are called *effects*, and denoted by lower-case latin letters, e.g. a . When it is appropriate, we will use the symbol $|\rho\rangle$ to denote a state, and $\langle a|$ to denote an effect. We will also use the circuit notation, where we denote states, transformations and effects by the symbols

$$\boxed{\rho} \xrightarrow{A}, \xrightarrow{A} \boxed{\mathcal{A}} \xrightarrow{B}, \xrightarrow{A} \boxed{a},$$

respectively. Sequential composition of $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket$ and $\mathcal{B} \in \llbracket B \rightarrow C \rrbracket$ is denoted by the diagram

$$\xrightarrow{A} \boxed{\mathcal{B} \mathcal{A}} \xrightarrow{C} = \xrightarrow{A} \boxed{\mathcal{A}} \xrightarrow{B} \boxed{\mathcal{B}} \xrightarrow{C}.$$

For composite systems we use diagrams with multiple wires, e.g.

$$\begin{array}{c} A \quad B \\ \hline \boxed{\mathcal{A}} \\ \hline C \quad D \end{array}.$$

The identity will be omitted: $\xrightarrow{A} \boxed{\mathcal{I}} \xrightarrow{A} = \xrightarrow{A}$. The swap $\mathcal{S}_{AB}$ and its inverse will be denoted as follows

$$\begin{array}{c} A \quad B \\ \hline \boxed{\mathcal{S}} \\ \hline B \quad A \end{array} = \begin{array}{c} A \quad B \\ \hline \text{swap} \\ \hline B \quad A \end{array}$$

$$\begin{array}{c} A \quad B \\ \hline \boxed{\mathcal{S}^*} \\ \hline B \quad A \end{array} = \begin{array}{c} A \quad B \\ \hline \text{swap} \\ \hline B \quad A \end{array}$$

In the present paper we will always assume that the theory under consideration is symmetric, however all the results will be straightforwardly generalisable. We will consequently draw the swap $\mathcal{S}_{AB}$ as

$$\begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{S}} \\ \hline \text{B} \quad \text{A} \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{S}^*} \\ \hline \text{B} \quad \text{A} \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \diagdown \quad \diagup \\ \text{B} \quad \text{A} \end{array}$$

An OPT Θ is specified by the collections of systems and tests, along with the parallel composition rule $\otimes$

$$\Theta \equiv (\text{Test}(\Theta), \text{Sys}(\Theta), \otimes).$$

Definition 1 (Equal transformations). *Let $\mathcal{A}, \mathcal{B} \in \llbracket A \rightarrow B \rrbracket$. Then we define $\mathcal{A} = \mathcal{B}$ if for every system C and every $\rho \in \llbracket AC \rrbracket$ and $a \in \llbracket \bar{B}\bar{C} \rrbracket$, one has*

$$\begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{A}} \\ \hline \text{C} \end{array} \begin{array}{c} \rho \\ \hline \end{array} \begin{array}{c} \text{B} \\ \hline \end{array} \begin{array}{c} a \\ \hline \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{B}} \\ \hline \text{C} \end{array} \begin{array}{c} \rho \\ \hline \end{array} \begin{array}{c} \text{B} \\ \hline \end{array} \begin{array}{c} a \\ \hline \end{array}$$

Notice that, since states separate effects and viceversa effects separate states, the above definition is equivalent to the two following equality criteria.

Lemma 1. *Let $\mathcal{A}, \mathcal{B} \in \llbracket A \rightarrow B \rrbracket$. Then the following conditions are equivalent.*

1. $\mathcal{A} = \mathcal{B}$.
2. For every C and every $\rho \in \llbracket AC \rrbracket$, one has

$$\begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{A}} \\ \hline \text{C} \end{array} \begin{array}{c} \rho \\ \hline \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{B}} \\ \hline \text{C} \end{array} \begin{array}{c} \rho \\ \hline \end{array} \quad (2)$$

3. For every C and every $a \in \llbracket \bar{B}\bar{C} \rrbracket$, one has

$$\begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{A}} \\ \hline \text{C} \end{array} \begin{array}{c} a \\ \hline \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \hline \boxed{\mathcal{B}} \\ \hline \text{C} \end{array} \begin{array}{c} a \\ \hline \end{array} \quad (3)$$

One can show [8] that events $\mathcal{T} \in \llbracket A \rightarrow B \rrbracket$ can be identified with a family of linear maps, one for every C , that characterize the action of $\mathcal{T} \otimes \mathcal{I}_C$ on $\llbracket AC \rrbracket_{\mathbb{R}}$ as a linear map to $\llbracket BC \rrbracket_{\mathbb{R}}$. As anticipated in the introductory paragraph, we remind the reader that some of the difficulties that we will be faced with in the remainder originate from the fact that in a general theory it is

not true that $\llbracket AB \rrbracket_{\mathbb{R}} = \llbracket A \rrbracket_{\mathbb{R}} \otimes \llbracket B \rrbracket_{\mathbb{R}}$, but only $\llbracket A \rrbracket_{\mathbb{R}} \otimes \llbracket B \rrbracket_{\mathbb{R}} \subseteq \llbracket AB \rrbracket_{\mathbb{R}}$. As a consequence, the linear map representing $\mathcal{T}$ on $\llbracket A \rrbracket_{\mathbb{R}}$ is not sufficient to determine the linear map representing $\mathcal{T} \otimes \mathcal{I}_C$ on $\llbracket AC \rrbracket_{\mathbb{R}}$.

One can easily prove that $\llbracket A \rightarrow B \rrbracket$ spans a real vector space $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$. Being $\llbracket A \rightarrow B \rrbracket$ spanning for $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$, the criteria of definition 1 and lemma 1 hold for $\mathcal{A}, \mathcal{B} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$. Elements of $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$ are called *generalized events*. Every space $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$ has a *zero event* $0_{A \rightarrow B} := 0_{I \rightarrow I} \otimes \mathcal{T} \in \llbracket A \rightarrow B \rrbracket$, where $\mathcal{T}$ is an arbitrary event in $\llbracket A \rightarrow B \rrbracket$. As a consequence of the coarse graining rule for tests on $\llbracket I \rightarrow I \rrbracket$, one can easily show that coarse graining of two or more transformations is represented by their sum. Precisely, given a test $\{\mathcal{T}_i\}_{i \in X} \subseteq \llbracket A \rightarrow B \rrbracket$, for $X_0 = \{0, 1\} \subseteq X$ one has $\mathcal{T}'_0 = \mathcal{T}_0 + \mathcal{T}_1$, namely for every system C it is $\mathcal{T}'_0 \otimes \mathcal{I}_C = \mathcal{T}_0 \otimes \mathcal{I}_C + \mathcal{T}_1 \otimes \mathcal{I}_C$. Finally, every set $\llbracket A \rightarrow B \rrbracket$ has a subset consisting in singleton events, that we denote by $\llbracket A \rightarrow B \rrbracket_1$, and call *deterministic*. For $A, B \neq I$, a deterministic event is called *channel*. A channel $\mathcal{U} \in \llbracket A \rightarrow B \rrbracket_1$ is *reversible* if there exists a channel $\mathcal{V} \in \llbracket B \rightarrow A \rrbracket_1$ such that $\mathcal{V}\mathcal{U} = \mathcal{I}_A$, $\mathcal{U}\mathcal{V} = \mathcal{I}_B$.

Let us now define the cones

$$\llbracket A \rightarrow B \rrbracket_+ := \{\lambda \mathcal{T} \mid \lambda \geq 0, \mathcal{T} \in \llbracket A \rightarrow B \rrbracket\}.$$

We will often write $\mathcal{A} \triangleright 0$ as a shorthand for $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_+$. The cone $\llbracket A \rightarrow B \rrbracket_+$ introduces a partial ordering in $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$, defined by

$$\mathcal{A} \triangleright \mathcal{B} \quad \Leftrightarrow \quad (\mathcal{A} - \mathcal{B}) \triangleright 0.$$

OPTs are assumed to have all sets $\llbracket A \rightarrow B \rrbracket$ (and thus also cones $\llbracket A \rightarrow B \rrbracket_+$) closed in the operational norm.

Two systems may be operationally equivalent if they can be mapped one onto the other via a reversible transformation. Clearly, in this case every processing of the first system is perfectly simulated by a processing of the other, and viceversa. We define operationally equivalent systems as follows.

Definition 2. *Let A, B be two systems. We say that A and B are operationally equivalent, in formula $A \cong B$, if there exists a reversible transformation $\mathcal{U} \in \llbracket A \rightarrow B \rrbracket_1$.*

Definition 3. *If A_1 and A_2 are operationally equivalent through $\mathcal{U}$ and B_1 and B_2 through $\mathcal{V}$,*

then, for every system C , $\mathcal{A}_1 \in \llbracket A_1 C \rightarrow B_1 C \rrbracket_{\mathbb{R}}$ and $\mathcal{A}_2 \in \llbracket A_2 C \rightarrow B_2 C \rrbracket_{\mathbb{R}}$ are operationally equivalent if $\mathcal{A}_2 = (\mathcal{V} \otimes \mathcal{I}_C) \mathcal{A}_1 (\mathcal{W}^{-1} \otimes \mathcal{I}_C)$. In particular, $\rho_1 \in \llbracket A_1 C \rrbracket_{\mathbb{R}}$ and $\rho_2 \in \llbracket A_2 C \rrbracket_{\mathbb{R}}$ are operationally equivalent if $\rho_2 = (\mathcal{W} \otimes \mathcal{I}_C) \rho_1$, and $a_1 \in \llbracket \bar{A}_1 C \rrbracket_{\mathbb{R}}$ and $a_2 \in \llbracket \bar{A}_2 C \rrbracket_{\mathbb{R}}$ are operationally equivalent if $a_2 = a_1 (\mathcal{W}^{-1} \otimes \mathcal{I}_C)$.

Lemma 2. *Let AB be operationally equivalent to A . Then B must be operationally equivalent to the trivial system I .*

Proof. First of all, since $D_{AB} \geq D_A D_B \geq D_A$, if $AB \cong A$ the chain of inequalities is saturated, and thus $D_B = 1$. This implies that every state of system AB is of the form $\tau \otimes 1_B$, where $\tau \in \llbracket A \rrbracket$ and 1_B is the unique deterministic state of system B . The same result holds for states of ABC for arbitrary C . Now, let $\mathcal{W} \in \llbracket AB \rightarrow A \rrbracket_1$ be the reversible channel implementing operational equivalence. Let $\eta \in \llbracket AC \rrbracket$, and $(\mathcal{W} \otimes \mathcal{I}_C)(1_B \otimes \eta) = \tilde{\eta} \in \llbracket AC \rrbracket$. One can choose $\mathcal{W}$ as to identify $\tilde{\eta} = \eta$. Indeed, there is a reversible transformation $\mathcal{V} \in \llbracket A \rightarrow A \rrbracket$ such that $(\mathcal{V} \otimes \mathcal{I}_C)\tilde{\eta} = \eta$. In particular, one can choose $\mathcal{V} := 1_B \mathcal{W}^{-1}$, where now 1 represents the unique deterministic effect of B . Right-reversibility of $\mathcal{V}$ is trivial. On the other hand, let $\sigma \in \llbracket AC \rrbracket$, and let $1_B \otimes \tau := (\mathcal{W}^{-1} \otimes \mathcal{I}_C)\sigma \in \llbracket ABC \rrbracket$. Now, after applying the deterministic effect 1_B one has $(\mathcal{V} \otimes \mathcal{I}_C)\sigma = \tau$. One can reverse $\mathcal{V}$ by composing τ with 1_B , obtaining $1_B \otimes \tau$, and then applying $\mathcal{W} \otimes \mathcal{I}_C$, finally obtaining $(\mathcal{W} \otimes \mathcal{I}_C)(1_B \otimes \tau) = \sigma$. With a suitable choice of $\mathcal{W}$, one then has $(\mathcal{W} \otimes \mathcal{I}_C)(1_B \otimes \eta) = \eta$. Let now $1_B \otimes \eta = \tau_{BC} \otimes \nu_A$, with $\nu \in \llbracket A \rrbracket_1$. Discarding system A on both sides via some deterministic effect $e_A \in \llbracket \bar{A} \rrbracket_1$, one obtains $\tau_{BC} = 1_B \otimes \theta$, with $\theta = (e_A \otimes \mathcal{I}_C)\eta$. Finally, this implies that every state of system BC is of the form $\tau_{BC} = 1_B \otimes \theta$, with $\theta \in \llbracket C \rrbracket$. Once we fix $\eta \in \llbracket A \rrbracket_1$, we can then define the reversible channel $\mathcal{W} \in \llbracket BC \rightarrow C \rrbracket$ given by $\mathcal{W} := e_A(\mathcal{W} \otimes \mathcal{I}_C)(\eta \otimes \mathcal{I}_{BC})$. Clearly, $\mathcal{W}(1_B \otimes \theta) = \theta$, and the inverse map $\mathcal{W}^{-1} \in \llbracket C \rightarrow BC \rrbracket$ is obtained as $\mathcal{W}^{-1} := e_A(\mathcal{W}^{-1} \eta \otimes \mathcal{I}_C)$. We then have $BC \simeq C$ for every C , i.e. $B \simeq I$. $\square$

3.2 Causality and the no-restriction hypothesis

In the remainder of the paper we will focus on *causal* theories. The causality property, that we define right away, characterizes those theories

where signals can propagate only in the direction defined by input and output of processes, within a cone of causal influence determined by interactions between systems. These are the only theories where one can consistently use information acquired in a set of tests to condition the choice of subsequent tests, where the partial ordering we are referring to is that determined by the input/output direction.

Definition 4 (Causal theories). *A theory $(\mathcal{T}, A)$ is causal if for every test $\{\mathcal{T}_i\}_{i \in X}$ and every collection of tests $\{\mathcal{S}_j^i\}_{j \in Y}$ labelled by $i \in Y$, the generalized test $\{\mathcal{C}_{i,j}\}_{(i,j) \in X \times Y}$ with*

$$\frac{A}{\boxed{\mathcal{C}_{i,j}}} \frac{C}{\quad} := \frac{A}{\boxed{\mathcal{T}_i}} \frac{B}{\quad} \frac{\boxed{\mathcal{S}_j^i}}{\quad} \frac{C}{\quad},$$

is a test of the theory.

Notice that this notion of causality is strictly stronger than the one usually adopted in the literature about OPTs, that can be summarised as *uniqueness of the deterministic effect* (see e.g. Refs. [8, 9, 15]). Indeed, a first result of crucial importance derives uniqueness of the deterministic effect for every system type in causal theories.

Theorem 1 ([8, 15]). *In a causal theory, for every system type A the set of deterministic effects $\llbracket A \rrbracket_1$ is the singleton $\{e_A\}$.*

A proof of the above theorem, that proceeds by contradiction, can be found in the mentioned references. We remark that, being the notion of causality in these references different from the one defined here, the theorem has a slightly different statement. In the same references, one can find the equivalence of uniqueness of the deterministic effect with non-signalling from output to input.

Theorem 2 ([8, 15]). *In an OPT, every system type A has a unique deterministic effect if and only if the marginal probabilities of preparation tests cannot depend on the choice of subsequent observation tests.*

As a consequence of our notion of causality, in a non-deterministic theory (i.e. a theory where $\llbracket I \rrbracket \neq \{0, 1\}$), every convex combination of tests is a test (see e.g. [15]). This implies that the sets $\llbracket A \rrbracket_*$, $\llbracket \bar{A} \rrbracket_*$, $\llbracket A \rightarrow B \rrbracket_*$ are convex, where $*$ can be replaced by nothing, 1 or $+$. Moreover, the sets with $*$ or $+$ are convex cones.

Another important consequence of causality is that a transformation $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket$ is deterministic if and only if it maps the deterministic effect e_B to the deterministic effect e_A .

Theorem 3 ([8, 15]). *In a causal theory, a transformation $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket$ is a channel iff*

$$\text{---} \boxed{\mathcal{C}} \text{---} \boxed{e} = \text{---} \boxed{e} \text{---}.$$

In causal theories, one can always assume that every state is proportional to a deterministic one. Indeed, including in the theory every state $\rho \in \llbracket A \rrbracket_+$ such that $(e_A|\rho) = 1$ does not introduce any inconsistency with the set of transformations, as we now prove. Let Θ be a causal OPT, and define the theory Θ' through the bijection $\kappa : \text{Sys}(\Theta) \rightarrow \text{Sys}(\Theta') :: A \mapsto A'$, with

$$\begin{aligned} \llbracket A' \rightarrow B' \rrbracket_{\mathbb{R}} &\equiv \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}, \\ \llbracket A' \rightarrow B' \rrbracket_1 &:= \{\mathcal{T} \triangleright 0 \mid (e_B|\mathcal{T}) = (e_A|)\}, \\ \llbracket A' \rightarrow B' \rrbracket &:= \{\mathcal{T} \triangleright 0 \mid \exists \mathcal{C} \in \llbracket A' \rightarrow B' \rrbracket_1, \mathcal{C} \triangleright \mathcal{T}\}, \\ \langle\langle A' \rightarrow B' \rangle\rangle &:= \\ &\{\mathsf{T} \subseteq \llbracket A' \rightarrow B' \rrbracket \mid \sum_{\mathcal{T} \in \mathsf{T}} \mathcal{T} \in \llbracket A' \rightarrow B' \rrbracket_1\}. \end{aligned}$$

where $\triangleright$ denotes the ordering induced by the cones of the theory Θ , and the cardinality of sets T in the last definition is implicitly assumed to be finite.

Theorem 4. *Let Θ be a causal OPT, and consider the theory Θ' defined above. Then Θ' is an OPT with a unique deterministic effect $e_{A'}$ for every A' .*

Proof. All the compositional structures of Θ are inherited by Θ' . The defining conditions in the special case of $\llbracket A' \rrbracket = \llbracket I' \rightarrow A' \rrbracket$ give

$$\llbracket A' \rrbracket = \{\rho \in \llbracket A \rrbracket_+ \mid (e_A|\rho) \leq 1\},$$

while for $\llbracket \bar{A}' \rrbracket = \llbracket A' \rightarrow I' \rrbracket$ they give

$$\llbracket \bar{A}' \rrbracket_1 = \{e_A\}.$$

What remains to be proved is that the set of transformations is closed under sequential and parallel composition. First of all, we observe that by theorem 3 $\llbracket A \rightarrow B \rrbracket_1 \subseteq \llbracket A' \rightarrow B' \rrbracket_1$, thus $\llbracket A \rightarrow B \rrbracket \subseteq \llbracket A' \rightarrow B' \rrbracket$. This makes $\llbracket A \rightarrow B \rrbracket_+ \subseteq \llbracket A' \rightarrow B' \rrbracket_+$. On the other hand, since by definition it is also $\llbracket A' \rightarrow B' \rrbracket \subseteq \llbracket A \rightarrow B \rrbracket_+$, we have $\llbracket A' \rightarrow B' \rrbracket_+ \subseteq \llbracket A \rightarrow B \rrbracket_+$, which makes

$\llbracket A' \rightarrow B' \rrbracket_+ \equiv \llbracket A \rightarrow B \rrbracket_+$. Thus the ordering $\triangleright$ is the same for the two theories. As a consequence, the preservation of cones under the two compositions is inherited by Θ' from the same property in Θ . Now, if $\mathcal{A} \in \llbracket A' \rightarrow B' \rrbracket_1$ and $\mathcal{B} \in \llbracket B' \rightarrow C' \rrbracket_1$, then

$$(e_C|\mathcal{B}\mathcal{A}) = (e_B|\mathcal{A}) = (e_A|,$$

thus $\mathcal{B}\mathcal{A} \in \llbracket A' \rightarrow C' \rrbracket_1$. Moreover, thanks to causality of the theory Θ one has $e_{AB} = e_A \otimes e_B$, thus for $\mathcal{A} \in \llbracket A' \rightarrow B' \rrbracket_1$ and $\mathcal{B} \in \llbracket B' \rightarrow C' \rrbracket_1$,

$$\begin{aligned} (e_{BD}|\mathcal{A} \otimes \mathcal{B}) &= (e_B|\mathcal{A}) \otimes (e_D|\mathcal{B}) \\ &= (e_A| \otimes (e_C|) = (e_{AC}|, \end{aligned}$$

and consequently $\mathcal{A} \otimes \mathcal{B} \in \llbracket A'C' \rightarrow B'D' \rrbracket_1$. Now, given events $\mathcal{A}, \mathcal{B}$ in the theory Θ' , by definition we have $\mathcal{C}, \mathcal{D}$ deterministic in Θ' such that $\mathcal{C} \triangleright \mathcal{A}$ and $\mathcal{D} \triangleright \mathcal{B}$, thus

$$\begin{aligned} \mathcal{F} &:= (\mathcal{D} - \mathcal{B})(\mathcal{C} - \mathcal{A}) + (\mathcal{D} - \mathcal{B})\mathcal{A} \\ &\quad + \mathcal{B}(\mathcal{C} - \mathcal{A}) \triangleright 0, \\ \mathcal{G} &:= (\mathcal{C} - \mathcal{A}) \otimes (\mathcal{D} - \mathcal{B}) \\ &\quad + (\mathcal{C} - \mathcal{A}) \otimes \mathcal{B} + \mathcal{A} \otimes (\mathcal{D} - \mathcal{B}) \triangleright 0. \end{aligned}$$

Finally, $\mathcal{F} + \mathcal{B}\mathcal{A} \triangleright \mathcal{B}\mathcal{A}$ and $\mathcal{G} + \mathcal{A} \otimes \mathcal{B} \triangleright \mathcal{A} \otimes \mathcal{B}$ are channels, and thus $\mathcal{B}\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{B}$ are events. $\square$

One can now show that the new theory Θ' is causal.

Corollary 1. *Let Θ be a causal OPT, and Θ' as in theorem 4. Then Θ' is a causal OPT.*

Proof. Let $\{\mathcal{T}_i\}_{i \in X}$ be a test in $\llbracket A' \rightarrow B' \rrbracket$, and $\{\mathcal{S}_j^i\}_{j \in Y}$ be tests in $\llbracket B' \rightarrow C' \rrbracket$ for every $i \in X$. Then

$$\forall i \in X \sum_{j \in Y} (e_{C'}|\mathcal{S}_j^i) = (e_{B'}|, \quad \sum_{i \in X} (e_{B'}|\mathcal{T}_i) = (e_{A'}|.$$

This implies that

$$\begin{aligned} \sum_{i \in X} \sum_{j \in Y} (e_{C'}|\mathcal{W}_{i,j}) &= (e_{A'}|, \\ \mathcal{W}_{i,j} &:= \mathcal{S}_j^i \mathcal{T}_i \in \llbracket A' \rightarrow C' \rrbracket \quad \forall i \in X, j \in Y. \end{aligned}$$

As a consequence, every conditional test $\mathcal{W}_{i,j} := \mathcal{S}_j^i \mathcal{T}_i \in \llbracket A' \rightarrow C' \rrbracket$ is admitted in the theory Θ' . $\square$

Remark 1. In the remainder, we will focus on theories satisfying causality and the further requirements

$$\begin{aligned} \llbracket A \rightarrow B \rrbracket_1 &= \{ \mathcal{T} \triangleright 0 \mid (e_B | \mathcal{T} = (e_A |) \}, \\ \llbracket A \rightarrow B \rrbracket &= \{ \mathcal{T} \triangleright 0 \mid \exists \mathcal{C} \in \llbracket A \rightarrow B \rrbracket_1, \mathcal{C} \triangleright \mathcal{T} \}, \\ \llbracket A \rightarrow B \rrbracket &= \{ \mathcal{T} \subseteq \llbracket A \rightarrow B \rrbracket \mid \sum_{\mathcal{T} \in \mathcal{T}} \mathcal{T} \in \llbracket A \rightarrow B \rrbracket_1 \}. \end{aligned}$$

In particular, this implies that $\llbracket A \rrbracket = \{ \rho \in \llbracket A \rrbracket_+ \mid (e_A | \rho) \leq 1 \}$. Thanks to corollary 1, this is not a significant restriction.

We now narrow down focus on theories that satisfy the *no-restriction hypothesis*. Let us consider the set of preparation-tests for every system of the theory Θ . Then we will complete the set of tests allowing for all those tests that transform preparation tests to preparation tests, even when applied locally. This requirement is the generalisation of the assumption made in quantum theory that a map is a transformation if and only if it is completely positive and trace non increasing, and a collection of transformations is a test if and only if the sum of its elements is a channel, i.e. completely positive and trace-preserving.

Assumption 1. Let $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$. The *no-restriction hypothesis* consists in the requirement that $\mathcal{T} \in \llbracket A \rightarrow B \rrbracket$ if and only if for every C and every $P \in \llbracket I \rightarrow AC \rrbracket$, one has $(\mathcal{T} \otimes I_C)P \in \llbracket I \rightarrow BC \rrbracket$.

We will write $\mathcal{A} \succeq 0$ if for every C and every $P \in \llbracket AC \rrbracket_+$, one has $(\mathcal{A} \otimes I_C)P \in \llbracket BC \rrbracket_+$. The set of transformations $\mathcal{A} \succeq 0$ is a cone, that we denote by

$$K(A \rightarrow B) := \{ \mathcal{A} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}} \mid \mathcal{A} \succeq 0 \}.$$

The above defined cone introduces a (partial) ordering in $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$, defined by

$$\mathcal{A} \succeq \mathcal{B} \Leftrightarrow \mathcal{A} - \mathcal{B} \succeq 0. \quad (4)$$

Notice that, under the no-restriction hypothesis, the following identity holds

$$K(A \rightarrow B) = \llbracket A \rightarrow B \rrbracket_+,$$

and thus for every $\mathcal{A}, \mathcal{B} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$,

$$\mathcal{A} \succeq \mathcal{B} \Leftrightarrow \mathcal{A} \triangleright \mathcal{B}. \quad (5)$$

We remark that, under assumption 1, the hypothesis of theorem 3 can be relaxed to $\mathcal{C} \in$

$\llbracket A \rightarrow B \rrbracket_+$, since the latter implies that $\mathcal{C} = \lambda \mathcal{C}_0$ for some $\mathcal{C}_0 \in \llbracket A \rightarrow B \rrbracket$ and $\lambda \geq 0$. Then, $\mathcal{C} \succeq 0$. Moreover, since $\mathcal{C}$ sends e_B to e_A , one has $(\mathcal{C} \otimes I_C)[AC]_1 \subseteq [BC]_1$, and thus $(\mathcal{C} \otimes I_C)[AC] \subseteq [BC]$. By the no-restriction hypothesis, this is tantamount to $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket$.

We can now prove a theorem that is a very important consequence of the no-restriction hypothesis, along with causality.

Theorem 5. In a theory satisfying the *no-restriction hypothesis*, let $\mathcal{T} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$. Then $\mathcal{T}, \mathcal{C} - \mathcal{T} \succeq 0$ for some channel $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket_1$ iff $\mathcal{T} \in \llbracket A \rightarrow B \rrbracket$.

Proof. The hypothesis $\mathcal{T}, \mathcal{C} - \mathcal{T} \succeq 0$ is equivalent to $\mathcal{T}, \mathcal{C} - \mathcal{T} \in \llbracket A \rightarrow B \rrbracket_+$. Equivalently, for every C and every $P \in \llbracket AC \rrbracket_+$, one has $(\mathcal{X} \otimes I_C)P \in \llbracket BC \rrbracket_+$ for $\mathcal{X} = \mathcal{T}, \mathcal{C} - \mathcal{T}$. Let now $P \in \llbracket AC \rrbracket$. Then one has

$$(e_{BC} | ((\mathcal{C} - \mathcal{T}) \otimes I_C) | P) \geq 0,$$

and thus

$$0 \leq (e_{BC} | (\mathcal{T} \otimes I_C) | P) \leq (e_{AC} | P) \leq 1.$$

This implies that $(\mathcal{T} \otimes I_C)P \in \llbracket AC \rrbracket$. Finally, by the no-restriction hypothesis, $\mathcal{T} \in \llbracket A \rightarrow B \rrbracket$. The converse statement is trivial. $\square$

In the literature [10, 11] one can often find a different requirement under the name “no-restriction hypothesis”, namely that for every system A the set $\llbracket \bar{A} \rrbracket$ coincides with the set of functionals a on $\llbracket A \rrbracket_{\mathbb{R}}$ that satisfy $0 \leq (a | \rho) \leq 1$ for every $\rho \in \llbracket A \rrbracket$. The last condition may be neither necessary nor sufficient for the no-restriction hypothesis as we state it, despite counterexamples are still unknown for both cases. Indeed, the no-restriction hypothesis for effects imposes the requirement that for every C and every state $P \in \llbracket AC \rrbracket$, one has

$$\left(\begin{array}{c} \text{A} \\ \hline P \quad \text{C} \end{array} \begin{array}{c} a \\ \hline \end{array} \right) \in \llbracket C \rrbracket. \quad (6)$$

3.3 Norms

As the first step in the present work is to construct infinite composite systems, we will need a thorough notion of sequences and limits. From a topological point of view the vector spaces that we constructed so far have no special structure,

however the operational procedures for discrimination of processes provide a natural definition of distance between events. Such a distance is related to the success probability in discriminating events. Making the space of events into a metric space, the operational distance immediately provides a topological structure through the induced operational norm. However, in order to make the space of events of infinite composite systems into a Banach algebra, a stronger norm is needed, that is introduced here: the sup-norm. In the quantum and classical case the two norms coincide.

The operational norm $\|\cdot\|_{\text{op}}$ for states is defined as follows [15].

Definition 5. *The operational norm on $\llbracket A \rrbracket_{\mathbb{R}}$ is*

$$\|\rho\|_{\text{op}} := \sup_{a \in \llbracket \bar{A} \rrbracket} (2a - e|\rho) = \sup_{\substack{a_0, a_1 \in \llbracket \bar{A} \rrbracket \\ a_0 + a_1 = e_A}} (a_0 - a_1|\rho). \quad (7)$$

The operational norm $\|\cdot\|_{\text{op}}$ for transformations $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$ on finite systems is then defined as

$$\begin{aligned} \|\mathcal{A}\|_{\text{op}} &:= \sup_{C, \Psi \in \llbracket AC \rrbracket_1} \|(\mathcal{A} \otimes \mathcal{I}_C)\Psi\|_{\text{op}} \\ &= \sup_{C, \Psi \in \llbracket AC \rrbracket} \sup_{a \in \llbracket \bar{B}\bar{C} \rrbracket} (2a - e|\mathcal{A} \otimes \mathcal{I}_C|\Psi). \end{aligned}$$

In the special case of effects $a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}}$ we have

$$\begin{aligned} \|a\|_{\text{op}} &:= \sup_{C, \Psi \in \llbracket AC \rrbracket} \|(a \otimes \mathcal{I}_C)\Psi\|_{\text{op}} \\ &= \sup_{C, \Psi \in \llbracket AC \rrbracket} \sup_{b \in \llbracket \bar{C} \rrbracket} (a \otimes [2b - e_C]|\Psi). \end{aligned}$$

As a consequence,

$$\begin{aligned} \|a\|_{\text{op}} &= \sup_{\substack{\rho_0, \rho_1 \in \llbracket A \rrbracket \\ \rho_0 + \rho_1 \in \llbracket A \rrbracket}} (a|\rho_0 - \rho_1) \\ &= \sup_{p \in [0,1]} \{p \sup_{\rho \in \llbracket A \rrbracket} (a|\rho) - (1-p) \inf_{\sigma \in \llbracket A \rrbracket} (a|\sigma)\} \\ &= \max\{ \sup_{\rho \in \llbracket A \rrbracket} (a|\rho), \inf_{\sigma \in \llbracket A \rrbracket} (a|\sigma) \} \\ &= \sup_{\rho \in \llbracket A \rrbracket} |(a|\rho)|. \end{aligned}$$

Here we only prove one new result about the operational norm, that we will use in the following.

Lemma 3. *Let $\rho \in \llbracket A \rrbracket_{\mathbb{R}}$ and $\sigma \in \llbracket B \rrbracket_1$. Then*

$$\|\rho \otimes \sigma\|_{\text{op}} = \|\rho\|_{\text{op}}. \quad (8)$$

Proof. By definition it is

$$\begin{aligned} \|\rho \otimes \sigma\|_{\text{op}} &= \sup_{(a_0, a_1)} (a_0 - a_1|\rho \otimes \sigma) \\ &= \sup_{(b_0, b_1) \in M} (b_0 - b_1|\rho), \end{aligned}$$

where $M := \{(b_0, b_1) \mid b_i = a_i(\mathcal{I}_A \otimes \sigma)\}$. Thus

$$\|\rho \otimes \sigma\|_{\text{op}} \leq \|\rho\|_{\text{op}}.$$

On the other hand, for every binary observation-test (a_0, a_1) in $\llbracket \bar{A} \rrbracket$ one has

$$a_i = (a_i \otimes e_B)(\mathcal{I}_A \otimes \sigma),$$

and thus M actually contains every possible binary observation-test on A . Finally, this implies that

$$\|\rho \otimes \sigma\|_{\text{op}} = \|\rho\|_{\text{op}}. \quad \square$$

For more details on $\|\cdot\|_{\text{op}}$ see Refs. [8, 15].

We now proceed to define the sup-norm.

Definition 6. *The sup-norm $\|\mathcal{A}\|_{\text{sup}}$ of $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$ is defined as*

$$\begin{aligned} \|\mathcal{A}\|_{\text{sup}} &:= \inf J(\mathcal{A}), \\ J(\mathcal{B}) &:= \{\lambda \mid \exists \mathcal{C} \in \llbracket A \rightarrow B \rrbracket_1, \lambda \mathcal{C} \succeq \mathcal{B} \succeq -\lambda \mathcal{C}\}. \end{aligned}$$

We now show that $\|\cdot\|_{\text{sup}}$ actually defines a norm.

Proposition 1. *The sup-norm on $\llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$ is well defined:*

1. $\|\mathcal{A}\|_{\text{sup}}$ is non-negative, and it is null iff $\mathcal{A} = 0$,
2. for $\mu \in \mathbb{R}$ one has $\|\mu \mathcal{A}\|_{\text{sup}} = |\mu| \|\mathcal{A}\|_{\text{sup}}$,
3. $\|\mathcal{A} + \mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}} + \|\mathcal{B}\|_{\text{sup}}$.

Proof. 1. Let $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_{\mathbb{R}}$. Suppose that $j := \inf J(\mathcal{A}) < 0$. Then there exists $0 < \varepsilon < |j|$ such that $j + \varepsilon \in J(\mathcal{A})$, namely

$$\begin{aligned} -(|j| - \varepsilon)\mathcal{C} &\succeq \mathcal{A} \succeq (|j| - \varepsilon)\mathcal{C} \\ \Rightarrow -2(|j| - \varepsilon)\mathcal{C} &\succeq 0, \end{aligned}$$

for some $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket_1$, which is absurd. Then $\inf J(\mathcal{A}) \geq 0$. Suppose now that $J(\mathcal{A}) = 0$. This implies that for every $n \in \mathbb{N}$ there exists $\mathcal{C}_n \in \llbracket A \rightarrow B \rrbracket_1$ such that $\frac{1}{n}\mathcal{C}_n \succeq \mathcal{A} \succeq -\frac{1}{n}\mathcal{C}_n$. Now, by the closure of $\llbracket A \rrbracket_+$ in the operational norm, and considering that the sequences $(\frac{1}{n}\mathcal{C}_n \pm$

$\mathcal{A}$) converge to $\pm \mathcal{A}$, it must be $\pm \mathcal{A} \in \llbracket \mathbf{A} \rrbracket_+$. This is possible if and only if $\mathcal{A} = 0$. 2. The proof is trivial for $\mu = 0$. Let then $\mu \neq 0$. If $x \in J(\mathcal{A})$, then $x\mathcal{C} \pm \mathcal{A} \succeq 0$ for some $\mathcal{C} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_1$, and thus $|\mu|(x\mathcal{C} \pm \mathcal{A}) \succeq 0$, namely $|\mu|x \in J(\mu\mathcal{A})$. Thus $\|\mu\mathcal{A}\|_{\text{sup}} \leq |\mu|\|\mathcal{A}\|_{\text{sup}}$. For the same reason, $\|\mathcal{A}\|_{\text{sup}} \leq (1/|\mu|)\|\mu\mathcal{A}\|_{\text{sup}}$, and finally $\|\mu\mathcal{A}\|_{\text{sup}} = |\mu|\|\mathcal{A}\|_{\text{sup}}$. 3. Let now $x \in J(\mathcal{A})$ and $y \in J(\mathcal{B})$. Then $x\mathcal{C} \pm \mathcal{A} \succeq 0$, and $y\mathcal{D} \pm \mathcal{B} \succeq 0$, for $\mathcal{C}, \mathcal{D} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_1$. Thus $(x+y)\mathcal{F} \pm (\mathcal{A} + \mathcal{B}) \succeq 0$, where $\mathcal{F} := x/(x+y)\mathcal{C} + y/(x+y)\mathcal{D} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_1$. Thus, $x+y \in J(\mathcal{A} + \mathcal{B})$, and then $\|\mathcal{A} + \mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}} + \|\mathcal{B}\|_{\text{sup}}$. $\square$

The following property makes $(\llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_{\mathbb{R}}, \|\cdot\|_{\text{sup}})$ a Banach algebra.

Proposition 2. For $\mathcal{A} \in \llbracket \mathbf{B} \rightarrow \mathbf{C} \rrbracket_{\mathbb{R}}$ and $\mathcal{B} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_{\mathbb{R}}$, $\|\mathcal{A}\mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{sup}}$.

Proof. Let $x \in J(\mathcal{A})$, and $y \in J(\mathcal{B})$. Then there are $\mathcal{C}, \mathcal{D}$ such that $x\mathcal{C} \pm \mathcal{A} \succeq 0$, $y\mathcal{D} \pm \mathcal{B} \succeq 0$. Now, we have

$$\begin{aligned} & \frac{1}{2}[(x\mathcal{C} + \mathcal{A})(y\mathcal{D} - \mathcal{B}) + (x\mathcal{C} - \mathcal{A})(y\mathcal{D} + \mathcal{B})] \\ &= xy\mathcal{C}\mathcal{D} - \mathcal{A}\mathcal{B} \succeq 0 \\ & \frac{1}{2}[(x\mathcal{C} + \mathcal{A})(y\mathcal{D} + \mathcal{B}) + (x\mathcal{C} - \mathcal{A})(y\mathcal{D} - \mathcal{B})] \\ &= xy\mathcal{C}\mathcal{D} + \mathcal{A}\mathcal{B} \succeq 0, \end{aligned}$$

and then $xy \in J(\mathcal{A}\mathcal{B})$. Thus, $\|\mathcal{A}\mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{sup}}$. $\square$

Lemma 4. Let $\mathcal{A} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_{\mathbb{R}}$, and for an arbitrary $\mathbf{C}$, let $\mathcal{D} \in \llbracket \mathbf{C} \rightarrow \mathbf{D} \rrbracket_1$. Then $\|\mathcal{A} \otimes \mathcal{D}\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}$.

Proof. Let $x \in J(\mathcal{A})$. Then by definition there exists $\mathcal{C} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_1$ such that $x\mathcal{C} \pm \mathcal{A} \succeq 0$. This implies that

$$x\mathcal{C} \otimes \mathcal{D} \pm \mathcal{A} \otimes \mathcal{D} \succeq 0,$$

namely $x \in J(\mathcal{A} \otimes \mathcal{D})$, and then $J(\mathcal{A}) \subseteq J(\mathcal{A} \otimes \mathcal{D})$. Now, let $y \in J(\mathcal{A} \otimes \mathcal{D})$. Then there exists $\mathcal{C}' \in \llbracket \mathbf{AC} \rightarrow \mathbf{BD} \rrbracket_1$ such that $y\mathcal{C}' \pm \mathcal{A} \otimes \mathcal{D} \succeq 0$. Composing the l.h.s. of the latter relation with $\psi \in \llbracket \mathbf{C} \rrbracket_1$ and $e_{\mathbf{D}}$, we obtain $y\mathcal{C}'' \pm \mathcal{A} \succeq 0$, where $\mathcal{C}'' := (e_{\mathbf{D}}|\mathcal{C}'|\psi)$ is a channel, and then $y \in J(\mathcal{A})$. Thus, $J(\mathcal{A} \otimes \mathcal{D}) \subseteq J(\mathcal{A})$. Finally, since $J(\mathcal{A} \otimes \mathcal{D}) = J(\mathcal{A})$, we have $\|\mathcal{A} \otimes \mathcal{D}\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}$. $\square$

Corollary 2. For $\mathcal{A} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_{\mathbb{R}}$, and for arbitrary $\mathbf{C}$, it is $\|\mathcal{A} \otimes \mathcal{I}_{\mathbf{C}}\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}$.

Corollary 3. For $\mathcal{A} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_{\mathbb{R}}$ and $\mathcal{B} \in \llbracket \mathbf{C} \rightarrow \mathbf{D} \rrbracket_{\mathbb{R}}$, $\|\mathcal{A} \otimes \mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{sup}}$.

Proof. The result follows straightforwardly from proposition 2 and corollary 2. $\square$

An important result regarding the sup-norm is provided by the following proposition.

Proposition 3. For $\mathcal{A} \in \llbracket \mathbf{B} \rightarrow \mathbf{C} \rrbracket_{\mathbb{R}}$ and $\mathcal{B} \in \llbracket \mathbf{A} \rightarrow \mathbf{B} \rrbracket_{\mathbb{R}}$, $\|\mathcal{A}\mathcal{B}\|_{\text{op}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{op}}$.

Proof. By definition we have

$$\begin{aligned} \|\mathcal{A}\mathcal{B}\|_{\text{op}} &= \sup_{\mathbf{D}, \Psi \in \llbracket \mathbf{AD} \rrbracket_1} \sup_{a \in \llbracket \bar{\mathbf{C}}\bar{\mathbf{D}} \rrbracket} (2a - e_{\mathbf{CD}}|\mathcal{A}\mathcal{B} \otimes \mathcal{I}_{\mathbf{D}}|\Psi). \end{aligned}$$

Now, for every $a \in \llbracket \bar{\mathbf{C}}\bar{\mathbf{D}} \rrbracket$, and $\lambda \in J(\mathcal{A})$, upon defining $a_0 := a$, $a_1 := e_{\mathbf{CD}} - a$, we have

$$\begin{aligned} & (a_0 - a_1|\mathcal{A} \otimes \mathcal{I}_{\mathbf{C}} \\ &= (a_0|[\mathcal{A} \otimes \mathcal{I}_{\mathbf{D}}] - (a_1|[\mathcal{A} \otimes \mathcal{I}_{\mathbf{D}}] \\ &= \lambda[(\tilde{a}_0| - (\tilde{a}_1|], \end{aligned}$$

where $(\tilde{a}_i| := \lambda^{-1}\{(a_i|[\mathcal{A} \otimes \mathcal{I}_{\mathbf{D}}] + \frac{1}{2}(e|[(\lambda\mathcal{C} - \mathcal{A}) \otimes \mathcal{I}_{\mathbf{D}}])\}$, with $\mathcal{C} : \llbracket \mathbf{B} \rightarrow \mathbf{C} \rrbracket_1$ such that $\lambda\mathcal{C} \pm \mathcal{A} \succeq 0$. Clearly,

$$\tilde{a}_0, \tilde{a}_1 \in \llbracket \bar{\mathbf{B}}\bar{\mathbf{D}} \rrbracket, \quad \tilde{a}_0 + \tilde{a}_1 = e_{\mathbf{BD}},$$

thus

$$\begin{aligned} & (a_0 - a_1|\mathcal{A}\mathcal{B} \otimes \mathcal{I}_{\mathbf{D}}|\Psi) \\ &= \lambda[(\tilde{a}_0| - (\tilde{a}_1|\mathcal{B} \otimes \mathcal{I}_{\mathbf{D}}|\Psi) \\ &\leq \lambda \sup_{b \in \llbracket \bar{\mathbf{B}}\bar{\mathbf{D}} \rrbracket} (2b - e_{\mathbf{BD}}|\mathcal{B} \otimes \mathcal{I}_{\mathbf{D}}|\Psi) \end{aligned}$$

This implies that $\|\mathcal{A}\mathcal{B}\|_{\text{op}} \leq \lambda \sup_{b \in \llbracket \bar{\mathbf{B}}\bar{\mathbf{D}} \rrbracket} (2b - e_{\mathbf{BD}}|\mathcal{B} \otimes \mathcal{I}_{\mathbf{D}}|\Psi)$, and then for every $\lambda \in J(\mathcal{A})$

$$\begin{aligned} \|\mathcal{A}\mathcal{B}\|_{\text{op}} &\leq \lambda \sup_{\mathbf{D}, \Psi \in \llbracket \mathbf{AD} \rrbracket_1} \sup_{b \in \llbracket \bar{\mathbf{B}}\bar{\mathbf{D}} \rrbracket} (2b - e_{\mathbf{BD}}|\mathcal{B} \otimes \mathcal{I}_{\mathbf{D}}|\Psi) \\ &= \lambda\|\mathcal{B}\|_{\text{op}}. \end{aligned}$$

Finally, taking the infimum over $\lambda \in J(\mathcal{A})$ we get

$$\|\mathcal{A}\mathcal{B}\|_{\text{op}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{op}}. \quad \square$$

Corollary 4. The sup-norm is stronger than the operational norm.

Proof. It is sufficient to observe that $\|\mathcal{A}\|_{\text{op}} = \|\mathcal{A}\mathcal{I}\|_{\text{op}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{I}\|_{\text{op}} = \|\mathcal{A}\|_{\text{sup}}$. $\square$

Lemma 5. Let $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_1$ be a channel. Then $\|\mathcal{A}\|_{\text{sup}} = 1$.

Proof. Clearly, $\mathcal{A} \succeq \mathcal{A} \succeq -\mathcal{A}$, thus $1 \in J(\mathcal{A})$, and $\inf J(\mathcal{A}) \leq 1$. On the other hand, suppose that there exists $1 > \lambda \in J(\mathcal{A})$. This implies that there exists a channel $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket_1$ such that $\mathcal{D} := \lambda\mathcal{C} - \mathcal{A} \succeq 0$. However, this implies that $(e|_B \mathcal{D} = -(1 - \lambda)(e|_A \succeq 0$. This is absurd, and then $\lambda \in J(\mathcal{A})$ must be $\lambda \geq 1$. Thus, $\inf J(\mathcal{A}) \geq 1$. Finally, this implies $\|\mathcal{A}\|_{\text{sup}} = 1$. $\square$

Corollary 5. The sup-norm of the identity channel $\mathcal{I} \in \llbracket A \rightarrow A \rrbracket_1$ is 1.

The sup-norm for effects is just the special case of the sup-norm of transformations with the output system equal to I .

Definition 7. Let $a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}}$, and let us define the half-line

$$J(a) := \{\lambda \in \mathbb{R}_+ \mid \lambda e_A \succeq a \succeq -\lambda e_A\}.$$

The sup-norm $\|a\|_{\text{sup}}$ is defined as

$$\|a\|_{\text{sup}} := \inf J(a).$$

Proposition 4. The sup-norm on $\llbracket \bar{A} \rrbracket_{\mathbb{R}}$ is well defined:

1. $\|a\|_{\text{sup}}$ is non-negative, and it is null iff $a = 0$,
2. for $\mu \in \mathbb{R}$ one has $\|\mu a\|_{\text{sup}} = |\mu| \|a\|_{\text{sup}}$,
3. $\|a + b\|_{\text{sup}} \leq \|a\|_{\text{sup}} + \|b\|_{\text{sup}}$.

Proof. The case of effects is just a special case of the result of Proposition 1. $\square$

As an immediate consequence of proposition 3, we have the following result.

Corollary 6. The sup-norm is stronger than the operational norm on $\llbracket \bar{A} \rrbracket_{\mathbb{R}}$.

In the special case of $\llbracket I \rightarrow I \rrbracket_{\mathbb{R}} = \mathbb{R}$, one has $\llbracket I \rightarrow I \rrbracket_1 = \{1\}$, and $\llbracket I \rightarrow I \rrbracket_+ = \mathbb{R}_+$, thus $J(x) = \{\lambda \in \mathbb{R} \mid \lambda \pm x \geq 0\}$. Clearly, $\|x\|_{\text{sup}} = \inf J(x) = |x|$. We now need the following lemma.

Lemma 6. Let $a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}}$. Then $\|a \otimes e_B\|_{\text{op}} = \|a\|_{\text{op}}$ and $\|a \otimes e_B\|_{\text{sup}} = \|a\|_{\text{sup}}$.

Proof. For $\|\cdot\|_{\text{op}}$, the equality trivially follows from the definition. For $\|\cdot\|_{\text{sup}}$, it is a special case of lemma 4. $\square$

From lemma 5, the following consequence follows.

Corollary 7. For the deterministic effect e_A one has $\|e_A\|_{\text{sup}} = \|e_A\|_{\text{op}} = 1$.

We now prove a result that will be very useful later.

Lemma 7. Let $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_+$, and $(a| := (e_B|\mathcal{A} \in \llbracket \bar{A} \rrbracket_+$. Then $\|\mathcal{A}\|_{\text{sup}} = \|a\|_{\text{sup}}$.

Proof. First of all, by proposition 2, one has $\|a\|_{\text{sup}} \leq \|e_A\|_{\text{sup}} \|\mathcal{A}\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}$. On the other hand, let $\lambda \in J(a)$. This implies that

$$b := \lambda e_A - a \succeq 0.$$

Let us then define $\mathcal{B} := |\rho\rangle(b| \in \llbracket A \rightarrow B \rrbracket_+$, for some $\rho \in \llbracket B \rrbracket_1$. By theorem 3, it is easy to check that $\mathcal{A} + \mathcal{B} = \lambda\mathcal{C}$ for $\mathcal{C} \in \llbracket A \rightarrow B \rrbracket_1$. Then,

$$\lambda\mathcal{C} - \mathcal{A} = \mathcal{B} \succeq 0, \quad (9)$$

which means that $\lambda \in J(\mathcal{A})$. Then, $J(a) \subseteq J(\mathcal{A})$, and finally $\|\mathcal{A}\|_{\text{sup}} \leq \|a\|_{\text{sup}}$. $\square$

In the case of Classical and Quantum Theory, phrasing the definitions of sup- and operational norm in terms of semidefinite programming problems, one can conclude that they are strongly dual. As a consequence, in these special cases they define the same norm.

4 The quasi-local algebra in OPTs

Following the definition of Ref. [31], we define a cellular automaton in a general OPT as a triple $(G, A, \mathcal{V})$, where G is a denumerable set of labels for the systems that compose the automaton—addresses of the memory cells—, A_G is a (possibly infinite) composite system corresponding to the collection of systems A_g labelled by elements $g \in G$ —i.e. the memory array—, and $\mathcal{V}$ is a reversible transformation on $\llbracket \bar{A}_G \rrbracket$, such that $\mathcal{V} \cdot \mathcal{V}^{-1}$ is an automorphism of $\llbracket A_G \rightarrow A_G \rrbracket$. However, this definition is incomplete, for many reasons. In the first place, most of the objects mentioned above are not thoroughly defined. The purpose of the present section is to set the ground for the rigorous definition of a cellular automaton.

We start fixing some notation. For every $R \subseteq G$ let

$$A_R := \bigotimes_{g \in R} A_g,$$

with the convention that $A_\emptyset := I$. The full system is then $A_G = \bigotimes_{g \in G} A_g$. This purely formal notion will now be thoroughly substantiated.

With a slight abuse of notation, we will often use $R = g$ instead of $R = \{g\}$, dropping the braces. The set of finite regions of G will be denoted as

$$R^{(G)} := \{R \subseteq G \mid |R| < \infty\},$$

while the set of arbitrary regions of G will be denoted as

$$\bar{R}^{(G)} := \{R \subseteq G\}.$$

Clearly $R^{(G)} \subseteq \bar{R}^{(G)}$. In the remainder, when we write e.g. e_R or $\mathcal{I}_R$ for $R \in R^{(G)}$, we mean the deterministic effect or the identity transformation for the system A_R , respectively.

The following construction of $\bigotimes_{g \in G} A_g$ is inspired by that of Refs. [49, 50], however with significant differences.

4.1 Quasi-local effects

In this subsection we start the mathematical construction of the system A_G , the parallel composition of infinitely many finite systems. The first object that we will define is the space of generalised effects, along with its convex cone of positive effects, and the convex set of effects. The construction is based on the notion of a local effect, that is an effect which acts non trivially only on finitely many systems labelled by $g \in R$, with $R \in R^{(G)}$. We will say that such an effect acts on the finite region R . We will then introduce a real vector space structure over the set of local effects, and a norm that is induced by the sup norm for local effects. The last step is then the topological closure of the space of local effects in the sup-norm. The Banach space thus obtained is the space of generalised quasi-local effects, and the positive cone along with the convex set of effects contain the limits of sequences of elements of local cones or convex sets of effects, respectively.

As we will see in the next section, the definition of the space of quasi-local states is much more involved than that of quasi-local effects. The latter

is particularly simple, thanks to causality, that provides us with a preferential local effect, the unique deterministic one, without the need of introducing any arbitrary choice of a reference local effect. Moreover, while the construction of the space of quasi-local states is not logically necessary, as we will find them as a subspace of bounded functionals on quasi-local effects, the same is not true of effects.

If we defined effects as the dual space of quasi-local states, we would end up with far more effects than needed, while lacking sectors of the state space, that are usually reached by the evolution of a quasi-local state through a cellular automaton. These are the main reasons why our construction begins with effects.

The first notion we introduce is that of a local effect. Let A_G denote the formal composition of countably many systems from a general OPT: $A_G := \bigotimes_{g \in G} A_g$. Intuitively speaking, a local effect of A_G is an event within a test that discards all the systems in G but the finite region R , where a non-trivial measurement is performed. We then define the *local effect* (a, R) as a pair made of an effect $a \in \llbracket \bar{A}_R \rrbracket$ and the region R . To lighten the notation, in the remainder of the paper we will use the symbol a_R instead of (a, R) . The set of local effects is denoted by

$$\begin{aligned} \text{Pre}[\bar{A}_G]_L \\ := \bigsqcup_{R \in R^{(G)}} \llbracket \bar{A}_R \rrbracket = \{a_R \mid R \in R^{(G)}, a \in \llbracket \bar{A}_R \rrbracket\}. \end{aligned}$$

The above definition can be widened encompassing generalised effects:

$$\text{Pre}[\bar{A}_G]_{L\mathbb{R}} := \bigsqcup_{R \in R^{(G)}} \llbracket \bar{A}_R \rrbracket_{\mathbb{R}}.$$

Let us now consider a partition of the region R into two disjoint regions $S_0 \cup S_1 = R$. Since in the definition of a local effect a_R we did not set constraints on the effect $a \in \llbracket \bar{A}_R \rrbracket_{\mathbb{R}}$, it might be that $a = e_{S_0} \otimes a'$, with $a' \in \llbracket \bar{A}_{S_1} \rrbracket_{\mathbb{R}}$. It is clear from our intuitive notion of a local effect that a_R and a'_{S_1} should represent the same effect. This observation leads us to the equivalence relation defined as follows.

Definition 8 (Equivalent local effects). *We say that the effects a_R and a'_S in $\text{Pre}[\bar{A}_G]_{L\mathbb{R}}$ are equivalent, and denote this as $a_R \sim a'_S$, if there*

exists $a_0 \in \llbracket \bar{A}_{R \cap S} \rrbracket_{\mathbb{R}}$ such that the following identities hold

$$\begin{cases} a = a_0 \otimes e_{S \setminus R}, \\ a' = a_0 \otimes e_{R \setminus S}. \end{cases} \quad (10)$$

It is clear that the real notion of a local effect is captured by the equivalence classes modulo the above equivalence relation. We thus quotient $\text{Pre}[\bar{A}_G]_{\mathbb{R}}$ and define the obtained set as the set of local effects of A_G .

Definition 9. A generalised local effect is an equivalence class $[a_R]_{\sim}$. The set of generalised local effects of A_G is

$$\llbracket \bar{A}_G \rrbracket_{L\mathbb{R}} := \text{Pre}[\bar{A}_G]_{L\mathbb{R}} / \sim.$$

With a slight abuse of notation, in the following we will write a_R instead of $[a_R]_{\sim}$, unless the context requires explicit distinction of the two symbols. One can easily prove the following result

Lemma 8. Let $a_R \in \text{Pre}[\bar{A}_G]_{L\mathbb{R}}$. Then for every finite region $H \in \mathcal{R}^{(G)}$ such that $H \cap R = \emptyset$, one has $(a \otimes e_H)_{R \cup H} \in \text{Pre}[\bar{A}_G]_{L\mathbb{R}}$, and $(a \otimes e_H)_{R \cup H} \sim a_R$.

The proof of the above lemma is straightforward, and we do not report it here.

Let us now come back to our initial goal, that is to define a local effect as an event that acts non-trivially only on a finite region R . Intuition leads again to figure out what is the preferred representative of the class of a local effect: within the equivalence class, it is the element defined on the smallest region. We now provide a formal definition of such a minimal representative, and show that it is well posed.

Definition 10. The minimal representative of the equivalence class a_R , denoted as $\tilde{a}_{R_a}$, is defined through

$$R_a := \bigcap_{S \in \mathcal{R}_{(a,R)}} S, \quad \tilde{a}_{R_a} \sim a_R, \quad (11)$$

where $\mathcal{R}_{(a,R)}$ is the set of all those finite regions $S \in \mathcal{R}^{(G)}$ for which there exists $b \in \llbracket \bar{A}_S \rrbracket_{\mathbb{R}}$ such that $b_S \sim a_R$.

Lemma 9. The minimal representative exists and is unique.

Proof. As to existence, we remark that, by Eq. (11), R_a is the set of all $g \in G$ such that for all regions $S \in \mathcal{R}_{(a,R)}$ one has $g \in S$. Thus, if $h \notin R_a$, there must exist $S \in \mathcal{R}_{(a,R)}$ such that $h \notin S$. This implies that i) by definition there exists $f_S \sim a_R$, and ii) by lemma 8 $c_{S \cup h} \sim a_R$, with

$$c = f \otimes e_h. \quad (12)$$

As a consequence, if $c_T \sim a_R$ and $h \in T$, but $h \notin R_a$, by definition (10) c_T must be of the form of Eq. (12), for some $f \in \llbracket \bar{A}_{T \setminus h} \rrbracket_{\mathbb{R}}$. Clearly, $R_a \in \mathcal{R}^{(G)}$, since for any $S \in \mathcal{R}_{(a,R)}$ one has $R_a \subseteq S$. Now, let $S \in \mathcal{R}_{(a,R)}$, and $c_S \sim a_R$. One has $S = R_a \cup S'$, where $S' := (S \setminus R_a) \in \mathcal{R}^{(G)}$, thus $R_a \cap S' = \emptyset$. By Eq. (12), we then have

$$c = b \otimes e_{S'}, \quad b \in \llbracket \bar{A}_{R_a} \rrbracket_{\mathbb{R}}. \quad (13)$$

We now define $\tilde{a}_{R_a} := b_{R_a}$, and finally, since Eq. 13 holds for any $c_S \sim a_R$, one can easily verify that $\tilde{a}_{R_a} \sim a_R$.

As to uniqueness, we remark that the region R_a is uniquely defined. Now, suppose that there were two different $b, c \in \llbracket \bar{A}_{R_a} \rrbracket_{\mathbb{R}}$ such that $b_{R_a}, c_{R_a} \sim a_R$. Then by Eq. (10) it must be $b = c$. $\square$

We now make local effects into a vector space.

Definition 11. Let $a_R, b_S \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$, and $h \in \mathbb{R}$. Then we define

$$\begin{aligned} ha_R &:= \begin{cases} (ha)_R & h \neq 0, \\ 0_{\emptyset} & h = 0, \end{cases} \\ a_R + b_S &:= c_{R \cup S} \\ c &:= a \otimes e_{S \setminus R} + b \otimes e_{R \setminus S}. \end{aligned}$$

Notice that it is not always true that $R_c = R_a \cup R_b$. As an example, consider $a = f_{g_1} \otimes f_{g_2}$ and $b = f_{g_1} \otimes (e - f_{g_2})$, with $f_{g_1} \neq e_{g_1}$, $0 \neq f_{g_2} \neq e_{g_2}$, and $R_a = R_b = \{g_1, g_2\}$. Then $c = a + b = f_{g_1} \otimes e_{g_2}$, and clearly $R_c = \{g_1\}$, which is strictly included in $R_a = R_b = R_a \cup R_b$.

It is easy to check that $\llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$ is a real vector space with null element given by the equivalence class of $0_{\emptyset}$.

We now equip the real vector space of local effects with a norm, and we then close it to obtain the Banach space of *quasi-local* effects. The definition is based on a norm for effects of finite

systems, as every local effect $a_R \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$ reduces to the effect $\tilde{a} \in \llbracket \bar{A}_{R_a} \rrbracket_{\mathbb{R}}$. A natural norm one might think of is then the *operational norm*, that induces the following definition.

Definition 12. *The operational norm $\|a_R\|_{\text{op}}$ of $a_R \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$ is defined by the following expression*

$$\|a_R\|_{\text{op}} := \|\tilde{a}\|_{\text{op}}, \quad (14)$$

where $\tilde{a} \in \llbracket \bar{A}_{R_a} \rrbracket_{\mathbb{R}}$.

Unfortunately, if one completes the space of local effects in the operational norm, in general the dual norm on the space of bounded linear functionals—which will be our state space—does not coincide with the operational norm on the state space, for those states that can be interpreted as quasi-local preparations. We will then choose a different norm on our space of effects. The new norm will be referred to as *sup-norm*, as it is the extension of the sup-norm to the infinite case.

As far as finite-dimensional systems are concerned, this choice does not represent a problem, as all norms are equivalent in finite-dimensional vector spaces. However, the sup-norm is stronger than the operational one—see corollary 4—and thus the space that we construct, completing our normed vector space of local effects with sup-norm Cauchy sequences, might contain distinct limits that are operationally equivalent. This point is a delicate one, and to avoid an unreasonable construction where there exist different effects that are operationally equivalent, we impose a sufficient constraint for the operational norm and the sup-norm to be equivalent also for infinite systems: we restrict attention to those theories where the following property holds: there exists a finite constant k such that for every system A and every $a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}}$

$$\|a\|_{\text{sup}} \leq k\|a\|_{\text{op}}. \quad (15)$$

This implies that not only the bound (15) holds for a fixed system A , but it holds with a fixed constant independent of the system A . In turn, a sufficient condition for (15) is that for every system A

$$[A]_+^* \equiv [\bar{A}]_+, \quad (16)$$

namely every positive functional on the cone of states is proportional to an effect by a positive

constant, and viceversa. In all the presently known theories the latter condition is satisfied. The two above conditions in Eqs. (15) and (16) are discussed in detail in Appendix A, where we prove that condition (16) implies condition (15).

Our new norm is an order-unit norm. As we will discuss in subsection 4.2, its dual coincides with the operational norm on quasi-local states.

Let us now see the definition of the sup-norm in detail.

Definition 13. *The sup-norm $\|a_R\|_{\text{sup}}$ of $a_R \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$ is defined by the following expression*

$$\|a_R\|_{\text{sup}} := \|\tilde{a}\|_{\text{sup}}, \quad (17)$$

where $\tilde{a} \in \llbracket \bar{A}_{R_a} \rrbracket_{\mathbb{R}}$.

We now want to prove that the operational norm and the sup-norm are well-defined norms on $\llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$.

Proposition 5. *Let $a_R \sim \tilde{a}_{R_a}$. Then $\|a_R\|_{\text{op}} = \|a\|_{\text{op}}$, and $\|a_R\|_{\text{sup}} = \|a\|_{\text{sup}}$.*

Proof. Let $a_R \sim \tilde{a}_{R_a}$, and $R = R_a \cup R'$, with $R_a \cap R' = \emptyset$. Then by Eq. (10) we have

$$a = \tilde{a} \otimes e_{R'},$$

and thus by definitions 12 and 13 and lemma 6, it is

$$\begin{aligned} \|a_R\|_{\text{op}} &= \|\tilde{a}\|_{\text{op}} = \|\tilde{a} \otimes e_{R'}\|_{\text{op}} = \|a\|_{\text{op}}, \\ \|a_R\|_{\text{sup}} &= \|\tilde{a}\|_{\text{sup}} = \|\tilde{a} \otimes e_{R'}\|_{\text{sup}} = \|a\|_{\text{sup}}. \quad \square \end{aligned}$$

We can then prove the desired result.

Proposition 6. *The functionals $\|\cdot\|_{\text{op}}$ and $\|\cdot\|_{\text{sup}}$ are norms on $\llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$.*

Proof. 1. For every $a_R \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$ it is clear that $\|a_R\|_{\text{op}} \geq 0$ and $\|a_R\|_{\text{sup}} \geq 0$. Now, $\|\tilde{a}\|_{\text{op}} = \|\tilde{a}\|_{\text{sup}} = 0$ if and only if $\tilde{a} = 0$, namely, reminding definition 11, $a_R = 0_{\emptyset}$. 2. For every $a_R \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$, by definitions 7, 11, and 12, one straightforwardly has that, for every $\mu \in \mathbb{R}$,

$$\begin{aligned} \|\mu a_R\|_{\text{op}} &= \|\mu \tilde{a}\|_{\text{op}} = |\mu| \|\tilde{a}\|_{\text{op}} = |\mu| \|a_R\|_{\text{op}}, \\ \|\mu a_R\|_{\text{sup}} &= \|\mu \tilde{a}\|_{\text{sup}} = |\mu| \|\tilde{a}\|_{\text{sup}} = |\mu| \|a_R\|_{\text{sup}}. \end{aligned}$$

3. Let now $c_{R \cup S} = a \otimes e_{S \setminus R} + e_{R \setminus S} \otimes b$ for $a \in \llbracket \bar{A}_R \rrbracket_{\mathbb{R}}$ and $b \in \llbracket \bar{A}_S \rrbracket_{\mathbb{R}}$, as from definition 11.

Then, by lemma 6, proposition 5 and by the triangle inequality, we have

$$\begin{aligned} \|a_R + b_S\|_{\text{op}} &= \|c_{R \cup S}\|_{\text{op}} = \|c\|_{\text{op}} \\ &\leq \|a\|_{\text{op}} + \|b\|_{\text{op}} = \|a_R\|_{\text{op}} + \|b_S\|_{\text{op}}, \\ \|a_R + b_S\|_{\text{sup}} &= \|c_{R \cup S}\|_{\text{sup}} = \|c\|_{\text{sup}} \\ &\leq \|a\|_{\text{sup}} + \|b\|_{\text{sup}} = \|a_R\|_{\text{sup}} + \|b_S\|_{\text{sup}}. \quad \square \end{aligned}$$

We remark that, in a theory that satisfies assumption (16), one has $\|\cdot\|_{\text{op}} \equiv \|\cdot\|_{\text{sup}}$ on $[\bar{A}]_{\mathbb{R}}$, and thus $\|\cdot\|_{\text{op}} = \|\cdot\|_{\text{sup}}$ on $[\bar{A}_G]_{\mathbb{R}}$ (the proof can be found in appendix A). In general, however, the sup-norm is stronger than the operational norm, as we now prove.

Lemma 10. *Let $a_R \in [\bar{A}_G]_{L\mathbb{R}}$. Then $\|a_R\|_{\text{op}} \leq \|a_R\|_{\text{sup}}$.*

Proof. Let us consider $\mu \in J(a)$. Then for every $\rho \in [A_R]_1$ we have

$$(\mu e_R \pm a|\rho) \geq 0,$$

i.e. $|(a|\rho)| \leq \mu$. Then, taking the supremum over states on l.h.s. we obtain $\|a_R\|_{\text{op}} \leq \mu$, and finally, taking the infimum over $J(a)$ we obtain the thesis. $\square$

The space of local effects that we constructed so far is a normed real vector space. We now make it into a Banach space, the space of quasi-local effects, by the usual completion procedure for normed spaces. We first introduce the space $[\bar{A}_G]_{C\mathbb{R}}$ of Cauchy sequences

$$a : \mathbb{N} \rightarrow [\bar{A}_G]_{L\mathbb{R}} :: n \mapsto a_{nR_n}.$$

We then define the equivalence relation between Cauchy sequences in $[\bar{A}_G]_{C\mathbb{R}}$ defined by $a \cong b$ iff $\lim_{n \rightarrow \infty} \|a_{nR_n} - b_{nS_n}\|_{\text{sup}} = 0$. Finally, we define the space $[\bar{A}_G]_{Q\mathbb{R}}$ of *generalised quasi-local effects*, by taking the quotient

$$[\bar{A}_G]_{Q\mathbb{R}} := [\bar{A}_G]_{C\mathbb{R}} / \cong. \quad (18)$$

The elements of this space will be denoted by $a = [a_{nR_n}]$. In this context, the class of a constant Cauchy sequence ($R_n = R$ and $a_{nR_n} = a_R$) will be denoted by $a_R := [a_{nR_n}]$. Clearly, since $\|a_{nR_n}\|_{\text{sup}} - \|a_{mR_m}\|_{\text{sup}} \leq \|a_{nR_n} - a_{mR_m}\|_{\text{sup}}$, for a Cauchy sequence $a = [a_{nR_n}]$ also $\|a_{nR_n}\|_{\text{sup}}$ is a Cauchy sequence, and we define $\|a\|_{\text{sup}} := \lim_{n \rightarrow \infty} \|a_{nR_n}\|_{\text{sup}}$. One can easily verify that

the sup-norm on $[\bar{A}_G]_{Q\mathbb{R}}$ thus defined is independent of the specific sequence within an equivalence class. The space $[\bar{A}_G]_{Q\mathbb{R}}$ is by construction a real Banach space, and $[\bar{A}_G]_{L\mathbb{R}}$ can be identified with the dense submanifold containing constant sequences $a_R = [a_R]$. Clearly, in this case $\|a\|_{\text{sup}} = \|a_R\|_{\text{sup}}$.

Definition 14. *Let $a \in [\bar{A}_G]_{Q\mathbb{R}}$. If there is a Cauchy sequence a_{nR_n} in the class defining a , such that, for some $n_0 \in \mathbb{N}$, for every $n \geq n_0$ one has $\tilde{a}_n \in [\bar{A}_{R_n}]$, then we call a a quasi-local effect. We denote the set of quasi-local effects by $[\bar{A}_G]_Q$.*

The cone $[\bar{A}_G]_{Q+}$ contains all elements that are proportional to an element in $[\bar{A}_G]_Q$ by a positive constant.

The deterministic quasi-local effect e_G is the class $e_G := 1_\emptyset$.

The cone $[\bar{A}_G]_{Q+}$ introduces a partial ordering in $[\bar{A}_G]_{Q\mathbb{R}}$, that we denote by

$$a \succeq b \Leftrightarrow a - b \in [\bar{A}_G]_{Q+}.$$

Quasi-local effects can be interpreted as effects that can be arbitrarily well approximated by local observation procedures. The effect e_G plays an important role, as it is the unique deterministic effect in $[\bar{A}_G]_Q$. This will be proved shortly. Let us start providing the first important property of e_G .

Lemma 11. *Let $a \in [\bar{A}_G]_Q$. Then also $e_G - a \in [\bar{A}_G]_Q$.*

Proof. Let $a = [a_{nR_n}]$. Let $a'_{nR_n} := (e - a_{nR_n})_{R_n} \in [\bar{A}_{R_n}]$. Then one has $(a'_{nR_n} - a'_{mR_m})_{R_n \cup R_m} = (a_m - a_n)_{R_m \cup R_n}$, and thus a'_{nR_n} is a Cauchy sequence whose class we call $a' \in [\bar{A}_G]_Q$. Finally, since $a + a'$ is the class $[a_{nR_n}] + [a'_{nR_n}] = [(a_n + a'_{nR_n})_{R_n}] = [1_\emptyset] = e_G$, we have that $a + a' = e_G$, namely $e_G - a = a' \in [\bar{A}_G]_Q$. $\square$

By the above lemma we know that not only $e_G \succeq a$ for every quasi-local effect a , but also that every quasi-local effect a can be complemented by a quasi-local effect a' such that $a + a' = e_G$. In other words $(a, e_G - a)$ is a binary quasi-local observation test.

We can also prove the converse result.

Lemma 12. *Let $a \in [\bar{A}_G]_{Q+}$, and $e_G - a \in [\bar{A}_G]_{Q+}$. Then $a, e_G - a \in [\bar{A}_G]_Q$.*

Proof. Indeed, the statement is true for finite systems thanks to the no-restriction hypothesis (see theorem 5), and thus it holds for a given Cauchy sequence in the class of a , namely if $a'_{nR_n} := (e - a_n)_{R_n} \in [\bar{A}_{R_n}]_+$ for all n , then $a_n, a'_n \in [\bar{A}_{R_n}]$. Thus, $a = \lim_{n \rightarrow \infty} a_n \in [\bar{A}_G]_Q$, and $a' = \lim_{n \rightarrow \infty} a'_n = e_G - a \in [\bar{A}_G]_Q$. $\square$

We now prove that e_G lies in the interior of $[\bar{A}_G]_{Q+}$, namely there is a ball of radius r around e_G in sup-norm that is contained in $[\bar{A}_G]_{Q+}$.

Lemma 13. *There exists an open ball $B_r(e_G)$ of radius r in $[\bar{A}_G]_{Q\mathbb{R}}$ that is fully contained in $[\bar{A}_G]_{Q+}$.*

Proof. Let $\|a - e_G\|_{\text{sup}} < r < 1/2$. Then, by definition, there exist a sequence a_{nR_n} such that $\lim_{n \rightarrow \infty} a_n = a$, and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\|a_n - e_G\|_{\text{sup}} \leq \|a_n - a\|_{\text{sup}} + \|a - e_G\|_{\text{sup}} < 2r.$$

This implies that

$$[2re + (a_n - e)]_{R_n} = [a_n - (1 - 2r)e]_{R_n} \succeq 0.$$

Now, this implies that for $n \geq n_0$ it is $a_n \succeq (1 - 2r)e_{R_n} \succeq 0$, and thus $a_n \in [\bar{A}_G]_{L+}$. Finally, by definition, one obtains $a \in [\bar{A}_G]_{Q+}$. Thus, the ball $B_r(e_G)$ is contained in $[\bar{A}_G]_{Q+}$. $\square$

Corollary 8. *Let $a \in [\bar{A}_G]_{Q+}$. Then for $\varepsilon > 0$, $B_{\varepsilon r}(\varepsilon e_G + a) \subseteq [\bar{A}_G]_{Q+}$ for some $r > 0$.*

Proof. Let $b \in B_{\varepsilon r}(\varepsilon e_G + a)$. Then

$$\begin{aligned} \|b - a - \varepsilon e_G\|_{\text{sup}} &< \varepsilon r \\ \Leftrightarrow \|(b - a)/\varepsilon - e_G\|_{\text{sup}} &< r. \end{aligned}$$

Then $(b - a)/\varepsilon = c \in [\bar{A}_G]_{Q+}$, and thus

$$b = \varepsilon c + a \in [\bar{A}_G]_{Q+}. \quad \square$$

We now prove that the deterministic effect e_G allows for an extension of the defining property of the sup-norm.

Lemma 14. *The sup-norm of $a \in [\bar{A}_G]_{Q\mathbb{R}}$ can be expressed as*

$$\begin{aligned} \|a\|_{\text{sup}} &= \inf J(a), \\ J(a) &:= \{\mu \geq 0 \mid \mu e_G \succeq a \succeq -\mu e_G\}. \end{aligned}$$

Proof. In the case of local effects $a = a_R$ the statement is a trivial recasting of the definition of sup-norm. Let us then consider the class $a \in [\bar{A}_G]_{Q\mathbb{R}}$ with $a = [a_{nR_n}]$. One has, by definition, $\|a\|_{\text{sup}} = \lim_{n \rightarrow \infty} \|a_n\|_{\text{sup}}$. Let us consider the sequence $\mu_n := \|a_n\|_{\text{sup}} + \varepsilon \in J(a_n)$. Clearly, $\lim_{n \rightarrow \infty} \mu_n = \|a\|_{\text{sup}} + \varepsilon$. Moreover, since

$$\begin{aligned} &\|(\mu_n - \mu_m)e \pm (a_n - a_m)\|_{R_n \cup R_m} \\ &\leq \|a_m\|_{\text{sup}} - \|a_n\|_{\text{sup}} + \|a_m - a_n\|_{\text{sup}} \leq 2\varepsilon, \end{aligned}$$

the sequence $\{(\mu_n + \varepsilon)e \pm a_n\}_{R_n}$ converges to $(\|a\|_{\text{sup}} + \varepsilon)e_G \pm a \in [\bar{A}_G]_{Q+}$, thus for every $\varepsilon \geq 0$ one has $\|a\|_{\text{sup}} + \varepsilon \in J(a)$. This implies that

$$\inf J(a) \leq \|a\|_{\text{sup}}.$$

On the other hand, let $\mu \in J(a)$, then

$$\mu e_G \pm a \succeq 0.$$

By corollary 8, we have that, for $\varepsilon > 0$,

$$(\varepsilon + \mu)e_G \pm a \in B_{\varepsilon r}(\varepsilon e_G + \mu e_G \pm a) \subseteq [\bar{A}_G]_{Q+}.$$

Now, this implies that there are two open balls, centred at $(\mu + \varepsilon)e_G \pm a$, that are fully contained in $[\bar{A}_G]_{Q+}$. There exists then $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ one has

$$(\mu + \varepsilon)e_G \pm a_n \succeq 0,$$

namely $(\mu + \varepsilon) \in J(a_n)$. Thus, $J(a) \subseteq J(a_n) - \varepsilon$, and

$$\inf J(a) \geq \|a_n\|_{\text{sup}} - \varepsilon. \quad (19)$$

Finally, this implies that $\inf J(a) \geq \|a\|_{\text{sup}}$. $\square$

As the cone $[\bar{A}_G]_{Q+}$ is closed, we can easily prove that the infimum in the expression of the sup norm is actually a minimum.

Lemma 15. *Let $a \in [\bar{A}_G]_{Q\mathbb{R}}$. Then*

$$\|a\|_{\text{sup}} e_G \pm a \succeq 0. \quad (20)$$

Proof. By lemma 14, for every $\varepsilon > 0$ one has

$$(\|a\|_{\text{sup}} + \varepsilon)e_G \pm a \succeq 0.$$

The sequences $b_n^\pm := (\|a\|_{\text{sup}} + 1/n)e_G \pm a$ are both Cauchy, and their limits are both in $[\bar{A}_G]_{Q+}$, then

$$\|a\|_{\text{sup}} e_G \pm a \succeq 0. \quad \square$$

Before concluding, we remark that the operational norm can be extended to $[\bar{A}_G]_{Q\mathbb{R}}$, by simply defining for $a = [a_{nR_n}]$

$$\|a\|_{\text{op}} := \lim_{n \rightarrow \infty} \|a_{nR_n}\|_{\text{op}}.$$

Indeed, since

$$\begin{aligned} \left| \|a_{nR_n}\|_{\text{op}} - \|a_{mR_m}\|_{\text{op}} \right| &\leq \|a_{nR_n} - a_{mR_m}\|_{\text{op}} \\ &\leq \|a_{nR_n} - a_{mR_m}\|_{\text{sup}}, \end{aligned}$$

the sequence $\|a_{nR_n}\|_{\text{op}}$ is Cauchy, and the limit is well defined. Moreover, also on $[\bar{A}_G]_{Q\mathbb{R}}$ the sup-norm is stronger than the operational norm, as can be straightforwardly concluded from lemma 10. However, in theories where there exists $k > 0$ such that $\|\cdot\|_{\text{sup}} \leq k\|\cdot\|_{\text{op}}$ in $[\bar{A}]_{\mathbb{R}}$ for every finite system A , the same inequality holds in the limit, and the operational and sup-norm are equivalent. In particular, this is the case under assumption (16).

We now introduce a diagrammatic notation for effects in $[\bar{A}]_{Q\mathbb{R}}$ that will make part of the subsequent proofs and arguments more intuitive. First of all, we denote a quasi-local effect $a \in [\bar{A}_G]_{Q\mathbb{R}}$ by the symbol

$$\text{---} \overset{G}{\boxed{a}} \text{---} . \quad (21)$$

In the case of a local effect $a_R \in [\bar{A}_G]_{L\mathbb{R}}$, we will draw

$$\text{---} \overset{G}{\boxed{a_R}} \text{---} = \frac{\overset{R}{\boxed{a}}}{\underset{G \setminus R}{\boxed{e}}} . \quad (22)$$

Notice that, since for $(a \otimes e_{R_1})_R$ with $a \in [\bar{A}_{R_0}]$ and $R_1 := R \setminus R_0$, it is $(a \otimes e_{R_1})_R \sim a_{R_0}$, we have

$$\frac{\overset{R_0}{\boxed{a}}}{\underset{G \setminus R}{\boxed{e}}} = \frac{\overset{R_0}{\boxed{a}}}{\underset{(G \setminus R) \cup R_1}{\boxed{e}}} . \quad (23)$$

From the above identity, we can intuitively conclude that

$$\frac{\underset{(G \setminus R) \cup R_1}{\boxed{e}}}{\underset{G \setminus R}{\boxed{e}}} = \frac{\overset{R_1}{\boxed{e}}}{\underset{G \setminus R}{\boxed{e}}} . \quad (24)$$

Indeed, let $G = R \cup H$, with $|R| < \infty$. Then $[e_R] = [1_\emptyset] = e_G$, which is the equation represented by the diagram in Eq. 24. Similarly, let $a_R \otimes b_H := \lim_{n \rightarrow \infty} (a \otimes b_n)$, where $[b_{nS_n}] = b \in [\bar{A}_H]_{Q\mathbb{R}}$. By corollary 3, one has $[(a \otimes b_n)_{R \cup S_n}] = (a_R \otimes b_H)$. Thus, in $G = R \cup H$, one has

$a_R = [a_R] = [(a_R \otimes e_H)] = a_R \otimes e_H$, which is the meaning of Eq. (22).

As a final remark, we observe that for every denumerable set G , and every subset $R \subseteq G$, including infinite ones, one can define $[\bar{A}_R^{(G)}]_{Q\mathbb{R}}$ as the closed subspace spanned by those quasi-local effects $a = a_{nR_n}$ such that, for every $n \in \mathbb{N}$, $R_n \subseteq R$. If one constructs the system $A_R := \bigotimes_{g \in R} A_g$, it is straightforward to construct an ordered Banach space isomorphism

$$\mathcal{J}_R^\dagger : [\bar{A}_R^{(G)}]_{Q\mathbb{R}} \rightarrow [\bar{A}_R]_{Q\mathbb{R}} :: [a_{nR_n}]_G \mapsto [a_{nR_n}]_R, \quad (25)$$

with the norm coinciding with the sup-norm in both spaces. The left-inverse of $\mathcal{J}_R^\dagger$ is

$$\mathcal{J}_R^{-1\dagger} : [\bar{A}_R]_{Q\mathbb{R}} \rightarrow [\bar{A}_R^{(G)}]_{Q\mathbb{R}}. \quad (26)$$

$\mathcal{J}_R^\dagger$ and $\mathcal{J}_R^{-1\dagger}$ can be diagrammatically denoted as

$$\text{---} \overset{R}{\boxed{\mathcal{J}_R^\dagger}} \overset{R}{\boxed{a}} \underset{G \setminus R}{\boxed{e}} \text{---} = \text{---} \overset{R}{\boxed{a}} \text{---}, \quad (27)$$

$$\text{---} \overset{G}{\boxed{\mathcal{J}_R^{-1\dagger}}} \overset{R}{\boxed{a}} \underset{G \setminus R}{\boxed{e}} \text{---} = \frac{\overset{R}{\boxed{a}}}{\underset{G \setminus R}{\boxed{e}}}. \quad (28)$$

4.2 Extended states

Now that we defined the space of quasi-local effects, we can define the space of states as the space of bounded linear functionals on $[\bar{A}_G]_{Q\mathbb{R}}$. The set of states is then defined considering those linear functionals that, acting on local effects of an arbitrary finite region R , behave as a state in $[A_R]$.

Definition 15 (Generalised extended states). *The space $[A_G]_{\mathbb{R}}$ of generalised extended states of A_G is the topological dual of $[\bar{A}_G]_{Q\mathbb{R}}$, i.e. the Banach space $[\bar{A}_G]_{Q\mathbb{R}}^*$ of bounded linear functionals on $[\bar{A}_G]_{Q\mathbb{R}}$, equipped with the norm*

$$\|\rho\|_* := \sup_{\|a\|_{\text{sup}}=1} |(a|\rho)|. \quad (29)$$

The operational norm on $[A_G]_{\mathbb{R}}$, defined as

$$\|\rho\|_{\text{op}} := \sup_{a \in [\bar{A}_G]_Q} (2a - e_G|\rho), \quad (30)$$

coincides with the norm $\|\cdot\|_*$, as we now prove.

Lemma 16. Let $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$. Then

$$\|\rho\|_* = \sup_{a \in \llbracket \bar{A}_G \rrbracket_Q} (2a - e_G|\rho). \quad (31)$$

Proof. Let $0 \leq \varepsilon \leq \|\rho\|_{\text{op}}$. We then have

$$0 \leq \|\rho\|_{\text{op}} - \varepsilon \leq (2a - e_G|\rho),$$

for some $a \in \llbracket \bar{A}_G \rrbracket_Q$. Invoking lemma 11, one has

$$\begin{aligned} e_G + (2a - e_G) &= 2a \succeq 0, \\ e_G - (2a - e_G) &= 2(e_G - a) \succeq 0, \end{aligned}$$

and, by lemma 14, $\|2a - e_G\|_{\text{sup}} \leq 1$. Then it is

$$\|\rho\|_{\text{op}} - \varepsilon \leq \frac{(2a - e_G|\rho)}{\|2a - e_G\|_{\text{sup}}} \leq \|\rho\|_*.$$

On the other hand, let us now pick $a \in \llbracket \bar{A}_G \rrbracket_{\mathbb{R}}$ with $\|a\|_{\text{sup}} = 1$. Then, by lemma 15, $e_G \pm a \succeq 0$. If we define $a_{\pm} := \frac{1}{2}(e_G \pm a)$, we have $a_{\pm} \succeq 0$, and

$$a_+ + a_- = e_G, \quad (a_+ - a_-) = a, \quad (32)$$

which by lemma 12 implies that $a_{\pm} \in \llbracket \bar{A}_G \rrbracket_Q$. Then

$$\pm(a|\rho) = (2a_{\pm} - e_G|\rho).$$

Thus, $|(a|\rho)| \leq \|\rho\|_{\text{op}}$, for every $\varepsilon > 0$, and taking the supremum on the l.h.s. we obtain $\|\rho\|_* \leq \|\rho\|_{\text{op}}$. $\square$

Technically speaking, the two norms $\|\cdot\|_{\text{sup}}$ and $\|\cdot\|_*$ are a base and order-unit norm pair [51].

What is missing now is the notion of a convex set of proper states, the *extended preparations* of our infinite system A_G . Let us then give the following definition.

Definition 16 (Local restriction). *Given a state $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$, the local restriction of ρ to $S \in \mathcal{R}^{(G)}$ is the functional $\rho|_S \in \llbracket A_S \rrbracket_{\mathbb{R}}$ defined through*

$$(a|\rho|_S) := (a_S|\rho), \quad \forall a \in \llbracket \bar{A}_S \rrbracket. \quad (33)$$

The notion of a restriction can be brought further, considering infinite regions, by invoking the isomorphism $\mathcal{J}_R$ defined in Eq. (25), as follows.

Definition 17 (Restriction). *Given a state $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$, the restriction of ρ to $S \in \bar{\mathcal{R}}^{(G)}$ is the functional $\rho|_S \in \llbracket A_S \rrbracket_{\mathbb{R}}$ defined through*

$$(a|\rho|_S) := (\mathcal{J}_S^{-1}a|\rho), \quad \forall a \in \llbracket \bar{A}_S \rrbracket. \quad (34)$$

Defining $\hat{\mathcal{J}}_S^{-1}$ by duality as

$$(a|\hat{\mathcal{J}}_S^{-1}\rho) := (\mathcal{J}_S^{-1}a|\rho), \quad (35)$$

we can then equivalently express Eq. (34) as

$$\rho|_S = \hat{\mathcal{J}}_S^{-1}\rho. \quad (36)$$

Representing generalised extended states through diagrams, and reminding eq. (28), the above equations can be recast as

$$\begin{aligned} \text{Diagram 1: } \rho|_S \text{ (box) } \xrightarrow{S} a \text{ (box)} &= \text{Diagram 2: } \rho \text{ (box) } \xrightarrow{R} \boxed{\mathcal{J}_S^{-1}} \xrightarrow{S} a \text{ (box)} \\ &= \text{Diagram 3: } \rho \text{ (box) } \xrightarrow{S} \begin{array}{c} a \text{ (box)} \\ G \setminus S \text{ (box)} \\ e \text{ (box)} \end{array} \end{aligned}$$

which can be taken as the definition of the equation

$$\rho|_S \text{ (box) } \xrightarrow{S} := \rho \text{ (box) } \xrightarrow{S} \begin{array}{c} a \text{ (box)} \\ G \setminus S \text{ (box)} \\ e \text{ (box)} \end{array}. \quad (37)$$

We can now introduce the set of states $\llbracket A_G \rrbracket$ as the special set of generalised extended states whose restrictions are all states. In other words, an element ρ of the space $\llbracket A_G \rrbracket_{\mathbb{R}}$ is a state if, restricted to any finite region $R \in \mathcal{R}^{(G)}$, it defines a state for that region.

Definition 18 (Extended states). *The set $\llbracket A_G \rrbracket$ of states in the space of generalised extended states $\llbracket A_G \rrbracket_{\mathbb{R}}$ is the set of those elements $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$ such that for every $R \in \mathcal{R}^{(G)}$ the local restriction of ρ to R is a state $\rho|_R \in \llbracket A_R \rrbracket$. Deterministic states, whose set is denoted by $\llbracket A_G \rrbracket_1$, are those states $\rho \in \llbracket A_G \rrbracket$ with $(e_G|\rho) = 1$.*

One can easily verify that the set $\llbracket A_G \rrbracket$ is convex, as a consequence of convexity of $\llbracket A \rrbracket$ for every finite system A . As we did for finite systems, we also define a positive cone generated by $\llbracket A_G \rrbracket$. Moreover, it is easy to check that the restriction of a state to an infinite region $S \in \bar{\mathcal{R}}^{(G)}$ is a state in $\llbracket A_S \rrbracket$.

Definition 19 (Extended positive cone). *The set $\llbracket A_G \rrbracket_+$ in the space of generalised extended states $\llbracket A_G \rrbracket_{\mathbb{R}}$ is the cone of those elements $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$ such that there exist $\bar{\rho} \in \llbracket A_G \rrbracket$ and $\mu \geq 0$ such that $\rho = \mu\bar{\rho}$.*

We can now show an important result about the restriction of states.

Lemma 17. *The restriction $\hat{\mathcal{J}}_R^{-1}[\mathbb{A}_G]$ of the set of states of A_G coincides with $[\mathbb{A}_R]$.*

Proof. It is a straightforward exercise to verify that $\hat{\mathcal{J}}_R^{-1}[\mathbb{A}_G] \subseteq [\mathbb{A}_R]$. For the converse, let $\rho \in [\mathbb{A}_R]$, and let us now construct a state $\sigma \in [\mathbb{A}_G]$ such that $\sigma|_R = \rho$. We define $\bar{R} := G \setminus R$. There are two possible situations: $\bar{R} \in \mathbf{R}^{(G)}$, or $\bar{R} \in \bar{\mathbf{R}}^{(G)}$. If $\bar{R} \in \mathbf{R}^{(G)}$, let $a_T \in [\bar{\mathbb{A}}_G]_{L\mathbb{R}}$, with $T \in \mathbf{R}^{(G)}$. We define

$$(a|\sigma) := (a|\rho|_{T \cap R} \otimes \nu|_{T \cap \bar{R}}), \quad (38)$$

for an arbitrary $\nu \in [\mathbb{A}_{\bar{R}}]$, where we introduced the notation $\sigma|_{S'}$ for $S' \subseteq S \in \mathbf{R}^{(G)}$, and σ a state of the composite system $\sigma \in [\mathbb{A}_S] = [\mathbb{A}_{S'}\mathbb{A}_{S \setminus S'}]$, with

$$\boxed{\sigma|_{S'}} \xrightarrow{S'} := \left(\sigma \begin{array}{c} \xrightarrow{S'} \\ \xrightarrow{S \setminus S'} \end{array} \begin{array}{c} \boxed{e} \end{array} \right).$$

Clearly, the functional σ is bounded, and thus it can be extended to a functional on the full space $[\bar{\mathbb{A}}_G]_{Q\mathbb{R}}$, i.e. $\sigma \in [\mathbb{A}_G]_{\mathbb{R}}$. Moreover one can easily verify that $\sigma|_T \in [\mathbb{A}_T]$ for every $T \in \mathbf{R}^{(G)}$, thus σ is the desired state in $[\mathbb{A}_G]$. A similar construction can be carried out for the case $\bar{R} \in \bar{\mathbf{R}}^{(G)}$, taking $\nu \in [\mathbb{A}_{\bar{R}}]$. By construction, also in this case $\sigma \in [\mathbb{A}_G]$, as can be straightforwardly checked. $\square$

The next result that we prove is that quasi-local effects $a \in [\bar{\mathbb{A}}_G]_Q$ are positive on the set $[\mathbb{A}_G]$.

Lemma 18. *Let $a \in [\bar{\mathbb{A}}_G]_Q$, and $\rho \in [\mathbb{A}_G]$. Then $(a|\rho) \geq 0$*

Proof. Let $a \in [\bar{\mathbb{A}}_G]_Q$ be a local effect $a_R = [a_R]$. Then, by definition, $(a_R|\rho) = (a|\rho|_R) \geq 0$. Now, let $a = [a_{nR_n}]$. We have $(a|\rho) = \lim_{n \rightarrow \infty} (a_{nR_n}|\rho) \geq 0$. $\square$

The following result draws a first analogy between the properties of the sup-norm for finite systems and that for infinite parallel compositions.

Lemma 19. *Let $\rho \in [\mathbb{A}_G]_+$. Then $\|\rho\|_* = (e_G|\rho)$.*

Proof. Let $\rho \in [\mathbb{A}_G]_+$. Then, by virtue of lemma 18, $(a|\rho) \geq 0$ for all $a \in [\bar{\mathbb{A}}_G]$. Let now $\|a\|_{\text{sup}} = 1$. By lemma 15, one has $e_G \pm a \geq 0$, and thus $(e_G|\rho) \geq |(a|\rho)|$. Then $(e_G|\rho) \geq \|\rho\|_*$.

On the other hand, being e_G a legitimate element of $[\bar{\mathbb{A}}_G]_Q$ with $\|e_G\|_{\text{sup}} = 1$, one has $\|\rho\|_* = \sup_{\|a\|_{\text{sup}}=1} |(a|\rho)| \geq (e_G|\rho)$. $\square$

Thanks to lemma 12, we have the following corollary.

Corollary 9. *Let $\rho \in [\mathbb{A}_G]$. Then $0 \leq (a|\rho) \leq (e_G|\rho)$ for every $a \in [\bar{\mathbb{A}}_G]_Q$.*

We can now show that also in the infinite case every state is proportional to a deterministic one. Let us first prove a preliminary lemma.

Lemma 20. *Let $\rho \in [\mathbb{A}_G]_{\mathbb{R}}$. If $(a|\rho) = 0$ for every $a \in [\bar{\mathbb{A}}_G]_Q$, then $\rho = 0$.*

Proof. If $(a|\rho) = 0$ for every $a \in [\bar{\mathbb{A}}_G]_Q$ then, using lemma 11, we have

$$(2a - e_G|\rho) = (a|\rho) - (e_G - a|\rho) = 0, \quad \forall a \in [\bar{\mathbb{A}}_G]_Q.$$

This implies that $\|\rho\|_* = 0$, and thus $\rho = 0$. $\square$

Proposition 7. *Every state $\rho \in [\mathbb{A}_G]$ is proportional to a deterministic state $\bar{\rho} \in [\mathbb{A}_G]_1$.*

Proof. Let $\rho \in [\mathbb{A}_G]$. If $\rho = 0$, then for every $\bar{\rho} \in [\mathbb{A}_G]_1$ it is $\rho = 0\bar{\rho}$. Let then $\rho \neq 0$. By lemmas 18 and 20, there must exist $a \in [\bar{\mathbb{A}}_G]$ such that $(a|\rho) > 0$. By corollary 9, one then has $(e_G|\rho) \geq (a|\rho) > 0$. By definition, for every $S \in \mathbf{R}^{(G)}$, we have

$$(e_{A_S}|\rho|_S) = (e_S|\rho) = (e_G|\rho) > 0.$$

Thus, if we set $\bar{\rho} := \rho/(e_G|\rho)$, for every $S \in \mathbf{R}^{(G)}$ we obtain

$$(e_{A_S}|\bar{\rho}|_S) = (e_S|\bar{\rho}) = 1,$$

implying that $\bar{\rho}|_S \in [\mathbb{A}_S]_1$ for every $S \in \mathbf{R}^{(G)}$. Then, $\bar{\rho} \in [\mathbb{A}_G]_1$, and $\bar{\rho}$ is such that $\rho = (e_G|\rho)\bar{\rho}$. $\square$

We now show that the operational norm is equivalently defined on $[\bar{\mathbb{A}}_G]_{Q\mathbb{R}}$ by

$$\|a\|_{\text{op}} := \sup_{\rho \in [\mathbb{A}_G]} |(a|\rho)|.$$

Indeed, let $a = [a_{nR_n}]$. Then

$$|(a|\rho)| = \lim_{n \rightarrow \infty} |(a_{nR_n}|\rho)| = \lim_{n \rightarrow \infty} |(a_n|\rho|_{R_n})|.$$

This implies that

$$|(a|\rho)| \leq |(a_n|\rho|_{R_n})| + \varepsilon \leq \|a_{nR_n}\|_{\text{op}} + \varepsilon,$$

thus $\sup_{\rho \in \llbracket A_G \rrbracket} |(a|\rho)| \leq \|a\|_{\text{op}}$. On the other hand, for every a_{nR_n} and every ε , there exists $\rho_\varepsilon \in \llbracket A_{R_n} \rrbracket$ such that $\|a_{nR_n}\|_{\text{op}} \leq |(a_n|\rho_\varepsilon)| + \varepsilon$. Now, by lemma 17, this implies that for every ε there exists $\tilde{\rho}_\varepsilon \in \llbracket A_G \rrbracket$ such that

$$\|a_{nR_n}\|_{\text{op}} - \varepsilon \leq |(a_{nR_n}|\tilde{\rho}_\varepsilon)|.$$

Finally, this implies that for every ε there exists $\rho_\varepsilon \in \llbracket A_G \rrbracket$ such that

$$\begin{aligned} \|a\|_{\text{op}} - 3\varepsilon &\leq \|a_{nR_n}\|_{\text{op}} - 2\varepsilon \\ &\leq |(a_{nR_n}|\tilde{\rho}_\varepsilon)| - \varepsilon \\ &\leq |(a|\tilde{\rho}_\varepsilon)|. \end{aligned} \quad (39)$$

Thus, in conclusion, $\|a\|_{\text{op}} \leq \sup_{\rho \in \llbracket A_G \rrbracket} |(a|\rho)|$.

A crucial property of $\llbracket A_G \rrbracket$ is that extended states are separating for $\llbracket \bar{A}_G \rrbracket_Q$, namely if for every state two generalised effects give the same value, then they coincide.

Theorem 6 (States separate effects). *Let $a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$. If $(a|\rho) = 0$ for all $\rho \in \llbracket A_G \rrbracket_1$, then $a = 0$.*

Proof. Let $a = [a_{nR_n}] \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$, and $(a|\rho) = 0$ for every $\rho \in \llbracket A_G \rrbracket_1$. Then for every $\rho \in \llbracket A_G \rrbracket_1$ one has

$$|(a_n|\rho)| = |(a_n - a|\rho)| \leq \|a_n - a\|_{\text{sup}},$$

and thus for every $\varepsilon > 0$ there exists n_0 such that, for $n \geq n_0$, $\|a_n\|_{\text{op}} \leq \varepsilon$. Thus, we have $\|a\|_{\text{op}} = 0$. As a consequence, for every $\varepsilon > 0$ there exists n_0 such that for $n \geq n_0$ one has

$$\|a_n\|_{\text{sup}} \leq k\|a_n\|_{\text{op}} \leq \varepsilon,$$

and thus $\|a\|_{\text{sup}} = 0$. Now, this implies that $a = 0$. $\square$

Remark 2. *The above result is made possible by requirement (15) that the sup-norm and the operational norm for effects of finite systems A are bounded as*

$$\forall a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}} \quad \|a\|_{\text{sup}} \leq k\|a\|_{\text{op}},$$

with k independent of the system A . In particular, this is true under the assumption (16), i.e. that $\llbracket A \rrbracket_+^ \equiv \llbracket A \rrbracket_+$.*

We can now prove the main consequence of uniqueness of the deterministic effect e_G for A_G , i.e. that e_G is the unique effect that amounts to 1 on $\llbracket A_G \rrbracket_1$.

Proposition 8. *Let $a \in \llbracket \bar{A}_G \rrbracket_Q$. Then $(a|\rho) = 1$ for all $\rho \in \llbracket A_G \rrbracket_1$ if and only if $a = e_G$.*

Proof. Notice that, since $e_G \succeq a$ for every $a \in \llbracket \bar{A}_G \rrbracket_Q$, and $(b|\rho) \geq 0$ for every $b \in \llbracket \bar{A}_G \rrbracket_Q$ and $\rho \in \llbracket A_G \rrbracket_1$, we have

$$0 \leq (e_G - a|\rho) = 1 - (a|\rho),$$

which implies $(a|\rho) \leq 1$ for every $\rho \in \llbracket A_G \rrbracket_1$. Now, if $(a|\rho) = 1$ for every $\rho \in \llbracket A_G \rrbracket_1$, we have

$$(e_G - a|\rho) = 0, \quad \forall \rho \in \llbracket A_G \rrbracket_1,$$

which by theorem 6 implies $a = e_G$. $\square$

A class of states of particular interest is that of quasi-local states, which are intuitively understood as states whose preparation can be arbitrarily approximated by local procedures. These live in small subspaces of the space $\llbracket A_G \rrbracket_{\mathbb{R}}$. Their construction is similar to that of quasi-local effects, but differs from it in a relevant respect, that is the necessity of defining arbitrary reference states. The construction is inspired by pioneering works on infinite tensor products by von Neumann and Murray [49], and can be found in appendix B.

4.3 Quasi-local transformations

Now we are going to define the quasi-local algebra $\llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$ of transformations. In usual approaches to quantum infinite systems, one usually introduces the *effect algebra*, that in the OPT case would correspond to the space of quasi-local effects. However, one can easily understand that, unlike effects in a general OPT, the quantum effect algebra contains far more information than the mere structure of the space of effects. Indeed, in the quantum effect algebra one can find every operator on the Hilbert space, and in turn, the operational role of a linear operator is to provide a Kraus representation of a transformation. In this perspective, one can also understand why multiplication of effects has a meaning at all: there is no point in multiplying effects, but multiplication of Kraus operators is the mathematical representation of sequential composition. Thus, the algebraic structure of effects conveniently summarises information about every kind of event in the theory: effects, with their coarse-graining represented by the sum, and transformations, with

sequential composition represented by multiplication.

Cellular automata are defined in quantum theory by specifying their action on the effect algebra. Such an action thus provides information about how both effects and transformations are transformed by the cellular automaton. For a general OPT, however, such a compact algebraic structure embodying all the relevant information is absent, and the definition of a CA on the space of effects is not sufficient: we need to specify how the CA transforms transformations.

For these reasons we construct now the Banach algebra of quasi-local transformations, in a way that is very closely reminiscent of the construction of quasi-local effects. Here, in order to define a local transformation, we recur to the notion of a transformation that acts non-trivially on finitely many systems, while it acts as the identity on the remaining ones. In other words, the role that is played by the deterministic effect in the case of local effects is played here by the identity transformation. Also in this case, the topological closure will be taken in the sup-norm, and this choice is of great relevance to ensure a Banach algebra structure for the closure.

Let us then introduce the Banach algebra of quasi-local transformations.

Definition 20. Let $R \subseteq G$ be an arbitrary finite region of G . We define local transformation $\mathcal{A}_R$ of A_G any pair $(\mathcal{A}, R)$, where $\mathcal{A} \in \llbracket A_R \rightarrow A_R \rrbracket_{\mathbb{R}}$. The action $\mathcal{A}_R^\dagger$ of $\mathcal{A}_R$ on $\text{Pre}[\bar{A}_G]_{L\mathbb{R}}$ is defined as follows

$$\frac{R \setminus S}{R \cap S} \mathcal{A}_R^\dagger a_S := \frac{R \setminus S}{R \cap S} \mathcal{A}_R \frac{R \setminus S}{R \cap S} e \quad (40)$$

We also define $\text{Pre}[A_G \rightarrow A_G]_{L\mathbb{R}}$ as the set of local transformations $\mathcal{A}_R$.

Lemma 21. Let $a_S \sim b_T$. Then $\mathcal{A}_R^\dagger a_S \sim \mathcal{A}_R^\dagger b_T$.

Proof. By hypothesis, there exists $c \in \llbracket \bar{A}_{S \cap T} \rrbracket_{\mathbb{R}}$ such that

$$a = c \otimes e_{S \setminus T}, \quad b = c \otimes e_{T \setminus S}.$$

Then we have

$$\begin{aligned} |\mathcal{A}_R^\dagger a_S| &= (c \otimes e_{S \setminus T} \otimes e_{R \setminus S}) |(\mathcal{A} \otimes \mathcal{I}_{S \setminus R}), \\ |\mathcal{A}_R^\dagger b_T| &= (c \otimes e_{T \setminus S} \otimes e_{R \setminus T}) |(\mathcal{A} \otimes \mathcal{I}_{T \setminus R}). \end{aligned}$$

By Eq. (10) the thesis follows. $\square$

We can now define an equivalence relation between local transformations as follows

$$\mathcal{A}_S \sim \mathcal{B}_T \Leftrightarrow \begin{cases} \mathcal{A} = \mathcal{C} \otimes \mathcal{I}_{S \setminus T}, \\ \mathcal{B} = \mathcal{C} \otimes \mathcal{I}_{T \setminus S}, \end{cases} \quad (41)$$

for some $\mathcal{C} \in \llbracket A_{S \cap T} \rrbracket_{\mathbb{R}}$. The set of local transformations is then defined as

$$\llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}} := \text{Pre}[A_G \rightarrow A_G]_{L\mathbb{R}} / \sim$$

Lemma 22. Let $\mathcal{A}_R \in \text{Pre}[A_G \rightarrow A_G]_{L\mathbb{R}}$. Then for every finite region $H \in \mathcal{R}^{(G)}$ such that $H \cap R = \emptyset$, one has $(\mathcal{A} \otimes \mathcal{I}_{A_H})_{R \cup H} \in \text{Pre}[A_G \rightarrow A_G]_{L\mathbb{R}}$, and $(\mathcal{A} \otimes \mathcal{I}_{A_H})_{R \cup H} \sim \mathcal{A}_R$.

Proof. It is straightforward to verify that $(\mathcal{A} \otimes \mathcal{I}_{A_H})_{R \cup H} \sim \mathcal{A}_R$ by direct inspection of the defining equation (41). $\square$

We can now provide a way to identify a canonical representative of the equivalence class of $\mathcal{A}_R$, which is defined in analogy to the case of effects as follows.

Definition 21. The minimal representative of the equivalence class $\mathcal{A}_R$, denoted as $\tilde{\mathcal{A}}_{R, \mathcal{A}}$, is defined through

$$R_{\mathcal{A}} := \bigcap_{S \in \mathcal{R}(\mathcal{A}, R)} S, \quad \tilde{\mathcal{A}}_{R, \mathcal{A}} \sim \mathcal{A}_R, \quad (42)$$

where $\mathcal{R}(\mathcal{A}, R)$ is the set of all those finite regions $S \subseteq G$ for which there exists $\mathcal{B} \in \llbracket A_S \rightarrow A_S \rrbracket_{\mathbb{R}}$ such that $\mathcal{B}_S \sim \mathcal{A}_R$.

Lemma 23. The minimal representative exists and is unique.

The proof follows step by step that of lemma 73.

The first result that we need to prove is the following.

Lemma 24. Let $\mathcal{A}_R \sim \mathcal{B}_S$. Then for every effect $a_T \in \text{Pre}[\bar{A}_G]_{L\mathbb{R}}$, one has

$$\mathcal{A}_R^\dagger a_T \sim \mathcal{B}_S^\dagger a_T. \quad (43)$$

Proof. If $(\tilde{\mathcal{C}})_{R_{\tilde{\mathcal{C}}}}$ is the minimal representative of $\mathcal{A}_R \sim \mathcal{B}_S$, by Eq. (41), one has

$$\begin{aligned} \mathcal{A} &= \tilde{\mathcal{C}} \otimes \mathcal{I}_{R \setminus R_{\tilde{\mathcal{C}}}}, \\ \mathcal{B} &= \tilde{\mathcal{C}} \otimes \mathcal{I}_{S \setminus R_{\tilde{\mathcal{C}}}}. \end{aligned}$$

Now, by the defining equation 40, we have that

$$\begin{aligned} (\mathcal{A}_R^\dagger a_T) &= (\tilde{a} \otimes e_{(T \cup R) \setminus R_a} | (\tilde{\mathcal{C}} \otimes \mathcal{I}_{(T \cup R) \setminus R_{\mathcal{C}}}) \\ &= (\tilde{a} \otimes e_{[T \cup (R \cap S)] \setminus R_a} | (\tilde{\mathcal{C}} \otimes \mathcal{I}_{[T \cup (R \cap S)] \setminus R_{\mathcal{C}}}) \\ &\quad \otimes (e_{R \setminus S} |, \\ (\mathcal{B}_S^\dagger a_T) &= (\tilde{a} \otimes e_{(T \cup S) \setminus R_a} | (\tilde{\mathcal{C}} \otimes \mathcal{I}_{(T \cup S) \setminus R_{\mathcal{C}}}) \\ &= (\tilde{a} \otimes e_{[T \cup (R \cap S)] \setminus R_a} | (\tilde{\mathcal{C}} \otimes \mathcal{I}_{[T \cup (R \cap S)] \setminus R_{\mathcal{C}}}) \\ &\quad \otimes (e_{S \setminus R} |, \end{aligned}$$

and finally by Eq. (123) this implies the thesis. $\square$

By virtue of lemmas 21 and 24, we can define the action of $\mathcal{A}_R \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$ on local effects $\llbracket \bar{A}_G \rrbracket_{\mathbb{R}}$ as

$$\mathcal{A}_R^\dagger[a_T] := [\mathcal{A}_R^\dagger a_T],$$

where we leave the square braces to denote equivalence classes for $\sim$ in $\text{Pre}\llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$, for the sake of clarity. We can now make local transformations into a vector space as we did for local effects, as follows.

Definition 22. Let $\mathcal{A}_R, \mathcal{B}_S \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$, and $h \in \mathbb{R}$. Then we define

$$\begin{aligned} h\mathcal{A}_R &:= \begin{cases} (h\mathcal{A})_R & h \neq 0, \\ 0_\emptyset & h = 0, \end{cases} \\ \mathcal{A}_R + \mathcal{B}_S &:= \mathcal{C}_{R \cup S} \\ \mathcal{C} &:= \mathcal{A} \otimes \mathcal{I}_{S \setminus R} + \mathcal{B} \otimes \mathcal{I}_{R \setminus S}. \end{aligned}$$

Moreover, the following operation makes the set of local transformations into an algebra.

Definition 23. Let $\mathcal{A}_R, \mathcal{B}_S \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$. Then we define

$$\mathcal{A}_R \mathcal{B}_S := (\{\mathcal{A} \otimes \mathcal{I}_{S \setminus R}\} \{\mathcal{B} \otimes \mathcal{I}_{R \setminus S}\})_{R \cup S}.$$

We omit the straightforward proof that the above definition is well defined, i.e. independent of the choice of representatives in the classes of $\mathcal{A}_R$ and $\mathcal{B}_S$.

Definition 24. The algebra of generalized local transformations is the unital algebra $\llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$ of finite real combinations of local transformations, with unit $1_\emptyset$ and null element $0_\emptyset$.

In order to close the algebra of local operations we introduce the topology given by the *sup-norm*, given in the following, in analogy to the case of effects. We also discuss the interplay of the sup-norm topology with that given by the *operational norm*, that we define right away.

Definition 25. The operational norm $\|\mathcal{A}_R\|_{\text{op}}$ of $\tilde{\mathcal{A}}_{R_{\mathcal{A}}} = \mathcal{A}_R \in \llbracket A \rightarrow A \rrbracket_{L\mathbb{R}}$ is defined by the following expression

$$\|\mathcal{A}_R\|_{\text{op}} := \|\tilde{\mathcal{A}}\|_{\text{op}}. \quad (44)$$

Proposition 9. Let $\mathcal{A}_R \in \llbracket A \rightarrow A \rrbracket_{L\mathbb{R}}$. Then $\|\mathcal{A}_R\|_{\text{op}} = \|\mathcal{A}\|_{\text{op}}$.

Proof. Let $\mathcal{A}_R \sim \tilde{\mathcal{A}}_{R_{\mathcal{A}}}$, and let $R = R_{\mathcal{A}} \cup R'$, with $R_{\mathcal{A}} \cap R' = \emptyset$. Then by Eq. (41) we have

$$\mathcal{A} = \tilde{\mathcal{A}} \otimes \mathcal{I}_{A_{R'}},$$

Now, by definition of $\|\cdot\|_{\text{op}}$ (see [8]) it straightforwardly follows that $\|\mathcal{B} \otimes \mathcal{I}_C\|_{\text{op}} = \|\mathcal{B}\|_{\text{op}}$. Thus, $\|\mathcal{A}_R\|_{\text{op}} = \|\tilde{\mathcal{A}}\|_{\text{op}} = \|\mathcal{A}\|_{\text{op}}$. $\square$

Unfortunately, the operational norm does not enjoy the basic property that would make the algebra of transformations into a Banach algebra, i.e. it is not true that $\|\mathcal{A} \mathcal{B}\|_{\text{op}} \leq \|\mathcal{A}\|_{\text{op}} \|\mathcal{B}\|_{\text{op}}$. For this reason, in analogy with the case of quasi-local effects, we introduce a second norm, the *sup-norm*, whose interplay with the operational norm will make it possible to define the closed Banach algebra of quasi-local transformations.

This is obtained extending the definition of sup-norm to the algebra of local transformations of A_G .

Definition 26. The sup-norm $\|\mathcal{A}_R\|_{\text{sup}}$ of $\mathcal{A}_R \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$ is defined as

$$\|\mathcal{A}_R\|_{\text{sup}} := \inf J(\tilde{\mathcal{A}}) = \|\tilde{\mathcal{A}}\|_{\text{sup}}. \quad (45)$$

Proposition 10. Let $\mathcal{A}_R \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$. Then $\|\mathcal{A}_R\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}$.

Proof. Let $\mathcal{A}_R \sim \tilde{\mathcal{A}}_{R_{\mathcal{A}}}$, and let $R = R_{\mathcal{A}} \cup R'$, with $R_{\mathcal{A}} \cap R' = \emptyset$. Then by Eq. (41) we have

$$\mathcal{A} = \tilde{\mathcal{A}} \otimes \mathcal{I}_{A_{R'}},$$

and by corollary 2, $\|\mathcal{A}\|_{\text{sup}} = \|\tilde{\mathcal{A}}\|_{\text{sup}} = \|\mathcal{A}_R\|_{\text{sup}}$. $\square$

Corollary 10. Let $\mathcal{A}, \mathcal{B} \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$. Then $\|\mathcal{A} \mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}} \|\mathcal{B}\|_{\text{sup}}$.

Proof. The result is a straightforward consequence of propositions 10 and 2. $\square$

Corollary 11. *On $[[A_G \rightarrow A_G]]_{L\mathbb{R}}$, one has $\|\mathcal{A}_R\|_{\text{op}} \leq \|\mathcal{A}_R\|_{\text{sup}}$.*

Proof. The result is a straightforward consequence of propositions 9 and 10, and corollary 4. $\square$

The above results allow us to extend the local algebra of events into the *quasi-local* algebra, which is a Banach algebra. The construction is analogous to that of quasi-local states, and proceeds as follows. First we define the algebra $[[A_G \rightarrow A_G]]_{C\mathbb{R}}$ of sup-Cauchy sequences of local transformations, i.e. its elements are sequences $\mathcal{A} : \mathbb{N} \rightarrow [[A_G \rightarrow A_G]]_{L\mathbb{R}} :: n \mapsto \mathcal{A}_{nR_n}$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, one has $\|\mathcal{A}_n - \mathcal{A}_m\|_{\text{sup}} < \varepsilon$. Now we define the equivalence relation between elements of $[[A_G \rightarrow A_G]]_{C\mathbb{R}}$: $\mathcal{A} \cong \mathcal{B}$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ one has $\|\mathcal{A}_n - \mathcal{B}_n\|_{\text{sup}} < \varepsilon$. Finally, we define the quasi-local algebra as follows.

Definition 27. *The space of quasi-local transformations of A is defined as*

$$[[A_G \rightarrow A_G]]_{Q\mathbb{R}} := [[A_G \rightarrow A_G]]_{C\mathbb{R}} / \cong.$$

Definition 28. *An element $\mathcal{A}$ in $[[A_G \rightarrow A_G]]_{Q\mathbb{R}}$ is an event if $\mathcal{A} = [\mathcal{A}_{nR_n}]$ for a sequence $\mathcal{A}_{nR_n}$ such that $\mathcal{A}_n \in [[A_{R_n} \rightarrow A_{R_n}]]$ for all $n \in \mathbb{N}$. The set of events will be denoted by $[[A_G \rightarrow A_G]]_Q$.*

An element $\mathcal{A} \in [[A_G \rightarrow A_G]]_Q$ is a channel if $\mathcal{A} = [\mathcal{A}_{nR_n}]$ for a sequence $\mathcal{A}_{nR_n}$ such that $\mathcal{A}_n \in [[A_{R_n} \rightarrow A_{R_n}]]_1$ for all $n \in \mathbb{N}$. The set of channels will be denoted by $[[A_G \rightarrow A_G]]_{Q1}$.

The subset of $[[A_G \rightarrow A_G]]_{Q\mathbb{R}}$ containing elements $\mathcal{A} = [\mathcal{A}_{nR_n}]$ for a sequence $\mathcal{A}_{nR_n}$ such that $\mathcal{A}_n \in [[A_{R_n} \rightarrow A_{R_n}]]_+$ for all $n \in \mathbb{N}$ will be denoted by $[[A_G \rightarrow A_G]]_{Q+}$. Equivalently, we will write $\mathcal{A} \succeq 0$ for $\mathcal{A} \in [[A_G \rightarrow A_G]]_{Q+}$.

In the following, in analogy with the finite case, we will write $\mathcal{A} \succeq \mathcal{B}$ if $\mathcal{A} - \mathcal{B} \succeq 0$. The algebra of quasi-local transformations $[[A_G \rightarrow A_G]]_{Q\mathbb{R}}$ is defined as follows.

Definition 29. *Let $\mathcal{A} = [\mathcal{A}_{nR_n}]$, $\mathcal{B} = [\mathcal{B}_{nS_n}]$. We define $\mathcal{A}\mathcal{B} := [\mathcal{A}_{nR_n}\mathcal{B}_{nS_n}]$.*

The product $\mathcal{A}\mathcal{B}$ is well defined. Indeed:

1. $\mathcal{A}_{nR_n}\mathcal{B}_{nS_n}$ is a Cauchy sequence.
2. For every $\mathcal{A}'_{nR'_n}$, $\mathcal{B}'_{nS'_n}$ in the equivalence classes defining $\mathcal{A}$ and $\mathcal{B}$, respectively, one has $\mathcal{A}_{nR_n}\mathcal{B}_{nS_n} \cong \mathcal{A}'_{nR'_n}\mathcal{B}'_{nS'_n}$.

Item 1 can be easily proved since we have

$$\begin{aligned} & \|\mathcal{A}_n\mathcal{B}_n - \mathcal{A}_m\mathcal{B}_m\|_{\text{sup}} \\ &= \|\mathcal{A}_n\mathcal{B}_n - \mathcal{A}_m\mathcal{B}_n + \mathcal{A}_m\mathcal{B}_n - \mathcal{A}_m\mathcal{B}_m\|_{\text{sup}} \\ &\leq \|\mathcal{A}_n - \mathcal{A}_m\|_{\text{sup}}\|\mathcal{B}_n\|_{\text{sup}} \\ &\quad + \|\mathcal{A}_m\|_{\text{sup}}\|\mathcal{B}_n - \mathcal{B}_m\|_{\text{sup}}. \end{aligned}$$

Similarly, for item 2 we have

$$\begin{aligned} & \|\mathcal{A}_n\mathcal{B}_n - \mathcal{A}'_n\mathcal{B}'_n\|_{\text{sup}} \\ &= \|\mathcal{A}_n\mathcal{B}_n - \mathcal{A}'_n\mathcal{B}_n + \mathcal{A}'_n\mathcal{B}_n - \mathcal{A}'_n\mathcal{B}'_n\|_{\text{sup}} \\ &\leq \|\mathcal{A}_n - \mathcal{A}'_n\|_{\text{sup}}\|\mathcal{B}_n\|_{\text{sup}} \\ &\quad + \|\mathcal{A}'_n\|_{\text{sup}}\|\mathcal{B}_n - \mathcal{B}'_n\|_{\text{sup}}. \end{aligned}$$

The quasi-local algebra is actually a real Banach algebra if equipped with the norm $\|\cdot\|_{\text{sup}}$, as we now prove.

Proposition 11. *The algebra $[[A_G \rightarrow A_G]]_{Q\mathbb{R}}$ equipped with the norm $\|\cdot\|_{\text{sup}}$ is a Banach algebra.*

Proof. We only need to prove that $\|\mathcal{A}\mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{sup}}$. Let $\mathcal{A} = [\mathcal{A}_{nR_n}]$ and $\mathcal{B} = [\mathcal{B}_{nS_n}]$, respectively. Now, for every $\mathcal{A}_{nR_n}$ and $\mathcal{B}_{nS_n}$ one has

$$\begin{aligned} & \|(\mathcal{A}_n \otimes \mathcal{I}_{S_n \setminus R_n})(\mathcal{B}_n \otimes \mathcal{I}_{R_n \setminus S_n})\|_{\text{sup}} \\ &\leq \|\mathcal{A}_n \otimes \mathcal{I}_{S_n \setminus R_n}\|_{\text{sup}}\|\mathcal{B}_n \otimes \mathcal{I}_{R_n \setminus S_n}\|_{\text{sup}}, \end{aligned}$$

and by proposition 10 $\|\mathcal{A}\mathcal{B}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}}\|\mathcal{B}\|_{\text{sup}}$. $\square$

The subsets $[[A_G \rightarrow A_G]]_Q$ and $[[A_G \rightarrow A_G]]_{Q1}$ are convex, and $[[A_G \rightarrow A_G]]_{Q+}$ is a convex cone. Moreover, they are closed in the sup-norm.

Proposition 12. *The sets $[[A_G \rightarrow A_G]]_Q$, $[[A_G \rightarrow A_G]]_{Q1}$, and $[[A_G \rightarrow A_G]]_{Q+}$ are closed in the sup-norm.*

Proof. The argument is the same in both cases. Let $\{\mathcal{A}_m\}_{m \in \mathbb{N}}$ be a Cauchy sequence with $\mathcal{A}_m \in [[A_G \rightarrow A_G]]_{Q*}$ for all m , with $*$ = “nothing”, 1, +. By definition, $\mathcal{A}_m = [\mathcal{A}_{mnR_{mn}}]$, with $\mathcal{A}_{mn} \in [[A_{R_{mn}} \rightarrow A_{R_{mn}}]]_*$. Then one easily proves that the sequence $\{\mathcal{A}_{mmR_{mm}}\}_{m \in \mathbb{N}}$ is Cauchy, and its limit is $\mathcal{A} := \lim_{m \rightarrow \infty} \mathcal{A}_{mmR_{mm}}$, which by definition is in $[[A_G \rightarrow A_G]]_{Q*}$. $\square$

Proposition 13. Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$. Then

$$\|\mathcal{A}\|_{\text{op}} \leq \|\mathcal{A}\|_{\text{sup}}. \quad (46)$$

Proof. By definition $\|\mathcal{A}\|_{\text{op}} = \lim_{n \rightarrow \infty} \|\mathcal{A}_n\|_{\text{op}}$ and $\|\mathcal{A}\|_{\text{sup}} = \lim_{n \rightarrow \infty} \|\mathcal{A}_n\|_{\text{sup}}$, where $\{\mathcal{A}_{nR_n}\}_{n \in \mathbb{N}} \subseteq \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$ is a Cauchy sequence converging to $\mathcal{A}$, thus by proposition 3 one has

$$\|\mathcal{A}_n\|_{\text{op}} \leq \|\mathcal{A}_n\|_{\text{sup}}, \quad \forall n \in \mathbb{N},$$

which implies the inequality in the limit. $\square$

Now, we can prove the following lemma.

Lemma 25. Let $a \in \llbracket \bar{A}_G \rrbracket_{C\mathbb{R}}$ be a Cauchy sequence, and let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$. We then have

$$\mathcal{A}^\dagger a \in \llbracket \bar{A}_G \rrbracket_{C\mathbb{R}}. \quad (47)$$

Proof. We just need to evaluate $\|\mathcal{A}^\dagger a_m - \mathcal{A}^\dagger a_n\|_{\text{sup}}$, and using proposition 2 one has

$$\|\mathcal{A}^\dagger(a_m - a_n)\|_{\text{sup}} \leq \|a_m - a_n\|_{\text{sup}} \|\mathcal{A}\|_{\text{sup}}.$$

Since the sequence a is Cauchy, also $\mathcal{A}^\dagger a$ is Cauchy. $\square$

Lemma 26. Let $a, b \in \llbracket \bar{A}_G \rrbracket_{C\mathbb{R}}$ be equivalent Cauchy sequences, and let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$. We then have

$$[\mathcal{A}^\dagger a_{nR_n}] = [\mathcal{A}^\dagger b_{nS_n}]. \quad (48)$$

Proof. We can bound $\|\mathcal{A}^\dagger a_n - \mathcal{A}^\dagger b_n\|_{\text{sup}}$ using proposition 2, obtaining

$$\|\mathcal{A}^\dagger(a_n - b_n)\|_{\text{sup}} \leq \|a_n - b_n\|_{\text{sup}} \|\mathcal{A}\|_{\text{sup}}.$$

Since $[a_{nR_n}] = [b_{nS_n}]$, the thesis follows. $\square$

As a consequence of the above results, all local transformations leave the space $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ invariant. We now prove a further important result: the above statement can be extended to all the quasi-local algebra.

Theorem 7. The quasi-local algebra $\llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$ leaves the space $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ invariant, and for every $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$ and every $a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$, it is

$$\|\mathcal{A}^\dagger a\|_{\text{sup}} \leq \|a\|_{\text{sup}} \|\mathcal{A}\|_{\text{sup}}. \quad (49)$$

Proof. By proposition 2 and lemma 25, the thesis is true for $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$ and $a \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$. Let now $a = [a_{pR_p}]$ and $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{L\mathbb{R}}$. Then

$$\|\mathcal{A}^\dagger a_p\|_{\text{sup}} \leq \|a_p\|_{\text{sup}} \|\mathcal{A}\|_{\text{sup}},$$

and taking the limit for $p \rightarrow \infty$ we obtain the thesis. Finally, let $\mathcal{A} = [\mathcal{A}_{nR_n}]$. In this case, by lemma 25 $\mathcal{A}_n^\dagger a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ for every $n \in \mathbb{N}$. Moreover, $\{\mathcal{A}_n^\dagger a\}_{n \in \mathbb{N}} \in \llbracket \bar{A}_G \rrbracket_{C\mathbb{R}}$. Indeed, by the result we just proved

$$\|(\mathcal{A}_n^\dagger - \mathcal{A}_m^\dagger)a\|_{\text{sup}} \leq \|a\|_{\text{sup}} \|\mathcal{A}_n - \mathcal{A}_m\|_{\text{sup}}.$$

Finally, by lemma 25, we have

$$\|\mathcal{A}_n^\dagger a\|_{\text{sup}} \leq \|a\|_{\text{sup}} \|\mathcal{A}_n\|_{\text{sup}},$$

and taking the limit for $n \rightarrow \infty$ we obtain the thesis. $\square$

A remarkable result that will be important in the following is given by the following lemma.

Lemma 27. Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$. Then for every $a \in \llbracket \bar{A}_G \rrbracket_Q$ one has $\mathcal{A}^\dagger a \in \llbracket \bar{A}_G \rrbracket_Q$.

Proof. By hypothesis, $\mathcal{A} = [\mathcal{A}_{nR_n}]$ and $a = [a_{mS_m}]$, with $\mathcal{A}_n \in \llbracket A_{R_n} \rightarrow A_{R_n} \rrbracket$ and $a_m \in \llbracket \bar{A}_{S_m} \rrbracket$. Thus $\mathcal{A}_n^\dagger a_n \in \llbracket \bar{A}_{R_n \cup S_n} \rrbracket$. Now, one can easily verify that $\lim_{n \rightarrow \infty} \mathcal{A}_n^\dagger a_n = \mathcal{A}^\dagger a$, which proves the statement. $\square$

In particular, one has the following condition.

Lemma 28. Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$. Then $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$ iff $\mathcal{A}^\dagger e_G \in \llbracket \bar{A}_G \rrbracket_Q$.

Proof. By definition, one has $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$ if and only if $\mathcal{A} = [\mathcal{A}_{nR_n}]$ with $\mathcal{A}_n \in \llbracket A_{R_n} \rightarrow A_{R_n} \rrbracket$, and by our assumptions the latter is equivalent to $a_n := \mathcal{A}_n^\dagger e_{A_{R_n}} \in \llbracket \bar{A}_{R_n} \rrbracket$. Now, since $\|a_{nR_n} - a_{mR_m}\|_{\text{sup}} \leq \|\mathcal{A}_{nR_n} - \mathcal{A}_{mR_m}\|_{\text{sup}}$, one has that $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$ if and only if $\mathcal{A}^\dagger e_G = [a_{nR_n}] \in \llbracket \bar{A}_G \rrbracket_Q$. $\square$

Lemma 29. Let $\mathcal{A} \in \llbracket A \rightarrow B \rrbracket_{Q+}$, and $a := \mathcal{A}^\dagger e_G$. Then $\|\mathcal{A}\|_{\text{sup}} = \|a\|_{\text{sup}}$.

Proof. By definition, $\mathcal{A} = [\mathcal{A}_{nR_n}]$, with $\mathcal{A}_n \in \llbracket A_{R_n} \rightarrow A_{R_n} \rrbracket_+$. Now, $\|\mathcal{A}_{nR_n}\|_{\text{sup}} = \|\mathcal{A}_n\|_{\text{sup}}$, and by lemma 7, $\|\mathcal{A}_n\|_{\text{sup}} = \|\mathcal{A}_n^\dagger e_{R_n}\|_{\text{sup}} = \|\mathcal{A}_{nR_n}^\dagger e_G\|_{\text{sup}}$. Then,

$$\|\mathcal{A}\|_{\text{sup}} = \lim_{n \rightarrow \infty} \|\mathcal{A}_{nR_n}^\dagger e_G\|_{\text{sup}} = \|\mathcal{A}^\dagger e_G\|_{\text{sup}}. \quad \square$$

On the same line, we can provide the following condition for $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$ to be a channel.

Lemma 30. *Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$. Then $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$ iff $\mathcal{A}^\dagger e_G = e_G$.*

Proof. First, let $\mathcal{A}$ be a channel. By definition it must be $\mathcal{A} = \lim_{n \rightarrow \infty} \mathcal{A}_{nR_n}$, where $\mathcal{A}_{nR_n}$ are local channels. Thus, by theorem 3, $\mathcal{A}_{nR_n}^\dagger e_G = e_G$. It follows that $\mathcal{A}^\dagger e_G = \lim_{n \rightarrow \infty} \mathcal{A}_{nR_n}^\dagger e_G = e_G$. Now, for the converse, by definition $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$ iff $\mathcal{A} = [\mathcal{A}_{nR_n}]$ with $\mathcal{A}_{nR_n} \in \llbracket A_{R_n} \rightarrow A_{R_n} \rrbracket_+$ for all $n \in \mathbb{N}$. Let us take $n \geq n_0$ so that $\|\mathcal{A}_{nR_n} - \mathcal{A}\|_{\text{sup}} \leq \varepsilon$. Then we have

$$|\|\mathcal{A}_{nR_n}\|_{\text{sup}} - \|\mathcal{A}\|_{\text{sup}}| \leq \varepsilon.$$

Now, by lemma 29, since $\mathcal{A}^\dagger e_G = e_G$, we have $\|\mathcal{A}\|_{\text{sup}} = 1$. Let us define $\mathcal{A}'_{nR_n} := \mathcal{A}_{nR_n} / \|\mathcal{A}_{nR_n}\|_{\text{sup}}$. Clearly,

$$\|\mathcal{A}'_{nR_n} - \mathcal{A}_{nR_n}\|_{\text{sup}} = \|\mathcal{A}_{nR_n}\|_{\text{sup}} |1 - 1/\|\mathcal{A}_{nR_n}\|_{\text{sup}}| \leq \varepsilon.$$

This implies that $\mathcal{A}'_{nR_n}$ is a sequence converging to $\mathcal{A}$, with $\|\mathcal{A}'_{nR_n}\|_{\text{sup}} = 1$. As a consequence, there exist channels $\mathcal{C}_{nR_n}$ such that $(\mathcal{C}_n - \mathcal{A}'_n)_{R_n} \succeq 0$. Again by lemma 7,

$$\begin{aligned} \|(\mathcal{C}_n - \mathcal{A}'_n)_{R_n}\|_{\text{sup}} &= \|(\mathcal{C}_n - \mathcal{A}'_n)_{R_n}^\dagger e_{R_n}\|_{\text{sup}} \\ &= \|e_G - \mathcal{A}'_{nR_n}^\dagger e_G\|_{\text{sup}} \\ &= \|(\mathcal{A} - \mathcal{A}'_{nR_n})^\dagger e_G\|_{\text{sup}} \\ &\leq \|(\mathcal{A} - \mathcal{A}'_{nR_n})\|_{\text{sup}} \leq \varepsilon. \end{aligned}$$

This implies that $\mathcal{A} = [\mathcal{A}'_{nR_n}] = [\mathcal{C}_{nR_n}]$, and then $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_1$. $\square$

The following weak version of the converse of lemma 28 can be easily proved.

Lemma 31. *Let $a \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$. Then there exists $\mathcal{A} \in \llbracket A_G \rrbracket_{L\mathbb{R}}$ such that*

$$a = \mathcal{A}^\dagger e_G. \quad (50)$$

Moreover, for $a \in \llbracket \bar{A}_G \rrbracket_L$, Eq. (50) is satisfied with $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_L$.

Proof. Let $a = a_R \in \llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$. Let us define $\mathcal{A}_R := |\sigma\rangle\langle a|$, where $\sigma \in \llbracket A_R \rrbracket_1$ is an arbitrary deterministic state. Then clearly

$$\mathcal{A}_R^\dagger e_G = a_R.$$

It is also clear that, for $a = a_R \in \llbracket \bar{A}_G \rrbracket_L$, the above defined transformation $\mathcal{A}_R$ is in $\llbracket A_G \rrbracket_L$. $\square$

One can take a further step, and prove the following result that will be of crucial importance in the remainder.

Corollary 12. *Let $a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$. Then there exists a sequence of quasi-local transformations $\mathcal{A}_{nR_n}$ such that $\lim_{n \rightarrow \infty} \mathcal{A}_{nR_n}^\dagger e_G = a$. If $a \in \llbracket \bar{A}_G \rrbracket_Q$ the sequence can be found in $\llbracket A_G \rightarrow A_G \rrbracket_Q$.*

Notice that the sequence $\mathcal{A}_{nR_n}$ is not necessarily Cauchy, thus one cannot conclude that for every $a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ there is $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$ such that $a = \mathcal{A}^\dagger e_G$. However, the following lemma completes the picture.

Proposition 14. *Let $a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$. Then there exists $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$ such that*

$$a = \mathcal{A}^\dagger e_G. \quad (51)$$

Moreover, for $a \in \llbracket \bar{A}_G \rrbracket_{Q}$ with $*$ $\in \{\text{"nothing"}, 1, +\}$ one can find the above $\mathcal{A}$ in the set $\llbracket A_G \rightarrow A_G \rrbracket_{Q*}$.*

Proof. Let us define the sets

$$\mathsf{T}_a^\varepsilon := \{\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}} \mid \|\mathcal{A}^\dagger e_G - a\|_{\text{sup}} \leq \varepsilon\}.$$

The sets T_a^ε are not empty for every a and every $\varepsilon > 0$, as a consequence of corollary 12. Moreover, they are closed. Indeed, let $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in T_a^ε . This implies that by definition $a_n := \mathcal{A}_n^\dagger e_G \in \bar{B}_\varepsilon(a)$. Since the ball $\bar{B}_\varepsilon(a)$ is closed, also $\lim_{n \rightarrow \infty} a_n = (\lim_{n \rightarrow \infty} \mathcal{A}_n)^\dagger e_G \in \bar{B}_\varepsilon(a)$, namely $\lim_{n \rightarrow \infty} \mathcal{A}_n \in \mathsf{T}_a^\varepsilon$. Finally, one clearly has that for $\varepsilon < \delta$ it is $\mathsf{T}_a^\varepsilon \subseteq \mathsf{T}_a^\delta$, and the diameter $d_\varepsilon := \sup\{\|\mathcal{A} - \mathcal{B}\|_{\text{sup}} \mid \mathcal{A}, \mathcal{B} \in \mathsf{T}_a^\varepsilon\}$ is a non-decreasing function of ε . Let then

$$\mathsf{T}_a^0 := \bigcap_{\varepsilon > 0} \mathsf{T}_a^\varepsilon.$$

Either $\lim_{\varepsilon \rightarrow 0^+} d_\varepsilon > 0$, in which case T_a^0 is non empty, and each of its elements satisfies $\mathcal{A}^\dagger e_G = a$, or the limit is 0, in which case, by Cantor's intersection theorem for complete metric spaces, $\mathsf{T}_a^0 = \{\mathcal{A}\}$ is a singleton, with $\mathcal{A}^\dagger e_G = a$. The same argument can be applied in each of the cases where $a \in \llbracket \bar{A}_G \rrbracket_{Q*}$, considering the closed sets $\mathsf{T}_a^{\varepsilon*} := \mathsf{T}_a^\varepsilon \cap \llbracket A_G \rightarrow A_G \rrbracket_{Q*}$. $\square$

We now prove a result that is analogous to lemma 12.

Lemma 32. Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$, and $\mathcal{C} - \mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$ for some $\mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$. Then $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$.

Proof. By definition, there are three sequences $\mathcal{A}_{nR_n}, \mathcal{A}'_{nS_n}$ in $\llbracket A_G \rightarrow A_G \rrbracket_L$ and $\mathcal{C}_{nT_n}$ in $\llbracket A_G \rightarrow A_G \rrbracket_{L1}$ such that $\lim_{n \rightarrow \infty} \mathcal{A}_n = \mathcal{A}$, $\lim_{n \rightarrow \infty} \mathcal{A}'_n = \mathcal{C} - \mathcal{A}$, and $\lim_{n \rightarrow \infty} \mathcal{C}_n = \mathcal{C}$. This implies that

$$\|\mathcal{C}_n - (\mathcal{A}_n + \mathcal{A}'_n)\|_{\sup} \leq \varepsilon,$$

and consequently there is a sequence of channels $\mathcal{D}_n$ in $\llbracket A_G \rightarrow A_G \rrbracket_{L1}$ such that

$$\varepsilon \mathcal{D}_n + \mathcal{C}_n - (\mathcal{A}_n + \mathcal{A}'_n) \succeq 0.$$

By the above equation, it is easy to check that

$$\mathcal{F}_n - \frac{1}{1+\varepsilon} \mathcal{A}_n \succeq 0, \quad \mathcal{F}_n - \frac{1}{1+\varepsilon} \mathcal{A}'_n \succeq 0,$$

where $\mathcal{F}_n$ is the channel $\varepsilon/(1+\varepsilon) \mathcal{D}_n + 1/(1+\varepsilon) \mathcal{C}_n$. Thus, by theorem 5, both $\mathcal{A}_n/(1+\varepsilon)$ and $\mathcal{A}'_n/(1+\varepsilon)$ belong to $\llbracket A_G \rightarrow A_G \rrbracket_L$. Let ε_m be a real Cauchy sequence converging to 0, and consider the corresponding sequences $\mathcal{B}_m := \mathcal{A}_{n_m}/(1+\varepsilon_m)$, $\mathcal{B}'_m := \mathcal{A}'_{n_m}/(1+\varepsilon_m)$. Since

$$\begin{aligned} \|\mathcal{B}_m - \mathcal{A}_{n_m}\|_{\sup} &= \left| \frac{\varepsilon_m}{1+\varepsilon_m} \right| \|\mathcal{A}_{n_m}\|_{\sup}, \\ \|\mathcal{B}'_m - \mathcal{A}'_{n_m}\|_{\sup} &= \left| \frac{\varepsilon_m}{1+\varepsilon_m} \right| \|\mathcal{A}'_{n_m}\|_{\sup}, \end{aligned}$$

one has $\mathcal{A} = [\mathcal{A}_{nR_n}] = [\mathcal{B}_{mR_m}]$ and $\mathcal{C} - \mathcal{A} = [\mathcal{A}'_{nS_n}] = [\mathcal{B}'_{mS_m}]$, with $\mathcal{B}_m$ and $\mathcal{B}'_m$ belonging to $\llbracket A_G \rightarrow A_G \rrbracket_L$. Thus, $\mathcal{A}, \mathcal{C} - \mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$. $\square$

We will now define the action of the quasi-local algebra of transformations on quasi-local states. The action is defined by duality as follows.

Definition 30 (Dual action of the quasi-local algebra). Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$. We define the map $\hat{\mathcal{A}} : \llbracket A_G \rrbracket_{\mathbb{R}} \rightarrow \llbracket A_G \rrbracket_{\mathbb{R}}$ as follows

$$(a|\hat{\mathcal{A}}\rho) := (\mathcal{A}^\dagger a|\rho), \quad \forall a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}. \quad (52)$$

The first thing we need to prove is that the map $\hat{\mathcal{A}} : \llbracket A_G \rrbracket_{\mathbb{R}} \rightarrow \llbracket A_G \rrbracket_{\mathbb{R}}$ is linear and bounded.

Proposition 15. For every $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$ the map $\hat{\mathcal{A}} : \llbracket A_G \rrbracket_{\mathbb{R}} \rightarrow \llbracket A_G \rrbracket_{\mathbb{R}}$ is linear and bounded.

Proof. As to linearity, it is sufficient to consider that

$$\begin{aligned} (a|\hat{\mathcal{A}}[x\rho_1 + y\rho_2]) &= (\mathcal{A}^\dagger a|x\rho_1 + y\rho_2) \\ &= x(\mathcal{A}^\dagger a|\rho_1) + y(\mathcal{A}^\dagger a|\rho_2) \\ &= (a|x\hat{\mathcal{A}}\rho_1) + (a|y\hat{\mathcal{A}}\rho_2), \end{aligned}$$

which by definition implies

$$\hat{\mathcal{A}}(x\rho_1 + y\rho_2) = x\hat{\mathcal{A}}\rho_1 + y\hat{\mathcal{A}}\rho_2.$$

In order to prove boundedness, it is sufficient to consider Eq. (49), to conclude that

$$\begin{aligned} |(a|\hat{\mathcal{A}}\rho)| &= |(\mathcal{A}^\dagger a|\rho)| \leq \|\mathcal{A}^\dagger a\|_{\sup} \|\rho\|_* \\ &\leq \|a\|_{\sup} \|\mathcal{A}\|_{\sup} \|\rho\|_*, \end{aligned}$$

which implies that

$$\|\hat{\mathcal{A}}\rho\|_* \leq \|\mathcal{A}\|_{\sup} \|\rho\|_*. \quad \square$$

We now prove that $\llbracket A_G \rrbracket_{Q\mathbb{R}}$ is invariant under the quasi-local algebra, and more precisely, every space $\llbracket A_G \rrbracket_{\rho_0 Q\mathbb{R}}^{(B)}$ is invariant.

Theorem 8. Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$. Then

$$\hat{\mathcal{A}}\llbracket A_G \rrbracket_{Q\mathbb{R}} \subseteq \llbracket A_G \rrbracket_{Q\mathbb{R}},$$

and for every (B, ρ_0) ,

$$\hat{\mathcal{A}}\llbracket A_G \rrbracket_{\rho_0 Q\mathbb{R}}^{(B)} \subseteq \llbracket A_G \rrbracket_{\rho_0 Q\mathbb{R}}^{(B)}.$$

Proof. We start observing that, by Eq. (40), when we apply $\mathcal{A}_R^\dagger a_S$ to a local state τ_T in $\llbracket A_G \rrbracket_{\rho_0 Q\mathbb{R}}^{(B)}$, we obtain

for every S and every $a_S \in \llbracket \bar{A}_S \rrbracket_{\mathbb{R}}$, which implies

$$\begin{aligned} &\hat{\mathcal{A}}_{RTT} \begin{array}{c} A_{B(R \setminus T)} \\ A_{R \cap T} \\ A_{S \setminus R} \end{array} \\ &= \begin{array}{c} \rho_0 B(R \setminus T) \\ \tau_T \end{array} \begin{array}{c} A_{B(R \setminus T) \setminus (R \setminus T)} \\ A_{R \setminus T} \\ A_{R \cap T} \\ A_{T \setminus R} \end{array} \begin{array}{c} A_{R \setminus T} \\ A_{R \cap T} \end{array}. \end{aligned} \quad (53)$$

This means that local states in $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ are mapped to local states in the same sector under $\hat{\mathcal{A}}_R$. Now, by proposition 15, Cauchy sequences are mapped to Cauchy sequences, so that $\hat{\mathcal{A}}_R \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)} \subseteq \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. Moreover, again by proposition 15, also Cauchy sequences of local transformations satisfy the same condition, thus $\hat{\mathcal{A}} \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)} \subseteq \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. Similarly, one can easily prove that $\hat{\mathcal{A}} \llbracket A_G \rrbracket_{Q \mathbb{R}} \subseteq \llbracket A_G \rrbracket_{Q \mathbb{R}}$. $\square$

As a consequence of the above theorem, the action of $\llbracket A \rightarrow A \rrbracket_Q$ on the space $\llbracket A_G \rrbracket_{Q \mathbb{R}}$ of quasi-local states is decomposed into many irreducible representations, one for every space $\llbracket A_G \rrbracket_{\rho_0 Q}^{(B)}$.

Remark 3. Notice that, given $\mathcal{A}, \mathcal{B} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q \mathbb{R}}$, one has

$$(\mathcal{A} \mathcal{B})^\dagger = \mathcal{B}^\dagger \mathcal{A}^\dagger, \quad \widehat{(\mathcal{A} \mathcal{B})} = \hat{\mathcal{A}} \hat{\mathcal{B}}. \quad (54)$$

Let $R \in \mathcal{R}^{(G)}$, and let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$. Suppose that $\mathcal{A} = [\mathcal{A}_{mR_m}]$ and for every $m \in \mathbb{N}$, $R \cap R_m = \emptyset$. This implies that, for $S_m = R_m \cup R$, $\mathcal{A} = [\mathcal{A}'_{mS_m}]$, with $\mathcal{A}'_{mS_m} = (\mathcal{A}_m \otimes \mathcal{I}_R)_{S_m}$, for $\mathcal{A}'_m \in \llbracket A_{S_m \setminus R} \rrbracket_{\mathbb{R}}$. We then write $\mathcal{A} = \mathcal{B} \otimes \mathcal{I}_R$. Notice that in this case, given $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$, one has

$$\begin{aligned} \|\rho|_R - (\hat{\mathcal{A}}\rho)|_R\|_{\text{op}} &= \|(\hat{\mathcal{A}}_m \rho)|_R - (\hat{\mathcal{A}}\rho)|_R\|_{\text{op}} \\ &\leq \|(\hat{\mathcal{A}}_m - \hat{\mathcal{A}})\rho\|_* \leq \varepsilon \|\rho\|_*, \end{aligned}$$

and then $(\hat{\mathcal{A}}\rho)|_R = \rho|_R$.

For $R \in \mathcal{R}^{(G)}$, there is a straightforward ordered Banach algebra isomorphism between $\llbracket A_R \rightarrow A_R \rrbracket_{Q \mathbb{R}}$ and $\llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}}$, where the latter is defined as

$$\llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}} := \{\mathcal{A}_R \mid \mathcal{A} \in \llbracket A_R \rightarrow A_R \rrbracket_{\mathbb{R}}\}.$$

A similar isomorphism can be found also in the case of infinite regions $R \in \bar{\mathcal{R}}^{(G)}$, where by definition $\llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}}$ is the closed subalgebra of Cauchy classes $[\mathcal{A}_{nR_n}]$ with $R_n \subseteq R$ for every $n \in \mathbb{N}$. The isomorphism in this case is given by

$$\llbracket A_R \rightarrow A_R \rrbracket_{Q \mathbb{R}}^\dagger = \mathcal{J}_R^\dagger \llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}}^\dagger \mathcal{J}_R^{-1\dagger}, \quad (55)$$

where $\mathcal{J}_R^\dagger : \llbracket \bar{A}_R^{(G)} \rrbracket_{Q \mathbb{R}} \rightarrow \llbracket \bar{A}_R \rrbracket_{Q \mathbb{R}}$ is defined in Eq. (25). In the following, when we want to specify that a given quasi-local transformation $\mathcal{A}$ is in the subalgebra $\llbracket A_R \rightarrow A_R \rrbracket_{Q \mathbb{R}}$, we will write $\mathcal{A}_R$.

Remark 4. Notice that every quasi-local transformation in $\llbracket A_R \rightarrow A_R \rrbracket_{Q \mathbb{R}}$ does not only represent a linear map on $\llbracket \bar{A}_R \rrbracket_{Q \mathbb{R}}$, but a family of maps that represent the same transformation acting on $\llbracket \bar{A}_G \rrbracket$ for every $R \subseteq G$. This is due to the isomorphism of $\llbracket A_R \rightarrow A_R \rrbracket_{Q \mathbb{R}}$ with the subalgebra $\llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}}$ given in Eq. (55).

The following result shows that $\llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}}$ preserves the subspace $\llbracket \bar{A}_R^{(G)} \rrbracket_{Q \mathbb{R}}$.

Lemma 33. Let $\mathcal{A} \in \llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q \mathbb{R}}$. Then

$$\mathcal{A}^\dagger \llbracket \bar{A}_R^{(G)} \rrbracket_{Q \mathbb{R}} \subseteq \llbracket \bar{A}_R^{(G)} \rrbracket_{Q \mathbb{R}}.$$

Proof. Let $\mathcal{A}_{nR_n}$ be a sequence in the class of $\mathcal{A}$, with $R_n \subseteq R$ for every n . Let a_{mS_m} be a sequence in the class of $a \in \llbracket \bar{A}_R^{(G)} \rrbracket_{Q \mathbb{R}}$, with $S_m \subseteq R$ for all m . Then $\mathcal{A}^\dagger a = [(\mathcal{A}_n^\dagger a_n)_{R_n \cup S_n}]$, and for every n , $R_n \cup S_n \subseteq R$. This implies that $\mathcal{A}^\dagger a \in \llbracket \bar{A}_R^{(G)} \rrbracket_{Q \mathbb{R}}$. $\square$

The following intuitive result states that an effect that is localised in the region $R \subseteq G$ is obtained by a quasi-local transformation that is localised in the same region. This has a straightforward proof that will be omitted.

Lemma 34. Let $a \in \llbracket \bar{A}_R^{(G)} \rrbracket_{Q*}$, with $*$ $\in$ $\{\text{"nothing"}, 1, +, \mathbb{R}\}$. Then

$$a = \mathcal{A}^\dagger e_G,$$

with $\mathcal{A} \in \llbracket A_R^{(G)} \rightarrow A_R^{(G)} \rrbracket_{Q*}$.

We conclude the section with a result that confirms, under restrictive hypotheses, an obvious expectation.

Lemma 35. Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q \mathbb{R}}$. Then $\|\mathcal{A}\|_{\text{sup}} = \inf J(\mathcal{A})$, where

$$\begin{aligned} J(\mathcal{A}) &:= \{\lambda \in \mathbb{R} \mid \exists \mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}, \lambda \mathcal{C} \pm \mathcal{A} \succeq 0\}. \end{aligned}$$

Proof. Let $\lambda \in J(\mathcal{A})$. Then there exists a quasi-local channel $\mathcal{C}$ such that $\lambda \mathcal{C} \pm \mathcal{A} \succeq 0$. By definition, this implies that $\lambda \mathcal{C} \pm \mathcal{A} = [\mathcal{B}_{nR_n^\pm}^\pm]$ with $\mathcal{B}^\pm \succeq 0$. Defining $\mathcal{A}_n := 1/2(\mathcal{B}_n^+ - \mathcal{B}_n^-)$ and $\mathcal{C}_n := 1/(2\lambda)(\mathcal{B}_n^+ + \mathcal{B}_n^-)$, by construction one has

$$[\mathcal{C}_{nS_n^+}] = \mathcal{C}, \quad [\mathcal{A}_{nS_n^-}] = \mathcal{A},$$

where the domains $S_n^\pm$ are suitably defined. On the other hand, $\mathcal{C} = [\mathcal{D}_{nT_n}]$ for $\mathcal{D}_{nT_n} \in \llbracket A_{T_n} \rightarrow A_{T_n} \rrbracket_1$. Thus, one has $\|\mathcal{C}_n - \mathcal{D}_n\|_{\text{sup}} \leq \varepsilon$, taking suitable care in defining the domain $U_n := S_n^+ \cup T_n$ of $\mathcal{C}_n - \mathcal{D}_n$. In other terms, one can find a channel $\mathcal{F}_n$ such that

$$\begin{aligned}\varepsilon \mathcal{F}_n + \mathcal{D}_n - \mathcal{C}_n &\succeq 0, \\ \varepsilon \mathcal{F}_n - \mathcal{D}_n + \mathcal{C}_n &\succeq 0.\end{aligned}$$

In turn, this implies that

$$\begin{aligned}(1 + \varepsilon)e_{U_n} - c_n &\succeq 0, \\ (\varepsilon - 1)e_{U_n} + c_n &\succeq 0,\end{aligned}$$

where $c_n := \mathcal{C}_n^\dagger e_{U_n}$. Considering the first of the above relations, we can define

$$\mathcal{G}_n := \frac{1}{1 + \varepsilon} \mathcal{C}_n + |\rho_n\rangle\langle d_n|,$$

where $d_n := e_{U_n} - 1/(1 + \varepsilon)c_n \succeq 0$ and ρ_n is an arbitrary state in $\llbracket A_{U_n} \rrbracket_1$. By our assumptions, since $\mathcal{G}_n \succeq 0$ and $\mathcal{G}_n^\dagger e_{U_n} = e_{U_n}$, $\mathcal{G}_n$ is a channel. Moreover, by construction we have that

$$\lambda(1 + \varepsilon)\mathcal{G}_n \pm \mathcal{A}_n = \mathcal{B}_n^\pm + \lambda(1 + \varepsilon)|\rho_n\rangle\langle d_n| \succeq 0,$$

thus $\lambda(1 + \varepsilon) \in J(\mathcal{A}_n)$, i.e. $\lambda(1 + \varepsilon) \geq \|\mathcal{A}_n\|_{\text{sup}}$. In the limit, we then have $\lambda \geq \|\mathcal{A}\|_{\text{sup}}$. On the other hand, if $\lambda \notin J(\mathcal{A})$, for every $\mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$ one has

$$\lambda \mathcal{C} + \mathcal{A} \not\succeq 0 \vee \lambda \mathcal{C} - \mathcal{A} \not\succeq 0. \quad (56)$$

Since the sequence $\lambda \mathcal{C} \pm \mathcal{A}_n$ converges to $\lambda \mathcal{C} \pm \mathcal{A}$ for every sequence $\mathcal{A}_n$ converging to $\mathcal{A}$, and the positive cone is closed, for every $\mathcal{C}$ and every $\mathcal{A}_n$ as above, there must be n_0 such that for $n \geq n_0$

$$\lambda \mathcal{C} + \mathcal{A}_n \not\succeq 0 \vee \lambda \mathcal{C} - \mathcal{A}_n \not\succeq 0. \quad (57)$$

This holds, in particular, for every local $\mathcal{C}_n$ with the same domain as $\mathcal{A}_n$, which implies $\lambda \notin J(\mathcal{A}_n)$. As a consequence, $\lambda < \|\mathcal{A}_n\|_{\text{sup}}$, and finally $\lambda \leq \|\mathcal{A}\|_{\text{sup}}$. It is now easy to conclude that $\inf J(\mathcal{A}) = \|\mathcal{A}\|_{\text{sup}}$. $\square$

We now prove that the infimum defining the sup norm is actually a minimum, analogously to the case of effects. The proof in this case is less straightforward, and we first need the following lemma.

Lemma 36. *Given $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$, let $S_{\mathcal{A}}^\lambda \subseteq \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$ be the set of channels $\mathcal{C}$ such that $\lambda \mathcal{C} \pm \mathcal{A} \succeq 0$. Then $S_{\mathcal{A}}^\lambda$ is closed in the sup-norm.*

Proof. The empty set is closed, so we will consider the case where $S_{\mathcal{A}}^\lambda$ is not empty. Let $\{\mathcal{C}_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $S_{\mathcal{A}}^\lambda$. Then $\mathcal{D}_n^\pm := \lambda \mathcal{C}_n \pm \mathcal{A} \succeq 0$ are both Cauchy sequences, that by definition converge to $\lambda \mathcal{C} \pm \mathcal{A} \succeq 0$. By proposition 12, $\mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$. $\square$

Proposition 16. *Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$. Then there exists a quasi local channel $\mathcal{C}$ such that*

$$\|\mathcal{A}\|_{\text{sup}} \mathcal{C} \pm \mathcal{A} \succeq 0. \quad (58)$$

Proof. By lemma 35, for every $\varepsilon > 0$ there exists $\mathcal{C}_\varepsilon$ such that $(\|\mathcal{A}\|_{\text{sup}} + \varepsilon)\mathcal{C}_\varepsilon \pm \mathcal{A} \succeq 0$. Let us then consider the non empty sets $S_\varepsilon := S_{\mathcal{A}}^{\|\mathcal{A}\|_{\text{sup}} + \varepsilon}$, that are closed by lemma 36. It is straightforward to verify that for $\delta < \varepsilon$ it is $S_\delta \subseteq S_\varepsilon$. Let d_ε be the diameter of S_ε , namely $d_\varepsilon := \sup\{\|\mathcal{C}_1 - \mathcal{C}_2\|_{\text{sup}} \mid \mathcal{C}_1, \mathcal{C}_2 \in S_\varepsilon\}$. Notice that $0 \leq d_\varepsilon \leq 2$. Being d_ε a non-decreasing function of ε , $d_0 = \lim_{\varepsilon \rightarrow 0^+} d_\varepsilon$ exists. The limit d_0 is the diameter of

$$S_0 := \bigcap_{\varepsilon > 0} S_\varepsilon.$$

If $d_0 > 0$, then S_0 is clearly not empty, and each of its elements is a channel $\mathcal{C}$ such that $\|\mathcal{A}\|_{\text{sup}} \mathcal{C} \pm \mathcal{A} \succeq 0$. If $d_0 = 0$, then by Cantor's intersection theorem for complete metric spaces the set S_0 is a singleton $S_0 = \{\mathcal{C}_\infty\}$, with $\|\mathcal{A}\|_{\text{sup}} \mathcal{C}_\infty \pm \mathcal{A} \succeq 0$. $\square$

We finally prove a crucial result, that is the converse of lemma 32. This shows that every quasi-local event is an element of a quasi-local test.

Proposition 17. *Let $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$. Then $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$ if and only if there exists $\mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$ such that $\mathcal{C} - \mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q+}$.*

Proof. Sufficiency is proved in lemma 32. Let us then prove that if $\mathcal{A} \in \llbracket A_G \rightarrow A_G \rrbracket_Q$ then there exists $\mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q1}$ such that $\mathcal{C} - \mathcal{A} \succeq 0$. First of all, notice that $\mathcal{A} = [\mathcal{A}_{nR_n}]$ with $\mathcal{A}_n \in \llbracket A_{R_n} \rightarrow A_{R_n} \rrbracket$, and then by

$$\begin{aligned}\|\mathcal{A}\|_{\text{sup}} &= \lim_{n \rightarrow \infty} \|\mathcal{A}_n\|_{\text{sup}} \\ &= \lim_{n \rightarrow \infty} \|\mathcal{A}_{nR_n}^\dagger e_G\|_{\text{sup}} \leq 1.\end{aligned}$$

By proposition 16, there exists $\mathcal{C} \in \llbracket A_G \rightarrow A_G \rrbracket_{Q_1}$ such that

$$\mathcal{C} - \mathcal{A} \succeq \|\mathcal{A}\|_{\sup} \mathcal{C} - \mathcal{A} \succeq 0. \quad \square$$

5 Global update rules

We have now set the theoretical background needed for the definition of a cellular automaton. In the present section we give the definition of an update rule (UR), that along with the system A_G —the array of cells, whose definition has been analyzed in detail in the previous sections—will provide the backbone of the notion of a cellular automaton. We then introduce the notion of a global update rule (GUR), as a family of update rules satisfying suitable admissibility conditions in order to represent a local action when extended to composite systems $A_G C$. A GUR thus describes the evolution occurring when we apply the global update rule on a joint state of A_G and C , leaving C unaffected. We then analyse the main features of GURs, and in particular we focus on the *causal influence* relation given by a global update rule, and the *block decomposition* that allows one to calculate the action of a GUR on local effects using only local transformations.

Remark 5. In the remainder we will assume that for every $g \in G$ the system A_g is not trivial, namely $A_g \not\equiv I$.

First of all let $\mathcal{V}^\dagger$ be a bounded automorphism of the space $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ of quasi-local effects. Then, with a notation that is reminiscent of the one adopted for quasi-local transformations, we will denote its action as

$$\mathcal{V}^\dagger : \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}} \rightarrow \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}} :: a \mapsto \mathcal{V}^\dagger a.$$

The action on effects allows us to define the *dual map* that acts on the space of extended states, as follows

$$(a|\hat{\mathcal{V}}\rho) := (\mathcal{V}^\dagger a|\rho), \quad \forall a \in \llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}. \quad (59)$$

One can easily verify that $\hat{\mathcal{V}}$ is linear and bounded, thanks to boundedness of the map $\mathcal{V}^\dagger$. Unlike the usual notion of a map in quantum theory or theories with local discriminability, however, the action of $\hat{\mathcal{V}}$ or $\mathcal{V}^\dagger$ is not sufficient to identify uniquely the corresponding transformation. In particular, those actions are not sufficient to determine that of $\mathcal{V} \otimes \mathcal{I}_C$, when system A_G is considered as a part of the composite

system $A_G C$. For this reason we need to define an automorphic transformation of $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ with care. One further delicate issue is the following. When one conjugates a quasi-local transformation $\mathcal{A}$ with an automorphism $\mathcal{V}^\dagger$, obtaining $\mathcal{A}' = \mathcal{V}^{-1\dagger} \mathcal{A} \mathcal{V}^\dagger$, the latter is a linear bounded map on $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$, but there is no guarantee that it is still an element of the quasi-local algebra $\llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$, which is surely a desideratum for candidate cellular automata. In the remainder, when we write $A_G C$ we mean to deal with a system $A_{G'}$ with $G' = G \cup h_0$ and $A_{h_0} \cong C$. In view of the above observations, we introduce the following notion

Definition 31 (Automorphic family). *An automorphic family of maps on $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ is a collection of automorphisms $\mathcal{V}_C^\dagger$ of $\llbracket \bar{A}_G C \rrbracket_{Q\mathbb{R}}$, one for every $C \in \text{Sys}(\Theta)$, with the properties i) for every $\mathcal{A} \in \llbracket A_C \rightarrow A_C \rrbracket_{\mathbb{R}}$ one has $\mathcal{V}_C^{-1\dagger} \mathcal{A}_{h_0}^\dagger \mathcal{V}_C^\dagger = \mathcal{A}_{h_0}^\dagger$; ii) for fixed C_0 and every choice of system C_1 , setting $C = C_0 C_1$ and $A_{G'} := A_G C_0$, and for every $\mathcal{A} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q\mathbb{R}}$, there exist $\mathcal{A}', \mathcal{A}'' \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q\mathbb{R}}$ such that*

$$\mathcal{V}_C^{-1\dagger} \mathcal{A}_{G'}^\dagger \mathcal{V}_C^\dagger = \mathcal{A}_{G'}^\dagger, \quad (60)$$

$$\mathcal{A}_{G'}^\dagger = \mathcal{V}_C^\dagger \mathcal{A}_{G'}''^\dagger \mathcal{V}_C^{-1\dagger}, \quad (61)$$

where $\mathcal{A}^\dagger = (\mathcal{V}_{C_0}^{-1\dagger} \mathcal{A}^\dagger \mathcal{V}_{C_0}^\dagger)$, and $\mathcal{A}''^\dagger = (\mathcal{V}_{C_0}^{-1\dagger} \mathcal{A}''^\dagger \mathcal{V}_{C_0}^\dagger)$

We remark that, as a special case of item ii) in definition 31, taking $C_0 \equiv I$ and $C = C_1$, for every choice of system C , and every $\mathcal{A} \in \llbracket A_G \rrbracket_{Q\mathbb{R}}$, one has

$$\mathcal{V}_C^{-1\dagger} \mathcal{A}_G^\dagger \mathcal{V}_C^\dagger = (\mathcal{V}_I^{-1\dagger} \mathcal{A}_G^\dagger \mathcal{V}_I^\dagger)_G,$$

$$\mathcal{V}_C^\dagger \mathcal{A}_G^\dagger \mathcal{V}_C^{-1\dagger} = (\mathcal{V}_I^\dagger \mathcal{A}_G^\dagger \mathcal{V}_I^{-1\dagger})_G.$$

When referring to an automorphic family we will use the shorthand $\mathcal{V}^\dagger$. One has to remind, however, that this symbol refers to a family of automorphisms. In particular, the definition is meant to allow for formulas such as $\mathcal{V} \mathcal{A} \mathcal{V}^{-1}$, $\mathcal{V} \llbracket A_G \rightarrow A_G \rrbracket_{\$} \mathcal{V}^{-1}$, for $\$ = Q, L$ and $* = \mathbb{R}, +, 1$ or nothing. The meaning of the two expressions above is given by the following definitions:

$$\mathcal{V} \mathcal{A} \mathcal{V}^{-1} := \{ \hat{\mathcal{V}}_{C_0 C} (\hat{\mathcal{A}} \otimes \hat{\mathcal{I}}_C) \hat{\mathcal{V}}_{C_0 C}^{-1} \mid C \in \text{Sys}(\Theta) \}, \quad (62)$$

$$\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{\$} \mathcal{V}^{-1} := \{ \mathcal{V} \mathcal{A} \mathcal{V}^{-1} \mid \mathcal{A} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{\$} \}. \quad (63)$$

Notice that by definition of an automorphic family and by Eq. (62), one has

$$\mathcal{V}\mathcal{A}\mathcal{V}^{-1} = \{(\hat{\mathcal{V}}_{C_0}\mathcal{A}\hat{\mathcal{V}}_{C_0}^{-1}) \otimes \hat{\mathcal{I}}_C \mid C \in \text{Sys}(\Theta)\}, \quad (64)$$

namely an automorphic family maps $[[A_{G'} \rightarrow A_{G'}]]_{Q\mathbb{R}}$ onto itself surjectively. In particular, this is true of $[[A_G \rightarrow A_G]]_{Q\mathbb{R}}$. This implies that for $\mathcal{A} \in [[A_G C \rightarrow A_G C]]_{Q\mathbb{R}}$, $\mathcal{V}\mathcal{A}\mathcal{V}^{-1}$ represents a quasi-local transformation in $[[A_G C \rightarrow A_G C]]_{Q\mathbb{R}}$.

An automorphic family is defined as to satisfy some necessary conditions that we require for the representatives of a transformation of the form $\mathcal{V} \otimes \mathcal{I}_C$. However, the definition of an automorphic family is not sufficient to ensure consistency. This fact must be taken into account in the next subsection. Sufficient conditions will be only provided through the notion of admissibility.

5.1 Update rule

We can now use the notion of automorphic family to introduce update rules. An update rule is defined as a family of automorphisms, one for each extended system $A_G C$, with the constraint that the rule must act reversibly on the set of states and, by conjugation, on the cone of positive local transformations.

Definition 32 (Update rule). *An update rule (UR) is a triple $(G, A, \mathcal{V}^\dagger)$, where $\mathcal{V}^\dagger$ is an automorphic family of isometric maps on $[[\bar{A}_G]]_{Q\mathbb{R}}$ such that*

1. *the maps $\hat{\mathcal{V}}_C$ leave $[[A_G C]]$ invariant, i.e.*

$$\hat{\mathcal{V}}_C[[A_G C]] = [[A_G C]]; \quad (65)$$

2. *the map $\mathcal{V} \cdot \mathcal{V}^{-1}$ leaves the cones $[[A_G C \rightarrow A_G C]]_{L+}$ invariant, i.e. for every $\mathcal{A} \in [[A_G C \rightarrow A_G C]]_{L+}$ there are $\mathcal{A}', \mathcal{A}'' \in [[A_G C \rightarrow A_G C]]_{L+}$ such that*

$$\mathcal{V}\mathcal{A}\mathcal{V}^{-1} = \mathcal{A}', \quad (66)$$

$$\mathcal{A} = \mathcal{V}\mathcal{A}''\mathcal{V}^{-1}. \quad (67)$$

The first result is that the inverse of a UR is a UR.

Lemma 37. *Let $(G, A, \mathcal{V}^\dagger)$ be a UR. Then $(G, A, \mathcal{V}^{-1\dagger})$ is a UR.*

Proof. In the first place, the family $\mathcal{V}_C^{-1}$ is an automorphic family: items i) and ii) in definition 31 are trivially proved. Now, if $\|\mathcal{V}_C^\dagger a\|_{\text{sup}} = \|a\|_{\text{sup}}$ for every $a \in [[\bar{A}_G \bar{C}]]_{Q\mathbb{R}}$, then $\|\mathcal{V}_C^{-1\dagger} a\|_{\text{sup}} = \|\mathcal{V}_C^\dagger \mathcal{V}_C^{-1\dagger} a\|_{\text{sup}} = \|a\|_{\text{sup}}$, which proves isometricity of $\mathcal{V}_C^{-1\dagger}$. From the definition in Eq. (59) one can easily verify that $(\hat{\mathcal{V}}_C^{-1}) = (\hat{\mathcal{V}}_C)^{-1}$. Now, it is a straightforward observation that by Eq. (65), $\hat{\mathcal{V}}_C^{-1}[[A_G C]] = [[A_G C]]$. Finally, multiplying both sides of Eq. (66) by $\mathcal{V}^{-1}$ to the left and by $\mathcal{V}$ to the right, for every $\mathcal{A} \in [[A_G C \rightarrow A_G C]]_{L+}$ one has

$$\mathcal{A} = \mathcal{V}^{-1}\mathcal{A}'\mathcal{V},$$

which expresses the same condition as in Eq. (67) for $\mathcal{V}^{-1}$, and similarly, multiplying both sides of Eq. (67) by $\mathcal{V}^{-1}$ to the left and by $\mathcal{V}$ to the right, one obtains the same condition as in Eq. (66) for $\mathcal{V}^{-1}$. $\square$

In the following we will often use $\mathcal{V}^\dagger$ to denote a UR, when G and A_G are clear from the context. The symbols $\hat{\mathcal{V}}$ and $\mathcal{V}^\dagger$ will be often used instead of $\hat{\mathcal{V}}_C$ and $\mathcal{V}_C^\dagger$, respectively. Accordingly, the symbols $a, b, \dots$ and $\mathcal{A}, \mathcal{B}, \dots$ will denote $a \otimes e_C, b \otimes e_C, \dots$ and $\mathcal{A} \otimes \mathcal{I}_C, \mathcal{B} \otimes \mathcal{I}_C, \dots$, respectively. Finally, we will sometimes write G' for $G \cup h_0$ with $A_{h_0} \cong C$, so that $A_{G'} = A_G C$.

The next result states that an update rule leaves the cone $[[A_G^{(G \cup h_0)} \rightarrow A_G^{(G \cup h_0)}]]_{L+}$ invariant.

Lemma 38. *Let $(G, A, \mathcal{V}^\dagger)$ be a UR, and define $A_{G'} := A_G C$ for arbitrary $C \in \text{Sys}(\Theta)$. Then the map $\mathcal{V} \cdot \mathcal{V}^{-1}$ leaves the cone $[[A_G^{(G')} \rightarrow A_G^{(G')}]]_{L+}$ invariant, namely for $\mathcal{A} \in [[A_G^{(G')} \rightarrow A_G^{(G')}]]_{L+}$ Eqs. (66) and (67) hold with $\mathcal{A}', \mathcal{A}'' \in [[A_G^{(G')} \rightarrow A_G^{(G')}]]_{L+}$.*

Proof. Let $\mathcal{A} \in [[A_G^{(G')} \rightarrow A_G^{(G')}]]_{L+} \subseteq [[A_{G'} \rightarrow A_{G'}]]_{L+}$. By definition of UR

$$\mathcal{V}\mathcal{A}\mathcal{V}^{-1} = \mathcal{A}' \in [[A_{G'} \rightarrow A_{G'}]]_{L+}.$$

Moreover, by definition of an automorphic family of maps and Eq. (64), we have

$$\mathcal{A}' \in [[A_G^{(G')} \rightarrow A_G^{(G')}]]_{L\mathbb{R}}.$$

The same holds exchanging $\mathcal{V}^{-1}$ and $\mathcal{V}$. $\square$

As a consequence of isometricity on $[[\bar{A}_G \bar{C}]]_{Q\mathbb{R}}$, a UR $\mathcal{V}^\dagger$ has a dual $\hat{\mathcal{V}}$ that acts isometrically on $[[A_G C]]_{\mathbb{R}}$.

Lemma 39. Let $\mathcal{V}^\dagger$ be a UR. The maps $\hat{\mathcal{V}}_C$ on $[\![A_{G'}]\!]_{\mathbb{R}}$ are isometric.

Proof. Let $\rho \in [\![A_G C]\!]_{\mathbb{R}}$, and remind that

$$\|\rho\|_* = \sup_{\|a\|_{\text{sup}}=1} |(a|\rho)|.$$

Then we have

$$\begin{aligned} \|\hat{\mathcal{V}}_C \rho\|_* &= \sup_{\|a\|_{\text{sup}}=1} |(a|\hat{\mathcal{V}}_C \rho)| = \sup_{\|a\|_{\text{sup}}=1} |(\mathcal{V}_C^\dagger a|\rho)| \\ &= \sup_{\|a'\|_{\text{sup}}=1} |(a'|\rho)| = \|\rho\|_*, \end{aligned}$$

thanks to isometricity and invertibility of $\mathcal{V}_C^\dagger$. $\square$

Corollary 13. Let $\mathcal{V}^\dagger$ be a UR. Then $\hat{\mathcal{V}}_C[\![A_G C]\!]_1 = [\![A_G C]\!]_1$

Proof. By eqs. (65) and (29) and isometricity of $\hat{\mathcal{V}}_C$, one has $\hat{\mathcal{V}}_C[\![A_G C]\!]_1 \subseteq [\![A_G C]\!]_1$. The same is true for $\hat{\mathcal{V}}_C^{-1}$, thus $\hat{\mathcal{V}}_C[\![A_G C]\!]_1 = [\![A_G C]\!]_1$. $\square$

The next results will allow us to conclude that a UR preserves $[\![\bar{A}_G]\!]_Q$. We start proving that an update rule $\mathcal{V}^\dagger$ preserves the unique deterministic effects $e_{G'}$.

Lemma 40. If $\mathcal{V}^\dagger$ is a UR, then $\mathcal{V}^\dagger e_{G'} = e_{G'}$, for $A_{G'} := A_G C$ and arbitrary $C \in \text{Sys}(\Theta)$.

Proof. By definition of a UR, for every $\rho \in [\![A_{G'}]\!]$ one has $\hat{\mathcal{V}}\rho \in [\![A_{G'}]\!]$, and lemma 19, gives for every $\rho \in [\![A_{G'}]\!]$

$$(\mathcal{V}^\dagger e_{G'}|\rho) = (e_{G'}|\hat{\mathcal{V}}\rho) = \|\hat{\mathcal{V}}\rho\|_* = \|\rho\|_* = (e_{G'}|\rho).$$

Thanks to theorem 6, we then have $\mathcal{V}^\dagger e_{G'} = e_{G'}$. $\square$

Corollary 14. Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V}^\dagger a_{h_0} = a_{h_0}$ for all $a_{h_0} \in [\![A_{h_0}^{(G')} \rightarrow A_{h_0}^{(G')}]_{Q\mathbb{R}}$.

Proof. By proposition 14, $a_{h_0} = \mathcal{A}_{h_0}^\dagger e_{G'}$. Thus, by definition 31 and lemmas 40 and 56, we have

$$\begin{aligned} \mathcal{V}^\dagger a_{h_0} &= \mathcal{V}^\dagger \mathcal{A}_{h_0}^\dagger \mathcal{V}^{-1\dagger} e_{G'} \\ &= \mathcal{A}_{h_0}^\dagger e_{G'} = a_{h_0}. \end{aligned} \quad \square$$

We now prove that the cones of effects $[\![\bar{A}_{G'}]\!]_{L+}$, $[\![\bar{A}_G]\!]_{L+}$, $[\![\bar{A}_{G'}]\!]_{Q+}$ and $[\![\bar{A}_G]\!]_{Q+}$ are invariant under the action of any UR.

Lemma 41. Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_{L+} = [\![\bar{A}_{G'}]\!]_{L+}$, and $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_{Q+} = [\![\bar{A}_{G'}]\!]_{Q+}$. In particular, for $C = I$ it is $\mathcal{V}^\dagger[\![\bar{A}_G]\!]_{Q+} = [\![\bar{A}_G]\!]_{Q+}$

Proof. Let $a \in [\![\bar{A}_{G'}]\!]_{L+}$. By proposition 14, there exists $\mathcal{A} \in [\![A_{G'} \rightarrow A_{G'}]\!]_{L+}$ such that $a = \mathcal{A}^\dagger e_{G'}$. Then

$$\begin{aligned} \mathcal{V}^\dagger a &= \mathcal{V}^\dagger \mathcal{A}^\dagger e_{G'} \\ &= \mathcal{V}^\dagger \mathcal{A}^\dagger \mathcal{V}^{-1\dagger} \mathcal{V}^\dagger e_{G'} \\ &= \mathcal{V}^\dagger \mathcal{A}^\dagger \mathcal{V}^{-1\dagger} e_{G'} \\ &= \mathcal{A}^\dagger e_{G'} = a'. \end{aligned}$$

We observe that, by definition of UR and by lemma 38, $\mathcal{V} \cdot \mathcal{V}^{-1}$ preserves the cones $[\![A_{G'} \rightarrow A_{G'}]\!]_{L+}$. Thus, $a' \in [\![\bar{A}_{G'}]\!]_{L+}$, and $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_{L+} \subseteq [\![\bar{A}_{G'}]\!]_{L+}$. Since also $\mathcal{V}^{-1\dagger}$ is a UR, we have $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_{L+} = [\![\bar{A}_{G'}]\!]_{L+}$. Now, isometricity grants that Cauchy sequences of local effects in $[\![\bar{A}_{G'}]\!]_{L+}$ are mapped to Cauchy sequences in $[\![\bar{A}_{G'}]\!]_{L+}$, and that $\lim_{n \rightarrow \infty} \mathcal{V}^\dagger a_n = \mathcal{V}^\dagger \lim_{n \rightarrow \infty} a_n$. Finally, by proposition 12, $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_{Q+} \subseteq [\![\bar{A}_{G'}]\!]_{Q+}$. Since also $\mathcal{V}^{-1\dagger}$ is a UR, we conclude that $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_{Q+} = [\![\bar{A}_{G'}]\!]_{Q+}$. $\square$

Proposition 18. Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_Q = [\![\bar{A}_{G'}]\!]_Q$. In particular, for $C = I$, $\mathcal{V}^\dagger[\![\bar{A}_G]\!]_Q = [\![\bar{A}_G]\!]_Q$

Proof. Let $a \in [\![\bar{A}_{G'}]\!]_Q$. Then $e_{G'} - a \in [\![\bar{A}_{G'}]\!]_{Q+}$, by lemma 12, and thus, by virtue of lemmas 40 and 41, $\mathcal{V}^\dagger a, e_{G'} - \mathcal{V}^\dagger a \in [\![\bar{A}_{G'}]\!]_{Q+}$. Again by lemma 12, this implies that $\mathcal{V}^\dagger a \in [\![\bar{A}_{G'}]\!]_Q$, and thus $\mathcal{V}^\dagger[\![\bar{A}_{G'}]\!]_Q \subseteq [\![\bar{A}_{G'}]\!]_Q$. Finally, since also $\mathcal{V}^{-1\dagger}$ is a UR, the thesis follows. $\square$

Let now $\mathcal{V}^\dagger$ be an update rule. By our definition, the map $\mathcal{V} \cdot \mathcal{V}^{-1}$ preserves the cones $[\![A_{G'} \rightarrow A_{G'}]\!]_{L+}$. We will now prove that $\mathcal{V} \cdot \mathcal{V}^{-1}$ also preserves the cones $[\![A_{G'} \rightarrow A_{G'}]\!]_{Q+}$, as well as the sets of transformations $[\![A_{G'} \rightarrow A_{G'}]\!]_L$ and $[\![A_{G'} \rightarrow A_{G'}]\!]_Q$.

We start with the following lemma

Lemma 42. Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V} \cdot \mathcal{V}^{-1}$ leaves the sets $[\![A_{G'} \rightarrow A_{G'}]\!]_{L1}$ invariant, namely for $\mathcal{A} \in [\![A_{G'} \rightarrow A_{G'}]\!]_{L1}$ Eqs. (66) and (67) hold, with $\mathcal{A}', \mathcal{A}'' \in [\![A_{G'} \rightarrow A_{G'}]\!]_{L1}$. In particular, for $C = I$ the thesis holds with $G' = G$.

Proof. Let $\mathcal{C} \in [\![A_{G'} \rightarrow A_{G'}]\!]_{L1}$. By definition and by lemma 38, $\mathcal{V}\mathcal{C}\mathcal{V}^{-1} = \mathcal{C}'$, and $\mathcal{C} = \mathcal{V}\mathcal{C}''\mathcal{V}^{-1}$, for $\mathcal{C}', \mathcal{C}'' \in [\![A_{G'} \rightarrow A_{G'}]\!]_{L+}$. By lemmas 37 and 40,

$$\begin{aligned} \mathcal{C}'^\dagger e_{G'} &= \mathcal{V}^\dagger \mathcal{C}^\dagger \mathcal{V}^{-1\dagger} e_{G'} = e_{G'}, \\ \mathcal{C}''^\dagger e_{G'} &= \mathcal{V}^{-1\dagger} \mathcal{C}^\dagger \mathcal{V}^\dagger e_{G'} = e_{G'}, \end{aligned}$$

and due to lemma 30, $\mathcal{C}', \mathcal{C}'' \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L1}$. $\square$

Given a UR $\mathcal{V}^\dagger$, the above result allows us to prove that $\mathcal{V} \cdot \mathcal{V}^{-1}$ maps local transformations to local transformations, and is isometric on $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$.

Lemma 43. *Let $\mathcal{V}^\dagger$ be a UR, and $\mathcal{A} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Then $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}} \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$, and*

$$\|\mathcal{V} \mathcal{A} \mathcal{V}^{-1}\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}. \quad (68)$$

Proof. By proposition 16, there exists $\mathcal{C} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L1}$ such that

$$\|\mathcal{A}\|_{\text{sup}} \mathcal{C} \pm \mathcal{A} \succeq 0.$$

By definition of a UR, we then have

$$\|\mathcal{A}\|_{\text{sup}} \mathcal{V} \mathcal{C} \mathcal{V}^{-1} \pm \mathcal{V} \mathcal{A} \mathcal{V}^{-1} \succeq 0, \quad (69)$$

and being $\mathcal{V} \mathcal{C} \mathcal{V}^{-1} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L1} \subseteq \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$, it follows that $\mathcal{V} \mathcal{A} \mathcal{V}^{-1} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Consequently, we have $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}} \mathcal{V}^{-1} \subseteq \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Finally, since $\mathcal{V}^{-1\dagger}$ is a UR, we have that $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}} \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Moreover, by lemma 42, Eq. (69) implies that $\|\mathcal{A}\|_{\text{sup}} \in J(\mathcal{V} \mathcal{A} \mathcal{V}^{-1})$. Thus $\|\mathcal{V} \mathcal{A} \mathcal{V}^{-1}\|_{\text{sup}} \leq \|\mathcal{A}\|_{\text{sup}}$. Now, being $\mathcal{V}^{-1\dagger}$ a UR, the equality $\|\mathcal{V} \mathcal{A} \mathcal{V}^{-1}\|_{\text{sup}} = \|\mathcal{A}\|_{\text{sup}}$ follows. $\square$

Lemma 44. *Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q\mathbb{R}} \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q\mathbb{R}}$, and*

$$\lim_{n \rightarrow \infty} \mathcal{V} \mathcal{A}_n \mathcal{V}^{-1} = \mathcal{V} \left(\lim_{n \rightarrow \infty} \mathcal{A}_n \right) \mathcal{V}^{-1}. \quad (70)$$

Proof. Since $\mathcal{V} \cdot \mathcal{V}^{-1}$ is isometric and surjective on the dense submanifolds $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$, by lemma 43 it maps Cauchy sequences to Cauchy sequences, and their equivalence classes to equivalence classes, thus it maps $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q\mathbb{R}}$ to itself. Surjectivity follows from surjectivity on $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Eq. (70) can be independently checked applying both sides to arbitrary $a \in \llbracket A_{G'} \rrbracket_{Q\mathbb{R}}$. $\square$

Lemma 45. *Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q1} \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q1}$.*

Proof. Let $\{\mathcal{C}_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L1}$. Then by definition of UR and lemma 44 we have that $\{\mathcal{V} \mathcal{C}_n \mathcal{V}^{-1}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L1}$, and its limit $\mathcal{V} \mathcal{C} \mathcal{V}^{-1}$ is in $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q1}$. $\square$

The last results finally allow us to prove that for a UR $\mathcal{V}^\dagger$, the map $\mathcal{V} \cdot \mathcal{V}^{-1}$ preserves the cones $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q+}$.

Lemma 46. *Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q+} \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q+}$.*

Proof. By lemmas 38 and 44, along with the definition of a UR, $\mathcal{V} \cdot \mathcal{V}^{-1}$ maps Cauchy sequences in $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L+}$ to Cauchy sequences in $\llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L+}$, thus $\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q+} \mathcal{V}^{-1} \subseteq \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q+}$. Since $\mathcal{V}^{-1\dagger}$ is a UR, the thesis follows. $\square$

Next, we show that for a UR $\mathcal{V}^\dagger$, $\mathcal{V}^\dagger \llbracket \bar{A}_{G'} \rrbracket_{L\mathbb{R}} = \llbracket \bar{A}_{G'} \rrbracket_{L\mathbb{R}}$.

Lemma 47. *Let $\mathcal{V}^\dagger$ be a UR. Then $\mathcal{V}^\dagger \llbracket \bar{A}_{G'} \rrbracket_{L\mathbb{R}} = \llbracket \bar{A}_{G'} \rrbracket_{L\mathbb{R}}$.*

Proof. Let $a_R \in \llbracket \bar{A}_{G'} \rrbracket_{L\mathbb{R}}$. Then by proposition 14

$$a_R = \mathcal{A}_R^\dagger e_{G'},$$

for some $\mathcal{A}_R \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Now, we have

$$\mathcal{V}^\dagger a_R = \mathcal{V}^\dagger \mathcal{A}_R^\dagger e_{G'} = \mathcal{V}^\dagger \mathcal{A}_R^\dagger \mathcal{V}^{-1\dagger} e_{G'},$$

and by lemma 43 $\mathcal{V}^\dagger \mathcal{A}_R^\dagger \mathcal{V}^{-1\dagger} = \mathcal{A}_{R'}^\dagger$ with $\mathcal{A}_{R'} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{L\mathbb{R}}$. Then $\mathcal{V}^\dagger a_R = a_{R'}$, with $R' \in R^{(G')}$. As to surjectivity, it can be straightforwardly proved exploiting lemma 37. $\square$

Finally, for a UR $\mathcal{V}^\dagger$, the map $\mathcal{V} \cdot \mathcal{V}^{-1}$ preserves the set of quasi-local transformations.

Lemma 48. *Let $\mathcal{V}^\dagger$ be a UR. Then one has*

$$\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_Q \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_Q, \quad (71)$$

$$\mathcal{V} \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_L \mathcal{V}^{-1} = \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_L. \quad (72)$$

Proof. Let $\mathcal{A} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_Q$. Then $\|\mathcal{A}\|_{\text{sup}} \leq 1$, and by proposition 17, there is $\mathcal{C} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q1}$ such that

$$\mathcal{C} - \mathcal{A} \succeq 0.$$

By lemma 46, $\mathcal{V} \mathcal{A} \mathcal{V}^{-1} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q+}$. Moreover, by lemma 42, $\mathcal{V} \mathcal{C} \mathcal{V}^{-1} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_{Q1}$. By the above observations, we have

$$\mathcal{V} \mathcal{A} \mathcal{V}^{-1} \succeq 0,$$

$$\mathcal{V} \mathcal{C} \mathcal{V}^{-1} - \mathcal{V} \mathcal{A} \mathcal{V}^{-1} \succeq 0.$$

Finally, by lemma 32, one has

$$\mathcal{V}\mathcal{A}\mathcal{V}^{-1}, \mathcal{V}\mathcal{C}\mathcal{V}^{-1} - \mathcal{V}\mathcal{A}\mathcal{V}^{-1} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_Q.$$

The same argument holds for $\mathcal{A} \in \llbracket A_{G'} \rightarrow A_{G'} \rrbracket_L$. Since $\mathcal{V}^{-1\dagger}$ is a UR, surjectivity on both sets is proved. $\square$

Clearly, given a UR $\mathcal{V}^\dagger$, the image of the algebra of local transformations of A_g under $\mathcal{V} \cdot \mathcal{V}^{-1}$, i.e.

$$\mathcal{V}\llbracket A_g^{(G')} \rightarrow A_g^{(G')} \rrbracket_{Q\mathbb{R}}\mathcal{V}^{-1} \subseteq \llbracket A_G^{(G')} \rightarrow A_G^{(G')} \rrbracket_{Q\mathbb{R}},$$

is a subalgebra of $\llbracket A_G^{(G')} \rightarrow A_G^{(G')} \rrbracket_{Q\mathbb{R}}$ isomorphic to $\llbracket A_g^{(G')} \rightarrow A_g^{(G')} \rrbracket_{Q\mathbb{R}}$.

5.2 Admissibility and local action

As we have seen, independently of the nature of system C , the map $\mathcal{V} \cdot \mathcal{V}^{-1}$ does more than mapping linear maps to linear maps: it sends transformations to transformations. This is an important result, as the definitions of automorphic family and UR are meant to embody necessary conditions for a family of maps to represent $\mathcal{V} \otimes \mathcal{I}_C$. However, URs are not yet consistent with such a factorisation. The reason for this is that local discriminability in our theory is not assumed. Thus, while knowledge of the linear maps $\mathcal{V}_C^\dagger$ is sufficient to determine the action of the UR on factorised effects $a_G \otimes b_C$ and, by conjugation, on generalised transformations $\mathcal{A}_G \otimes \mathcal{I}_C$, and such actions are consistent with a factorised action of the form $\mathcal{V} \otimes \mathcal{I}_C$, still we do not have enough information to determine the action of $\mathcal{V}_C^\dagger$ on those elements of $\llbracket \bar{A}_G \bar{C} \rrbracket_{Q\mathbb{R}}$ that lie outside the subspace $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}} \otimes \llbracket \bar{C} \rrbracket_{Q\mathbb{R}}$ (and analogously for the conjugate action on transformations). For this purpose we introduce in this section the notion of admissibility.

We remind that G' stands for $G \cup h_0$, with $A_{h_0} \cong C$ for some system C . Let $R \in \mathcal{R}^{(G')}$ be a finite region. Since $\llbracket \bar{A}_R^{(G')} \rrbracket_{Q\mathbb{R}}$ is isomorphic to $\llbracket \bar{A}_R \rrbracket_{\mathbb{R}}$, whose size is $D_R := D_{A_R}$, we can find a basis $\{b_{jR}\}_{j=1}^{D_R} \subseteq \llbracket \bar{A}_R^{(G')} \rrbracket_{Q\mathbb{R}}$. Thus, by lemma 47, $\mathcal{V}^\dagger \llbracket \bar{A}_R^{(G')} \rrbracket_{Q\mathbb{R}} \subseteq \llbracket \bar{A}_{R'}^{(G')} \rrbracket_{Q\mathbb{R}}$ for some finite region $R' \in \mathcal{R}^{(G')}$, that can be obtained as

$$R' := \bigcup_{j=1}^{D_R} R'_j, \quad \mathcal{V}^\dagger b_{jR} \in \llbracket \bar{A}_{R'_j}^{(G')} \rrbracket_{Q\mathbb{R}}.$$

This implies that, for every finite region $R \in \mathcal{R}^{(G')}$, the UR $\mathcal{V}^\dagger$ induces a linear map $\mathcal{V}(R)^\dagger : \llbracket \bar{A}_R \rrbracket_{\mathbb{R}} \rightarrow \llbracket \bar{A}_{R'} \rrbracket_{\mathbb{R}}$. The following definition of admissible UR is now given in terms of the properties of the induced maps $\mathcal{V}(R)^\dagger$.

Definition 33. Let $(G, A, \mathcal{V}^\dagger)$ be a UR, and let $\mathcal{V}(R)^\dagger$ be the induced maps on finite regions $R \subseteq G'$, for arbitrary choice of $A_{h_0} \cong C$. We call the UR admissible if for every finite $S \in \mathcal{R}^{(G)}$ the family of maps $\mathcal{V}(S \cup h_0)^\dagger$ represents a transformation $\mathcal{V}_S \in \llbracket A_{S'} \rightarrow A_S \rrbracket_{\mathbb{R}}$, i.e.

$$\mathcal{V}(S \cup h_0)^\dagger = \mathcal{V}_S^\dagger \otimes \mathcal{I}_C^\dagger. \quad (73)$$

Consider now a partition $G = R \cup \bar{R}$, where $\bar{R} := G \setminus R$. We will write $A_G = A_R \otimes A_{\bar{R}}$. By definition, the space $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}}$ of quasi-local effects contains the spaces $\llbracket \bar{A}_R^{(G)} \rrbracket_{Q\mathbb{R}}$ and $\llbracket \bar{A}_{\bar{R}}^{(G)} \rrbracket_{Q\mathbb{R}}$ as closed subspaces. However, if the theory does not satisfy local discriminability, it is not true in general that $\llbracket \bar{A}_G \rrbracket_{Q\mathbb{R}} = \llbracket \bar{A}_R^{(G)} \rrbracket_{Q\mathbb{R}} \otimes \llbracket \bar{A}_{\bar{R}}^{(G)} \rrbracket_{Q\mathbb{R}}$. This makes it difficult to define a UR of the form $\mathcal{V} \otimes \mathcal{W}$.

In the remainder of the section we close this gap. For this purpose, we start defining what it means, for a UR $\mathcal{V}^\dagger$ on A_G , to act locally on R —in symbols $\mathcal{V}^\dagger = \mathcal{V}^\dagger_R \otimes \mathcal{I}_{\bar{R}}^\dagger$. For our purpose, we then need to provide a suitable definition of the identity $\mathcal{V}^\dagger = \mathcal{V}^\dagger_R \otimes \mathcal{I}_{\bar{R}}^\dagger$, which satisfies the necessary (but not sufficient) condition

$$(\mathcal{V}^\dagger_R \otimes \mathcal{I}_{\bar{R}}^\dagger)(a_S \otimes b_T) = (\mathcal{V}^\dagger_R a_S) \otimes (b_T),$$

for every $S, T \in \mathcal{R}^{(G)}$ such that $S \subseteq R$ and $T \subseteq \bar{R}$.

Definition 34. Let $\mathcal{V}^\dagger$ be a UR for A_G . We say that $\mathcal{V}^\dagger$ acts locally on the region $R \subseteq G$ if for every C and every finite region $S \in \mathcal{R}^{(G')}$ with $G' = G \cup h_0$ and $A_{h_0} \cong C$, the induced map $\mathcal{V}(S)^\dagger$ has the form

$$\mathcal{V}^\dagger(S) = \mathcal{W}_{S \cap R}^\dagger \otimes \mathcal{I}_{S \setminus R}^\dagger. \quad (74)$$

As a consequence of the above definition we now prove the following result, for which we do not provide the proof, that is straightforward.

Lemma 49. Let $(G, A, \mathcal{V}^\dagger)$ be a UR acting locally on R . Then

$$\mathcal{V}^\dagger(a_S \otimes b_T) = (\mathcal{W}_S^\dagger a_S) \otimes b_T, \quad (75)$$

for $S \subseteq R$ and $T \subseteq \bar{R}$. In particular, $\mathcal{V}^\dagger \llbracket \bar{A}_R^{(G')} \rrbracket_{Q\mathbb{R}} = \llbracket \bar{A}_R^{(G')} \rrbracket_{Q\mathbb{R}}$, and $\mathcal{V}^\dagger b = b$ for every $b \in \llbracket \bar{A}_{\bar{R}}^{(G')} \rrbracket_{Q\mathbb{R}}$.

Lemma 50. Let $(G, A, \mathcal{V}^\dagger)$ be a UR acting locally on R . Then there exists a UR $(R, A, \mathcal{V}'^\dagger)$ such that $\mathcal{V}^\dagger \mathcal{J}_{R'}^{-1\dagger} a = \mathcal{J}_{R'}^{-1\dagger} \mathcal{V}'^\dagger a$ and $\mathcal{V}^{-1\dagger} \mathcal{J}_{R'}^{-1\dagger} a = \mathcal{J}_{R'}^{-1\dagger} \mathcal{V}'^{-1\dagger} a$ for every $a \in [\bar{A}_{R'}]_{Q\mathbb{R}}$, with $A_{R'} := A_R C$.

Proof. Linearity and isometricity of the restriction $\mathcal{V}_{R'}^\dagger$ of $\mathcal{V}^\dagger$ to $[\bar{A}_{R'}^{(G')}]_{Q\mathbb{R}}$ are straightforward by lemma 49. Thus, also $\mathcal{V}'^\dagger := \mathcal{J}_{R'}^\dagger \mathcal{V}_{R'}^\dagger \mathcal{J}_{R'}^{-1\dagger}$ is linear and isometric. Let now $\mathcal{A} \in [A_{R'} \rightarrow A_{R'}]_Q$. Then one has

$$\begin{aligned} \mathcal{V}'^\dagger \mathcal{A}^\dagger \mathcal{V}'^{-1\dagger} &= \mathcal{J}_{R'}^\dagger \mathcal{V}_{R'}^\dagger \mathcal{J}_{R'}^{-1\dagger} \mathcal{A}^\dagger \mathcal{J}_{R'}^\dagger \mathcal{V}_{R'}^{-1\dagger} \mathcal{J}_{R'}^{-1\dagger} \\ &= \mathcal{J}_{R'}^\dagger \mathcal{V}_{R'}^\dagger \mathcal{A}^\dagger \mathcal{V}_{R'}^{-1\dagger} \mathcal{J}_{R'}^{-1\dagger} \\ &= \mathcal{J}_{R'}^\dagger (\mathcal{V}^\dagger \mathcal{A}^\dagger \mathcal{V}^{-1\dagger})_{R'} \mathcal{J}_{R'}^{-1\dagger}, \end{aligned}$$

where $\mathcal{A}_{R'}^\dagger := \mathcal{J}_{R'}^{-1\dagger} \mathcal{A}^\dagger \mathcal{J}_{R'}^\dagger$. In the above chain of equalities we used the fact that, since $\mathcal{V}^\dagger$, $\mathcal{V}^{-1\dagger}$ and $\mathcal{A}_{R'}^\dagger$ preserve the space $[\bar{A}_{R'}^{(G')}]_{Q\mathbb{R}}$, then also $\mathcal{V}^\dagger \mathcal{A}_{R'}^\dagger \mathcal{V}^{-1\dagger}$ does, and one can thus define its restriction $(\mathcal{V}^\dagger \mathcal{A}_{R'}^\dagger \mathcal{V}^{-1\dagger})_{R'}$. Finally, since $\mathcal{V}^\dagger$ is a UR, one has $\mathcal{V}^\dagger \mathcal{A}_{R'}^\dagger \mathcal{V}^{-1\dagger} = \mathcal{A}'^\dagger$ with $\mathcal{A}' \in [A_{R'}^{(G')} \rightarrow A_{R'}^{(G')}]_Q$. Thus, $\mathcal{V}'^{-1\dagger} \mathcal{A} \mathcal{V}'^\dagger \in [A_{R'} \rightarrow A_{R'}]_Q$. A similar argument leads to the existence of $\mathcal{A}'' \in [A_{R'} \rightarrow A_{R'}]_Q$ such that $\mathcal{A} = \mathcal{V}'^{-1\dagger} \mathcal{A}'' \mathcal{V}'^\dagger$. Now, let $\rho \in [A_G]$. Then ρ defines a state $\rho_{|R'}$ on $[\bar{A}_{R'}]_{Q\mathbb{R}}$, by lemma 17, with

$$(a|\rho_{|R'}) = (a_{R'}|\rho) = (\mathcal{J}_{R'}^{-1\dagger} a|\rho),$$

for every $a_{R'} \in [\bar{A}_{R'}^{(G')}]_{Q\mathbb{R}}$. Now, we have

$$\begin{aligned} (a|[\hat{\mathcal{V}}\rho]_{|R'}) &= (\mathcal{J}_{R'}^{-1\dagger} a|\hat{\mathcal{V}}\rho) = (\mathcal{V}^\dagger \mathcal{J}_{R'}^{-1\dagger} a|\rho) \\ &= (\mathcal{V}_{R'}^\dagger \mathcal{J}_{R'}^{-1\dagger} a|\rho) = (\mathcal{J}_{R'}^{-1\dagger} \mathcal{J}_{R'}^\dagger \mathcal{V}_{R'}^\dagger \mathcal{J}_{R'}^{-1\dagger} a|\rho) \\ &= (\mathcal{J}_{R'}^{-1\dagger} \mathcal{V}'^\dagger a|\rho) = (a|\hat{\mathcal{V}}'\rho_{|R'}). \end{aligned}$$

We then showed that $[\hat{\mathcal{V}}\rho]_{|R'} = \hat{\mathcal{V}}'(\rho_{|R'})$. Moreover, since $\mathcal{V}$ preserves the set of states, $[\hat{\mathcal{V}}\rho]_{|R'}$ is a state. Thus, $\hat{\mathcal{V}}'[A_{R'}] \subseteq [A_{R'}]$. The same argument for $\mathcal{V}'^{-1}$ brings to the conclusion that $\hat{\mathcal{V}}'[A_{R'}] = [A_{R'}]$. Thus, $\mathcal{V}'^\dagger$ is a UR for $A_{R'}$. Moreover

$$\mathcal{V}^\dagger \mathcal{J}_{R'}^{-1\dagger} a = \mathcal{V}_{R'}^\dagger \mathcal{J}_{R'}^{-1\dagger} a = \mathcal{J}_{R'}^{-1\dagger} \mathcal{V}'^\dagger a,$$

and the same holds for $\mathcal{V}^{-1}$. $\square$

In the following, under the hypothesis of lemma 50, we will write $\mathcal{V}^\dagger = \mathcal{V}'^\dagger \otimes \mathcal{J}_{R'}^\dagger$.

Lemma 51. Let $\mathcal{V}^\dagger$ be a UR for $[\bar{A}_G]_{\mathbb{R}}$ that acts locally on R , and $\mathcal{W}^\dagger$ a UR that acts locally on $\bar{R}$. Then $\mathcal{V}'^\dagger \otimes \mathcal{W}'^\dagger := \mathcal{V}^\dagger \mathcal{W}^\dagger = \mathcal{W}^\dagger \mathcal{V}^\dagger$.

We can now define a global update rule as follows.

Definition 35 (Global update rule). Let $(G, A, \mathcal{V}^\dagger)$ be a UR. We call $(G, A, \mathcal{V}^\dagger)$ a Global Update Rule (GUR), or global rule for short, if $(G, A, \mathcal{V}^\dagger)$ and $(G, A, \mathcal{V}^{-1\dagger})$ are admissible.

Remark 6. The set of GURs $(G, A, \mathcal{V}^\dagger)$ is a group. The proof of closure under composition and associativity are tedious but straightforward, and we omit them here. Moreover, given $(G, A, \mathcal{V}^\dagger)$ and $(F, A, \mathcal{W}^\dagger)$ one can define their parallel composition as $(G \cup F, A, \mathcal{V}^\dagger \otimes \mathcal{W}^\dagger)$. The analysis of properties of parallel composition is beyond the scope of the present work, and will be the subject of further studies.

We now provide an important example of GUR. Let $G = H \times \{0, 1\}$, and let $H_i := (H, i)$ for $i = 0, 1$. Let $a_R \in [\bar{A}_R^{(G')}]_{Q\mathbb{R}}$ with $A_{h_0} \cong C$. Let $R \cap G = (R_0, 0) \cup (R_1, 1)$. Then we define $\tilde{R} := R_0 \cup R_1 \subseteq H$, and $\bar{R} := \tilde{R} \times \{0, 1\} \cup R$. Clearly, $R \subseteq \bar{R}$. If $(h, i) \in \bar{R}$, then also $(h, i \oplus 1) \in \bar{R}$. Let then $\bar{a}_{\bar{R}} := a_R \otimes e_{\bar{R} \setminus R}$.

Now, for every $T \in \bar{R}^{(H)}$ let us define the GUR $(G, A, \mathcal{S}_T^\dagger)$ by setting

$$\mathcal{S}_T^\dagger[a_R] := \left[\left(\mathcal{S}_{\bar{R} \cap T}^\dagger \bar{a}_{\bar{R}} \right)_{\bar{R}} \right], \quad (76)$$

where $\mathcal{S}_X^\dagger := \bigotimes_{h \in X} \mathcal{S}_h^\dagger$ for $X \in \mathcal{R}^{(H)}$, and $\mathcal{S}_h$ is the map that swaps $A_{(h,0)}$ and $A_{(h,1)}$. It is straightforward to verify that, for every $T \in \bar{R}^{(H)}$, $\mathcal{S}_T$ is a UR. Being defined in terms of induced maps, it is immediate to verify that $\mathcal{S}_T^\dagger$ is admissible. Moreover, $\mathcal{S}_T$ is involutive, i.e. $\mathcal{S}_T^{-1} = \mathcal{S}_T$. Of particular interest for the following is the GUR $(G, A, \mathcal{S}_H^\dagger)$.

Notice that, for a finite disjoint partition $H = \cup_{i=1}^k S_k$, we have $\mathcal{S}_H^\dagger = \bigotimes_{i=1}^k \mathcal{S}_{S_k}^\dagger$. Thus, by lemma 51, for every GUR $(G, A, \mathcal{V}^\dagger)$, and for every i, j , we have

$$\begin{aligned} &\mathcal{V}(\mathcal{S}_{S_i \cup S_j} \otimes \mathcal{S}_{H \setminus (S_i \cup S_j)}) \mathcal{V}^{-1} \\ &= \mathcal{V}(\mathcal{S}_{S_i} \otimes \mathcal{S}_{H \setminus S_i}) \mathcal{V}^{-1} \mathcal{V}(\mathcal{S}_{S_j} \otimes \mathcal{S}_{H \setminus S_j}) \mathcal{V}^{-1} \\ &= \mathcal{V}(\mathcal{S}_{S_j} \otimes \mathcal{S}_{H \setminus S_j}) \mathcal{V}^{-1} \mathcal{V}(\mathcal{S}_{S_i} \otimes \mathcal{S}_{H \setminus S_i}) \mathcal{V}^{-1}, \end{aligned}$$

where $\mathcal{S}_S$ denotes $\mathcal{S}_{(S,0)} \otimes \mathcal{S}_{(S,1)}$.

For $\mathcal{A} \in \llbracket A_R \rightarrow A_R \rrbracket$, it is easily verified that

$$\begin{aligned}\mathcal{A}_{(R,1)} &= \mathcal{S}_H(\mathcal{A}_{(R,0)})\mathcal{S}_H = \mathcal{S}_R(\mathcal{A} \otimes \mathcal{I}_R)\mathcal{S}_R, \\ \mathcal{A}_{(R,0)} &= \mathcal{S}_H(\mathcal{A}_{(R,1)})\mathcal{S}_H = \mathcal{S}_R(\mathcal{I}_R \otimes \mathcal{A})\mathcal{S}_R.\end{aligned}\quad (77)$$

Finally, let us consider a GUR $(G, A, \mathcal{U}^\dagger \otimes \mathcal{V}^\dagger)$. One can prove that $\mathcal{S}_H(\mathcal{U} \otimes \mathcal{V})\mathcal{S}_H = \mathcal{V} \otimes \mathcal{U}$, by the following argument. Let us consider $a \in \llbracket A_R^{(G')} \rrbracket_{Q\mathbb{R}}$. Then

$$\begin{aligned}\mathcal{S}_H^\dagger(\mathcal{U} \otimes \mathcal{V})^\dagger \mathcal{S}_H^\dagger a_R &= \mathcal{S}_H^\dagger(\mathcal{U} \otimes \mathcal{V})^\dagger (\mathcal{S}_R^\dagger a)_{\bar{R}} \\ &= \mathcal{S}_H^\dagger[(\mathcal{U}(\tilde{R}, 0)^\dagger \otimes \mathcal{V}(\tilde{R}, 1)^\dagger)(\mathcal{S}_R^\dagger a)]_{\bar{R}} \\ &= [\mathcal{S}_R^\dagger(\mathcal{U}(\tilde{R}, 0)^\dagger \otimes \mathcal{V}(\tilde{R}, 1)^\dagger)(\mathcal{S}_R^\dagger a)]_{\bar{R}} \\ &= [\mathcal{V}(\tilde{R}, 0)^\dagger \otimes \mathcal{U}(\tilde{R}, 1)^\dagger a]_{\bar{R}} \\ &= (\mathcal{V} \otimes \mathcal{U})^\dagger a_R.\end{aligned}$$

5.3 Causal influence

In the present subsection we define the notion of causal influence that establishes, given a GUR, whether the evolution allows interventions on one system to affect other systems after one step, or more generally how many steps are needed for such an influence to occur. We will often use the notation $\llbracket A_R^{(G)}C \rightarrow A_R^{(G)}C \rrbracket_{**}$ to denote $\llbracket A_R^{(G')} \rightarrow A_R^{(G')} \rrbracket_{**}$, with $G' = G \cup h_0$ and $A_{h_0} \cong C$. In the following we also adopt the convention that for $R \in \bar{R}^{(G)}$, $\bar{R}$ denotes the complementary region $\bar{R} := G \setminus R$. Notice that, consistently with the notation $R = g$, we will write $\bar{g}$ to denote $G \setminus g$.

Definition 36 (Causal influence). *We say that, according to the global rule $\mathcal{V}$, system g does not causally influence g' if for every external system C*

$$\mathcal{V} \llbracket A_g^{(G)}C \rightarrow A_g^{(G)}C \rrbracket_{Q\mathbb{R}} \mathcal{V}^{-1} \subseteq \llbracket A_{g'}^{(G)}C \rightarrow A_{g'}^{(G)}C \rrbracket_{Q\mathbb{R}}.$$

On the other hand, we will say that according to the global rule $\mathcal{V}$ system g causally influences system g' , and write $g \rightarrow g'$, in the opposite case, i.e. when

$$\begin{aligned}\exists C, \mathcal{F} \in \llbracket A_g^{(G)}C \rightarrow A_g^{(G)}C \rrbracket_{Q\mathbb{R}} : \\ (\mathcal{V} \otimes \mathcal{I}_C)\mathcal{F}(\mathcal{V}^{-1} \otimes \mathcal{I}_C) \notin \llbracket A_{g'}^{(G)}C \rightarrow A_{g'}^{(G)}C \rrbracket_{Q\mathbb{R}}.\end{aligned}$$

The above definition gets a clear meaning as a diagram: according to $\mathcal{V}$ there is causal influence

from g to g' if there exists $\mathcal{F} \in \llbracket A_gC \rightarrow A_gC \rrbracket$ such that

$$\begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \begin{array}{c} \boxed{\mathcal{V}^{-1}} \\ \hline \end{array} \begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \begin{array}{c} \boxed{\mathcal{F}} \\ \hline \end{array} \begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \neq \begin{array}{c} C \\ \hline \bar{g}' \\ \hline g' \end{array} \begin{array}{c} \boxed{\mathcal{F}'} \\ \hline \end{array} \begin{array}{c} C \\ \hline \bar{g}' \\ \hline g' \end{array}, \quad (78)$$

and even better in the form

$$\begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \begin{array}{c} \boxed{\mathcal{F}} \\ \hline \end{array} \begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \neq \begin{array}{c} C \\ \hline \bar{g}' \\ \hline g' \end{array} \begin{array}{c} \boxed{\mathcal{V}} \\ \hline \end{array} \begin{array}{c} C \\ \hline \bar{g}' \\ \hline g' \end{array} \begin{array}{c} \boxed{\mathcal{F}'} \\ \hline \end{array} \begin{array}{c} C \\ \hline \bar{g}' \\ \hline g' \end{array}.$$

Definition 37. *We define forward and backward g -neighbourhoods the sets $N_g^+ := \{g' | g \rightarrow g'\}$ and $N_g^- := \{g' | g' \rightarrow g\}$, respectively.*

In terms of diagrams, the forward g -neighbourhood $A_{N_g^+}$ of g can be understood as the smallest region $K = \bigotimes_{k \in K} A_{g'_k}$ such that for every $k \in K$ there exists C and $\mathcal{F} \in \llbracket A_g^{(G)}C \rightarrow A_g^{(G)}C \rrbracket_{Q\mathbb{R}}$ such that

$$\begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \begin{array}{c} \boxed{\mathcal{F}} \\ \hline \end{array} \begin{array}{c} C \\ \hline g \\ \hline \bar{g} \end{array} \quad (79)$$

$$= \begin{array}{c} C \\ \hline N_g^+ \\ \hline \bar{N}_g^+ \end{array} \begin{array}{c} \boxed{\mathcal{V}} \\ \hline \end{array} \begin{array}{c} C \\ \hline N_g^+ \\ \hline \bar{N}_g^+ \end{array} \begin{array}{c} \boxed{\mathcal{F}'} \\ \hline \end{array} \begin{array}{c} C \\ \hline N_g^+ \\ \hline \bar{N}_g^+ \end{array}.$$

with $\mathcal{F}'$ acting non trivially on g'_k , namely $\mathcal{F}' \notin \llbracket A_{g'_k}^{(G)}C \rightarrow A_{g'_k}^{(G)}C \rrbracket_{Q\mathbb{R}}$. Notice that g may or may not belong to N_g^+ .

Analogously to the definition of $N_g^\pm$, one can give a definition of $N_R^\pm$ for every finite region $R \subseteq G$.

Definition 38. *We say that the system g' belongs to the forward neighbourhood of R , and write $g' \in N_R^+$, if*

$$\begin{aligned}\exists C, \mathcal{F} \in \llbracket A_R^{(G)}C \rightarrow A_R^{(G)}C \rrbracket_{Q\mathbb{R}} : \\ (\mathcal{V} \otimes \mathcal{I}_C)\mathcal{F}(\mathcal{V}^{-1} \otimes \mathcal{I}_C) \notin \llbracket A_{g'}^{(G)}C \rightarrow A_{g'}^{(G)}C \rrbracket_{Q\mathbb{R}}.\end{aligned}$$

The backward neighbourhood N_R^- of R is defined as

$$N_R^- := \{g' \in G \mid N_g^+ \cap R \neq \emptyset\}. \quad (80)$$

Clearly, one has

$$N_R^+ \supseteq \bigcup_{g \in R} N_g^+. \quad (81)$$

We will prove that the latter is actually an equality in the next section.

One might define causal influence without invoking any external system C , as follows.

Definition 39 (Individual influence). *We say that the system g does not individually influence g' if*

$$\begin{aligned} \mathcal{V} \llbracket A_g^{(G)} \rightarrow A_{g'}^{(G)} \rrbracket_{Q\mathbb{R}} \mathcal{V}^{-1} \\ \subseteq \llbracket A_{\bar{g}'}^{(G)} \rightarrow A_{\bar{g}}^{(G)} \rrbracket_{Q\mathbb{R}}. \end{aligned}$$

On the other hand, we will say that according to the global rule $\mathcal{V}$ system g individually influences system g' , and write $g \rightarrow_g g'$, in the opposite case, i.e. when

$$\begin{aligned} \exists \mathcal{F} \in \llbracket A_g^{(G)} \rightarrow A_{g'}^{(G)} \rrbracket_{Q\mathbb{R}} : \\ \mathcal{V} \mathcal{F} \mathcal{V}^{-1} \notin \llbracket A_{\bar{g}'}^{(G)} \rightarrow A_{\bar{g}}^{(G)} \rrbracket_{Q\mathbb{R}}. \end{aligned}$$

The individual forward neighbourhood of g is

$$N_g^+(g) := \{g' \in G \mid g \rightarrow_g g'\}.$$

A second definition that can be given without referring to external systems C is the following.

Definition 40 (Cooperative influence). *We say that the region S does not cooperatively influence g' if*

$$\mathcal{V} \llbracket A_S^{(G)} \rightarrow A_{g'}^{(G)} \rrbracket_{Q\mathbb{R}} \mathcal{V}^{-1} \subseteq \llbracket A_{\bar{g}'}^{(G)} \rightarrow A_{\bar{g}}^{(G)} \rrbracket_{Q\mathbb{R}}.$$

In the opposite case we will write $S \rightarrow_S g'$. The cooperative forward neighbourhood of a region S is defined as

$$N_S^+(S) := \{g' \in G \mid S \rightarrow_S g'\}.$$

Notice now that in a theory without local discriminability it may happen that a system g does not individually influence g' but it does cooperatively, namely one can have $g' \in N_S^+(S)$, but $g' \notin N_g^+(g)$ for any $g \in S$. In other words, $N_S^+(S) \supset \bigcup_{h \in S} N_h^+(h)$. We call this phenomenon *non-local activation of causal influence*. If such a phenomenon occurs, however, it can be hard to establish which of the “cooperating” systems $h \in S$ is responsible for causal influence on $g' \in N_S^+(S)$. This is the reason why we prefer

the former definition of causal influence, which provides also this piece of information.

Given a GUR $(G, A, \mathcal{V}^\dagger)$, we can now construct a graph $\Gamma(G, E)$ with vertices $g \in G$ and edges $E := \{(g, g') \in G \times G \mid g \rightarrow g'\}$. We call such a graph the *graph of causal influence* of the GUR $\mathcal{V}^\dagger$. In the next section we will prove that, if we denote by $N_g^{\pm*}$ the neighbourhoods for the GUR $\mathcal{V}^{-1}$, the following relations hold

$$N_f^\pm = N_f^{\mp*}. \quad (82)$$

We also define the *local perspective present* P_R^+ of $R \in \mathbb{R}^{(G)}$ as the set $P_R^+ := N_{N_R^+}^- \supseteq R$, namely the set of systems that can causally influence the one-step future N_R^+ of R . Similarly one can define the *local retrospective present* of R as $P_R^- := N_{N_R^-}^+ \supseteq R$, namely the set of systems that can be influenced by the one-step past N_R^- of R . In general, $P_R^+ \neq P_R^-$. We will also need to refer to the sets $P_R^{\pm*} := N_{N_R^\pm}^{\mp*}$.

5.3.1 Relation with signalling

The term *influence* that we used above is precisely defined in Eq. (78). In the present subsection we will make a clear point that this definition is stronger than the notion of *signalling* that is customary in the literature on quantum information theory [52, 53, 54] or in general operational probabilistic theories [15]. In that context, a channel $\mathcal{C} \in \llbracket AB \rightarrow A'B' \rrbracket$ is called non-signalling (or semicausal) from B to A' if there exists $\mathcal{F} \in \llbracket A \rightarrow A' \rrbracket$ such that

$$\begin{array}{c} \text{A} \\ \text{B} \end{array} \boxed{\mathcal{C}} \begin{array}{c} \text{A}' \\ \text{B}' \end{array} \begin{array}{c} \text{e} \end{array} = \begin{array}{c} \text{A} \\ \text{B} \end{array} \boxed{\mathcal{F}} \begin{array}{c} \text{A}' \\ \text{e} \end{array}. \quad (83)$$

We will now show that, under the hypothesis that $N_g^{\pm*}$ is finite for every $g \in G$, the condition of Eq. 79, in a suitably defined sense, implies that of Eq. (83). Let us consider the state $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$. Then $(a_S | \hat{\mathcal{A}}_T \rho) := (a'_{S \cup T} | \rho) = (\hat{\mathcal{A}}_T^\dagger a_S | \rho)$. In the case where $a_S = e_G$, $(\hat{\mathcal{A}}_T^\dagger e_G | \rho) = (b_T | \rho)$, where $b = \hat{\mathcal{A}}^\dagger e_T$. Let us now consider $(\hat{\mathcal{V}} \rho)_{|R}$, which by definition is the state in $\llbracket A_R \rrbracket_{\mathbb{R}}$ such that, for every $a \in \llbracket \bar{A}_R \rrbracket_{\mathbb{R}}$

$$(a | [\hat{\mathcal{V}} \rho]_{|R}) = (\mathcal{J}_R^\dagger a | \hat{\mathcal{V}} \rho) = (a_R | \hat{\mathcal{V}} \rho).$$

Now, by lemma 31, one has

$$\begin{aligned}
(a_R|\hat{\mathcal{V}}\rho) &= (e_G|\hat{\mathcal{A}}_R\hat{\mathcal{V}}\rho) \\
&= (e_G|\hat{\mathcal{V}}\hat{\mathcal{V}}^{-1}\hat{\mathcal{A}}_R\hat{\mathcal{V}}\rho) \\
&= (e_G|\hat{\mathcal{V}}\hat{\mathcal{A}}'_{N_R^{+*}}\rho) \\
&= (e_G|\hat{\mathcal{A}}'_{N_R^{+*}}\rho) \\
&= (b_{N_R^{+*}}|\rho) = (b|\rho_{|N_R^{+*}}).
\end{aligned}$$

In order to determine $[\hat{\mathcal{V}}\rho]_R$ it is sufficient to know $\rho_{|N_R^{+*}}$. Indeed, the linear map $\mathcal{V}_R : \llbracket A_{N_R^{+*}} \rrbracket_{\mathbb{R}} \rightarrow \llbracket A_R \rrbracket_{\mathbb{R}}$ gives

$$(\hat{\mathcal{V}}\rho)_R = \mathcal{V}_R(\rho_{|N_R^{+*}}). \quad (84)$$

Diagrammatically, the equality can be recast, with a slight abuse of notation, as in Eq. 83, as follows

$$\begin{array}{c} \overline{N_R^{+*}} \\ \overline{N_R^{+*}} \end{array} \boxed{\mathcal{V}} \begin{array}{c} R \\ \bar{R} \end{array} \boxed{e} = \begin{array}{c} \overline{N_R^{+*}} \\ \overline{N_R^{+*}} \end{array} \boxed{\mathcal{V}_R} \begin{array}{c} R \\ \bar{R} \end{array} \boxed{e}.$$

Notice that, by the no-restriction hypothesis, for an admissible $\mathcal{V}$ the map $\mathcal{V}_R$ is linear and admissible, and is thus a transformation in $\llbracket A_{N_R^{+*}}^{(G)} \rightarrow A_R^{(G)} \rrbracket$. Moreover,

$$\mathcal{V}^\dagger a_R = (\mathcal{V}_R^\dagger a)_{N_R^{+*}}. \quad (85)$$

Finally, notice that by definition $\mathcal{V}_R^\dagger e_R = e_{N_R^{+*}}$, namely $\mathcal{V}_R \in \llbracket A_{N_R^{+*}}^{(G)} \rightarrow A_R^{(G)} \rrbracket_1$. The argument can be repeated for the case where N_g^{+*} is not finite, as long as $N_g^{+*} \subset G$.

We remark that in the case of Quantum or Fermionic cellular automata, the condition for no causal influence is equivalent to no-signalling, while in the case of Classical cellular automata no causal influence is strictly stronger than no-signalling. In a different context, a difference between the two notions was pointed out in Ref. [55]

5.4 Block decomposition

Let us now consider the GUR $(G, A, \mathcal{W}^\dagger)$ where we set $G = H \times \{0, 1\}$, and $\mathcal{W} = \mathcal{V}_{H_0} \otimes \mathcal{V}_{H_1}^{-1}$, for an arbitrary GUR $(H, A, \mathcal{V}^\dagger)$, H_i denoting a shorthand for (H, i) . We observe that

$$\mathcal{W} = \mathcal{V}_{H_0} \otimes \mathcal{V}_{H_1}^{-1} = (\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1}) \mathcal{S}_H (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1}) \mathcal{S}_H.$$

Then, by Eq. (76), given $R \subseteq A_G$ and $a \in \llbracket \bar{A}_R^{(G)} \bar{C} \rrbracket_{Q\mathbb{R}}$, with $R \cap G = (R_0, 0) \cup (R_1, 1)$,

$$\begin{aligned}
\mathcal{W}^\dagger a_{RC} &= (\mathcal{W}^\dagger a)_{R'C} \\
&= [\mathcal{S}_H^\dagger (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_H^\dagger (\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1})^\dagger] a_{RC} \\
&= [\mathcal{S}_H^\dagger (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_H^\dagger] a'_{R''C} \\
&= [\mathcal{S}_H^\dagger (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_{N_{R_0}^{+*} \cup R_1}^\dagger] a'_{R''C} \\
&= [\mathcal{S}_H^\dagger (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_{N_{R_0}^{+*} \cup R_1}^\dagger (\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1})^\dagger] a_{RC},
\end{aligned}$$

where $a' := (\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1})^\dagger a$, $R' := (N_{R_0}^{+*}, 0) \cup (N_{R_1}^{+*}, 1)$, and $R'' := (N_{R_0}^{+*}, 0) \cup (R_1, 1)$.

Now, defining for $S \in \bar{R}^{(H)}$

$$\mathcal{S}'_S := \prod_{h \in S} \mathcal{S}'_h, \quad (86)$$

$$\mathcal{S}'_h := (\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1}) \mathcal{S}_h (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1}), \quad (87)$$

thanks to eq. (77) we have

$$\begin{aligned}
&(\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_{N_{R_0}^{+*} \cup R_1}^\dagger (\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1})^\dagger \\
&= \mathcal{S}_{N_{R_0}^{+*} \cup R_1}^{\prime\dagger}.
\end{aligned}$$

We remark that, since $(\mathcal{V}_{H_0} \otimes \mathcal{I}_{H_1}) \cdot (\mathcal{V}_{H_0}^{-1} \otimes \mathcal{I}_{H_1})$ is an automorphism of $\llbracket A_G \rightarrow A_G \rrbracket_{Q\mathbb{R}}$, one has

$$\mathcal{S}'_f \mathcal{S}'_g = \mathcal{S}'_g \mathcal{S}'_f, \quad \forall f, g \in H. \quad (88)$$

Thus, the product $\prod_{h \in N_R^{+*}} \mathcal{S}'_h$ in Eq. (86) is well defined. Notice also that, by definition

$$\mathcal{S}'_h \mathcal{S}'_h = \mathcal{I}_G. \quad (89)$$

Since $\mathcal{S}'_h \in \llbracket A_{(N_h^{+*}, 0) \cup (h, 1)}^{(G)} \rightarrow A_{(N_h^{+*}, 0) \cup (h, 1)}^{(G)} \rrbracket_{Q1}$ (see Fig. 1), one has that $\mathcal{S}'_{N_{R_0}^{+*} \cup R_1} \in \llbracket A_S^{(G)} \rightarrow A_S^{(G)} \rrbracket_{Q1}$, with

$$S := (N_{N_{R_0}^{+*}}^{+*} \cup N_{R_1}^{+*}, 0) \cup (N_{R_0}^{+*} \cup R_1, 1).$$

Then we can write

$$\begin{aligned}
\mathcal{W}^\dagger a_{RC} &= \mathcal{S}_H^\dagger \mathcal{S}_{N_{R_0}^{+*} \cup R_1}^{\prime\dagger} a_{RC} \\
&= \mathcal{S}_{N_{N_{R_0}^{+*}}^{+*} \cup N_{R_1}^{+*} \cup N_{R_0}^{+*} \cup R_1}^\dagger \mathcal{S}_{N_{R_0}^{+*} \cup R_1}^{\prime\dagger} a_{RC}.
\end{aligned}$$

However, due to the structure of $\mathcal{W}^\dagger$, one actually has

$$(\mathcal{S}'_{N_{R_0}^{+*} \cup R_1} \otimes \mathcal{I}_C)^\dagger a_{RC} = a''_{(N_{R_1}^{+*}, 0) \cup (N_{R_0}^{+*}, 1)C},$$

Figure 1: In the top picture we provide a graphical representation of the transformation $\mathcal{S}'_g$, with input wires on the top left and output on the bottom right. In the bottom picture we provide a graphical illustration of the identity $\mathcal{S}'_g \mathcal{S}'_{g'} = \mathcal{S}'_{g'} \mathcal{S}'_g$.

and finally this implies that

$$\mathcal{W}^\dagger a_{RC} = (\mathcal{S}'_{N_{R_0}^{+*} \cup N_{R_1}^{+*}} \mathcal{S}'_{N_{R_0}^{+*} \cup R_1} \otimes \mathcal{I}_C^\dagger) a_{RC}. \quad (90)$$

We now observe that

$$\mathcal{W}^{-1} = \mathcal{V}_{H_0}^{-1} \otimes \mathcal{V}_{H_1} = \mathcal{S}_H \mathcal{W} \mathcal{S}_H,$$

and thus, by Eq. (90), it is

$$\begin{aligned} \mathcal{W}^{-1\dagger} a_{RC} &= (\mathcal{S}'_{N_{R_1}^{+*} \cup R_0} \otimes \mathcal{I}_C)^\dagger a'_{R'C} \\ &= (\mathcal{S}'_{N_{R_1}^{+*} \cup R_0} \mathcal{S}'_{R_0 \cup R_1} \otimes \mathcal{I}_C^\dagger) a_{RC}, \end{aligned} \quad (91)$$

where $a'_{R'C} = [\mathcal{S}^\dagger a]_{(R_1,0) \cup (R_0,1)C}$. Let us then consider $R_1 = \emptyset$. In this case for $[a_{(R,0)C}] \in [\bar{A}_{(R,0)}^{(G)} \bar{C}]$, being $R = (R, 0) \cup (\emptyset, 1)$, it holds that

$$[(\mathcal{V}^\dagger a)_{(N_R^{+*}, 0)C}] = \mathcal{S}_{N_R^{+*}}^\dagger \mathcal{S}_{N_R^{+*}}^\dagger \otimes \mathcal{I}_C^\dagger [a_{(R,0)C}], \quad (92)$$

$$[(\mathcal{V}^{-1\dagger} a)_{(N_R^{+*}, 0)C}] = \mathcal{S}_R'^\dagger \mathcal{S}_R^\dagger \otimes \mathcal{I}_C^\dagger [a_{(R,0)C}]. \quad (93)$$

In view of the above results, we now consider the maps $\mathcal{V} \mathcal{A} \mathcal{V}^{-1}$ for $\mathcal{A} \in [\mathbb{A}_{(R,0)} C \rightarrow \mathbb{A}_{(R,0)} C]_{\mathbb{R}}$, with $R \in \mathbb{R}^{(H)}$. First of all, we observe that

$$(\mathcal{V} \mathcal{A} \mathcal{V}^{-1}) \otimes \mathcal{I}_{H_1} = \mathcal{W}(\mathcal{A} \otimes \mathcal{I}_{H_1}) \mathcal{W}^{-1}$$

is a local transformation in $\in [\mathbb{A}_{(N_R^{+*}, 0)}^{(G)} C \rightarrow \mathbb{A}_{(N_R^{+*}, 0)}^{(G)} C]_{Q\mathbb{R}}$. Let then $a \in [\bar{A}_{\bar{S}}^{(G)} \bar{C}]_{\mathbb{R}}$ for $S \in$

$\mathbb{R}^{(H)}$. We have the following (see Appendix C)

$$\begin{aligned} & [(\mathcal{V} \mathcal{A} \mathcal{V}^{-1}) \otimes \mathcal{I}_{H_1}]^\dagger a_{\bar{S}C} \\ &= [\mathcal{W}(\mathcal{A} \otimes \mathcal{I}_{H_1}) \mathcal{W}^{-1}]^\dagger a_{\bar{S}C} \\ &= (\mathcal{S}_R'^\dagger \mathcal{S}_R^\dagger \otimes \mathcal{I}_C^\dagger)(\mathcal{A} \otimes \mathcal{I}_{H_0})^\dagger (\mathcal{S}_R'^\dagger \mathcal{S}_R^\dagger \otimes \mathcal{I}_C^\dagger) a_{\bar{S}C}. \end{aligned}$$

Finally, since this holds for every a , we have that for $\mathcal{A} \in [\mathbb{A}_{(R,0)} C \rightarrow \mathbb{A}_{(R,0)} C]_{\mathbb{R}}$

$$\begin{aligned} & (\mathcal{V} \mathcal{A} \mathcal{V}^{-1}) \otimes \mathcal{I}_{H_1} \\ &= (\mathcal{S}_R' \otimes \mathcal{I}_C)(\mathcal{I}_{H_0} \otimes \mathcal{A})(\mathcal{S}_R' \otimes \mathcal{I}_C). \end{aligned} \quad (94)$$

The above relation, considering Eq. (86), shows that

$$N_R^+ \subseteq \bigcup_{g \in R} N_g^+.$$

Along with Eq. (81) the latter proves that actually

$$N_R^+ = \bigcup_{g \in R} N_g^+. \quad (95)$$

Analogously, since $N_R^- = \{g' \mid \exists g \in R, g \in N_{g'}^+\}$, it is

$$N_R^- = \{g' \mid \exists g \in R, g' \in N_g^-\} = \bigcup_{g \in R} N_g^-. \quad (96)$$

Let us finally consider the maps $(\mathcal{V}^{-1} \mathcal{B} \mathcal{V})$ for $\mathcal{B} \in [\mathbb{A}_{(R,1)} C \rightarrow \mathbb{A}_{(R,1)} C]_{\mathbb{R}}$, with $R \in \mathbb{R}^{(H)}$. In this case, using Eq. (89) and manipulating Eq. (94) we obtain

$$\begin{aligned} & \mathcal{I}_{H_0} \otimes (\mathcal{V}^{-1} \mathcal{B} \mathcal{V}) \\ &= (\mathcal{S}_{N_R^{+*}}' \otimes \mathcal{I}_C^\dagger)(\mathcal{B} \otimes \mathcal{I}_{H_1})(\mathcal{S}_{N_R^{+*}}' \otimes \mathcal{I}_C). \end{aligned} \quad (97)$$

Since the $\mathcal{S}'_h$'s that do not commute with $\mathcal{B} \otimes \mathcal{I}_{H_1}$ in Eq. (97) correspond to $h \in N_g^-$ for some $g \in R$, namely $h \in N_R^-$, we can conclude that

$$N_R^{+*} \subseteq N_R^-. \quad (98)$$

This observation is sufficient to prove that $N_R^{\pm*} = N_R^\mp$, as anticipated in the previous section.

Lemma 52. *Let $(G, A, \mathcal{V}^\dagger)$ be a GUR. Then*

$$N_R^{\pm*} = N_R^\mp. \quad (99)$$

Proof. We remind that $g \in N_f^\pm$ iff $f \in N_g^\mp$. Now, considering Eq. (98) for $R = g$ we have that

$$g' \in N_g^{-*} \Leftrightarrow g \in N_{g'}^{+*} \Rightarrow g \in N_{g'}^- \Leftrightarrow g' \in N_g^+.$$

This implies that $N_g^{-*} \subseteq N_g^+$. Exchanging $\mathcal{V}$ and $\mathcal{V}^{-1}$, we can conclude that $N_g^{+*} \subseteq N_g^- \subseteq N_g^{-*}$, i.e. $N_g^{+*} = N_g^-$. The thesis follows invoking Eqs. (95) and (96), which then give us

$$N_R^- = N_R^{+*}. \quad \square$$

In the remainder we will then use $N_R^\pm$ and $P_R^\pm$, and abandon the symbols $N_R^{\pm*}$ and $P_R^{\pm*}$.

We observe that the *block decomposition* of $\mathcal{V} \otimes \mathcal{V}^{-1}$ that we achieved through the block transformations $\mathcal{S}'_g$ is a generalisation of a result that was proved for quantum cellular automata in Ref. [32]. Local rules of other kinds are used in the literature, with nice properties that the present block decomposition lacks, such as composability (see e.g. Ref [56]). However, for later purposes the block form of Eq. (86) is particularly suitable in our case, to cope with update rules in theories without local discriminability.

6 Homogeneity

The principle of homogeneity, as it is usually formulated in physics, regards equivalence of points in space-time. Here, however, space-time is not the background for occurrence of physical events, on the contrary it emerges as a convenient description of relations between information processing events. We then provide a definition of homogeneity for a global update rule, that regards the way in which different systems are treated by the evolution. In particular, roughly speaking, every system must be treated in the same way by the GUR.

It would be easy to state the homogeneity principle if we knew in advance that the set G has some geometric structure that is invariant under the action of a group—technically speaking, if G were a symmetric space—so that we could define pairs of *homologous regions* as those mapped into one another by some group element. In this case, we could provide a criterion for the evolution to treat every system in the same way, consisting in the existence of some representation of the group of symmetries of the set G that connects the evolution of every two homologous regions. The group in this case would provide the

two following ingredients that are needed for the definition of homogeneity: i) a notion of “running the same test on different regions”, and ii) a criterion for equivalence of two homologous regions provided by the requirement that the events of the same test in the two regions have equal probabilities, if the regions are prepared in the same state.

However, we are in a situation where the structure of G must emerge from the evolution given by the global rule $\mathcal{V}$. The detailed idea is thus to turn the above argument on its head, and define two regions to be homologous if they are “treated in the same way” by $\mathcal{V}$. The precise construction is the following.

First of all, in order for the dynamics to treat two regions in the same way, the two regions need to have the same structure, in such a way that we can make sense of “performing the same range of tests in the two different regions”. This concept is captured by the extension of the notion of operational equivalence (see definition 2) to regions. Intuitively, two operationally equivalent regions must correspond to operationally equivalent systems. However, they also need to “have the same structure”, where the term structure refers to the fact that a system A_R associated with a region R is decomposed in a preferred way into subsystems A_{g_i} corresponding to the cells that compose the region $R = \{g_1, g_2, \dots, g_k\}$. The notion of operational equivalence of regions must then also capture the idea that two equivalent regions admit equivalent decompositions.

The definition is thus recursive, starting from elementary regions $R = \{g\}$, and then defining equivalence of R_1 and R_2 as the equivalence of any *strict subregion* $S_1 \subset R_1$ with a strict subregion $S_2 \subset R_2$, and viceversa.

We recall Remark 5, namely that for the remainder every elementary region $g \in G$ corresponds to some non-trivial system, $A_g \not\cong I$.

Definition 41. *Two elementary regions g and h are operationally equivalent if $A_g \cong A_h$. Two finite regions $R_1, R_2 \subseteq G$ are operationally equivalent if $A_{R_1} \simeq A_{R_2}$, and for every $S_1 \subset R_1$ there exists $S_2 \subset R_2$, and viceversa for every $S_2 \subset R_2$ there exists $S_1 \subset R_1$, such that A_{S_1} and A_{S_2} are operationally equivalent.*

Two operationally equivalent regions have the same structure, namely they can be decomposed

into pairwise equivalent subsystems, as we now prove.

Lemma 53. *An elementary region $R_1 = \{g\}$ cannot be operationally equivalent to a non-elementary one $R_2 = \{h_1, \dots, h_k\}$.*

Proof. The statement is trivial, since $R_1 = \{g_1\}$ does not have proper subregions, while any non-elementary region R_2 does, so operational equivalence cannot hold. $\square$

Lemma 54. *The regions R_1 and R_2 are operationally equivalent if and only if $R_1 = \{g_1, g_2, \dots, g_k\}$, $R_2 = \{h_1, h_2, \dots, h_k\}$, and $A_{g_i} \cong A_{h_i}$.*

Proof. The condition is clearly sufficient, as one can immediately realise by considering the reversible transformation $\mathcal{T}_S \in \llbracket A_{S_1} \rightarrow A_{S_2} \rrbracket_1$ defined as $\mathcal{T}_S := \bigotimes_{g_j \in S_1} \mathcal{T}_j$, with $\mathcal{T}_j \in \llbracket A_{g_j} \rightarrow A_{h_j} \rrbracket_1$ reversible for every $j = 1, \dots, k$. We now prove that the condition is necessary. Let R_1 and R_2 be operationally equivalent. By definition, and by lemma 53, for every $g_l \in R_1$ there is $h_m \in R_2$ such that $A_{g_l} \cong A_{h_m}$, and viceversa. We now introduce the equivalence relation

$$g_i \sim g_j \Leftrightarrow A_{g_i} \cong A_{g_j}.$$

We can then partition R_1 and R_2 into equivalence classes $[g_i]$ and $[h_j]$, respectively, obtaining $R_1 = \bigcup_{g_i} [g_i]$ and $R_2 = \bigcup_{h_i} [h_i]$. Let us start considering the case where R_1 contains a unique equivalence class: $R_1 = [g_1]$. Then R_2 must contain at least one element h_1 with $A_{h_1} \cong A_{g_1}$. Moreover, R_2 cannot contain h_0 such that $h_0 \not\sim h_1$, because h_0 must be equivalent to an elementary subregion of R_1 , by lemma 53. Then, $R_2 = [h_1]$. Finally, since $A_{g_1}^{\otimes k} \cong A_{R_2} \cong A_{R_1} \cong A_{g_1}^{\otimes k}$, by lemma 2 it must be $h = k$. In the general case where R_1 contains more than one equivalence class, one can simply consider the subregions $S_{1_l} := [g_{i_l}]$. Clearly, for each S_{1_l} there must be $S_{2_l} = [h_{j_l}]$, with $|S_{2_l}| = |S_{1_l}|$. $\square$

Remark 7. *Notice that, as a consequence of the above result, two operationally equivalent regions $R_1, R_2 \in \mathcal{R}^{(G)}$ have the same cardinality $|R_1| = |R_2|$.*

Given two operationally equivalent regions $R_1, R_2 \subseteq G$, we now have a way of defining the notion of “running the same test”, that resorts to the identification of a canonical isomorphism of the two regions.

Definition 42. *Given two operationally equivalent regions $R_1 = \{g_1, g_2, \dots, g_k\}$, $R_2 = \{h_1, h_2, \dots, h_k\} \in \mathcal{R}^{(G)}$, let us choose a special reversible transformation $\mathcal{U} \in \llbracket A_{R_1} \rightarrow A_{R_2} \rrbracket$ such that $\mathcal{U} = \bigotimes_{i=1}^k \mathcal{U}_i$, where $\mathcal{U}_i \in \llbracket A_{g_i} \rightarrow A_{h_i} \rrbracket$ is reversible for every i . We then say that R_1 is operationally equivalent to R_2 through $\mathcal{U}$, and that $\{\mathcal{B}_l^{(2)}\}_{l=1}^n \subseteq \llbracket A_{R_2} C \rightarrow A_{R_2} C \rrbracket$ represents the same test as $\{\mathcal{B}_l^{(1)}\}_{l=1}^n \subseteq \llbracket A_{R_1} C \rightarrow A_{R_1} C \rrbracket$ if for every $1 \leq l \leq n$ one has*

$$\mathcal{B}_l^{(2)} = (\mathcal{U} \otimes \mathcal{I}_C) \mathcal{B}_l^{(1)} (\mathcal{U}^{-1} \otimes \mathcal{I}_C). \quad (100)$$

In order to discriminate the way in which two systems A_{g_1} and A_{g_2} evolve, one needs to perform a testing-scheme, consisting of a stimulus-test $\{\mathcal{A}_k^{(x)}\}_{k=1}^m \subseteq \llbracket A_{R_x} C \rightarrow A_{R_x} C \rrbracket_Q$ on the region $R_x \ni g_x$, possibly involving an ancillary system C , and, after the evolution step $\mathcal{V}$, a control-test $\{\mathcal{B}_l^{(x)}\}_{l=1}^n \subseteq \llbracket A_{N_{R_x}^+} C \rightarrow A_{N_{R_x}^+} C \rrbracket_Q$ on the region $N_{R_x}^+$ (where $x \in \{1, 2\}$) and the ancilla C , in order to detect a “response”.

Two operationally equivalent regions are homologous if there is a GUR $\mathcal{T}$ that interchanges them while preserving equivalence of local tests of finite regions. In rigorous words, we have the following definition.

Definition 43 (Homologous regions). *Let $(G, A, \mathcal{V}^\dagger)$ be a GUR, and let $R_1, R_2 \in \mathcal{R}^{(G)}$ be operationally equivalent regions through $\mathcal{T}_i \in \llbracket A_{R_1} \rightarrow A_{R_2} \rrbracket$, such that $N_{R_1}^+$ and $N_{R_2}^+$ are operationally equivalent through $\mathcal{T}_o \in \llbracket A_{R_1'} \rightarrow A_{R_2'} \rrbracket$. We say that R_2 is homologous to R_1 if there exists an admissible GUR $(G, A, \mathcal{T}^\dagger)$, such that for all schemes $(\{\mathcal{A}_k^{(i)}\}_{k=1}^m, \{\mathcal{B}_l^{(i)}\}_{l=1}^n) \subseteq \llbracket A_{R_i} C \rightarrow A_{R_i} C \rrbracket_Q \times \llbracket A_{N_{R_i}^+} C \rightarrow A_{N_{R_i}^+} C \rrbracket_Q$ representing the same pair of tests for $i = 1, 2$, one has $\mathcal{B}_l^{(1)} \mathcal{V} \mathcal{A}_k^{(1)} = \mathcal{T}^{-1} \mathcal{B}_l^{(2)} \mathcal{V} \mathcal{A}_k^{(2)} \mathcal{T}$. We denote this relation as $R_2 \bowtie R_1$.*

We remark that, while the relation $\bowtie$ defined above clearly depends on $\mathcal{V}$, we will generally not make this fact explicit in the notation, as we will always consider contexts where a given $\mathcal{V}$ is considered, and there will not be room for ambiguity. On the other hand, when we want to specify the GUR $\mathcal{T}$ that makes R_1 homologous to R_2 , we write $R_2 \bowtie_{\mathcal{T}} R_1$.

Lemma 55. *If $R_2 \bowtie_{\mathcal{T}} R_1$ then $R_1 \bowtie_{\mathcal{T}^{-1}} R_2$.*

Proof. For every scheme $(\{\mathcal{A}_k^{(i)}\}_{k=1}^m, \{\mathcal{B}_l^{(i)}\}_{l=1}^n)$ with $\mathcal{A}_k^{(1)} = (\mathcal{T}_i^{-1} \otimes \mathcal{I}_C) \mathcal{A}_k^{(2)} (\mathcal{T}_i \otimes \mathcal{I}_C)$ and $\mathcal{B}_l^{(1)} = (\mathcal{T}_o^{-1} \otimes \mathcal{I}_C) \mathcal{B}_l^{(2)} (\mathcal{T}_o \otimes \mathcal{I}_C)$, one has $\mathcal{B}_l^{(1)} \mathcal{V} \mathcal{A}_k^{(1)} = \mathcal{T}^{-1} \mathcal{B}_l^{(2)} \mathcal{V} \mathcal{A}_k^{(2)} \mathcal{T}$. Now, inverting $\mathcal{T}$ on the right and $\mathcal{T}^{-1}$ on the left, one obtains $\mathcal{B}_l^{(2)} \mathcal{V} \mathcal{A}_k^{(2)} = \mathcal{T} \mathcal{B}_l^{(1)} \mathcal{V} \mathcal{A}_k^{(1)} \mathcal{T}^{-1}$. $\square$

Lemma 56. *Let $\mathcal{V}$ be a global rule, and let the region R_2 be homologous to R_1 through $\mathcal{T}$. Then $\mathcal{T}^{-1} \mathcal{V} \mathcal{T} = \mathcal{V}$.*

Proof. It is sufficient to consider the special case where the test scheme is $(\{\mathcal{A}_0^{(1)}\}, \{\mathcal{B}_0^{(1)}\})$, with $\mathcal{A}_0^{(1)} = \mathcal{I}_{R_1 C}$ and $\mathcal{B}_0^{(1)} = \mathcal{I}_{N_{R_1}^+ C}$. Then, we have $\mathcal{A}_0^{(2)} = \mathcal{I}_{R_2 C}$, $\mathcal{B}_0^{(2)} = \mathcal{I}_{N_{R_2}^+ C}$, and $\mathcal{V} = \mathcal{T}^{-1} \mathcal{V} \mathcal{T}$. $\square$

Lemma 57. *Let $\mathcal{V}$ be a global rule, and let $R_2 \bowtie R_1$ for some suitable $\mathcal{T}$. Then*

$$\begin{aligned} \mathcal{T} [\mathcal{A}_{R_1}^{(G)} C \rightarrow \mathcal{A}_{R_1}^{(G)} C] \mathcal{T}^{-1} &= [\mathcal{A}_{R_2}^{(G)} C \rightarrow \mathcal{A}_{R_2}^{(G)} C], \\ \mathcal{T} [\mathcal{A}_{N_{R_1}^+}^{(G)} C \rightarrow \mathcal{A}_{N_{R_1}^+}^{(G)} C] \mathcal{T}^{-1} &= [\mathcal{A}_{N_{R_2}^+}^{(G)} C \rightarrow \mathcal{A}_{N_{R_2}^+}^{(G)} C]. \end{aligned}$$

Proof. Let us choose the test scheme as $(\{\mathcal{A}_k^{(1)}\}_{k=1}^m, \{\mathcal{B}_0^{(1)}\})$, with $\mathcal{B}_0^{(1)} = \mathcal{I}_{R_1' C}$. Then, we have $\mathcal{B}_0^{(2)} = \mathcal{I}_{R_2' C}$, and

$$\begin{aligned} \mathcal{V} \mathcal{A}_k^{(1)} &= \mathcal{T}^{-1} \mathcal{V} \mathcal{A}_k^{(2)} \mathcal{T} \\ &= \mathcal{T}^{-1} \mathcal{V} (\mathcal{T}_i \otimes \mathcal{I}_C) \mathcal{A}_k^{(1)} (\mathcal{T}_i^{-1} \otimes \mathcal{I}_C) \mathcal{T} \\ &= \mathcal{T}^{-1} \mathcal{V} \mathcal{T} \mathcal{T}^{-1} (\mathcal{T}_i \otimes \mathcal{I}_C) \mathcal{A}_k^{(1)} (\mathcal{T}_i^{-1} \otimes \mathcal{I}_C) \mathcal{T} \\ &= \mathcal{V} \mathcal{T}^{-1} (\mathcal{T}_i \otimes \mathcal{I}_C) \mathcal{A}_k^{(1)} (\mathcal{T}_i^{-1} \otimes \mathcal{I}_C) \mathcal{T}, \end{aligned}$$

where in the last equality we used the result of lemma 56. If we now invert $\mathcal{V}$ to the left on both sides, we obtain

$$\mathcal{A}_k^{(1)} = \mathcal{T}^{-1} (\mathcal{T}_i \otimes \mathcal{I}_C) \mathcal{A}_k^{(1)} (\mathcal{T}_i^{-1} \otimes \mathcal{I}_C) \mathcal{T},$$

and finally

$$\begin{aligned} \mathcal{T} \mathcal{A}_k^{(1)} \mathcal{T}^{-1} \\ &= (\mathcal{T}_i \otimes \mathcal{I}_C) \mathcal{A}_k^{(1)} (\mathcal{T}_i^{-1} \otimes \mathcal{I}_C) = \mathcal{A}_k^{(2)}. \end{aligned}$$

A similar argument leads us to conclude that

$$\begin{aligned} \mathcal{T} \mathcal{B}_k^{(1)} \mathcal{T}^{-1} \\ &= (\mathcal{T}_o \otimes \mathcal{I}_C) \mathcal{B}_k^{(1)} (\mathcal{T}_o^{-1} \otimes \mathcal{I}_C) = \mathcal{B}_k^{(2)}. \quad \square \end{aligned}$$

We remark that, straightforwardly,

$$(R_2 \bowtie R_1) \Rightarrow (\mathcal{A}_{R_1} \cong \mathcal{A}_{R_2}). \quad (101)$$

Now, the discrimination of systems g_1 and g_2 is successful if there exists a region $R_1 \ni g_1$ such that for any operationally equivalent region $R_2 \ni g_2$, the region R_2 is not homologous to R_1 . In precise terms, we have the following definition.

Definition 44 (Absolute discrimination). *The evolution given by the global rule $\mathcal{V}$ discriminates two systems $g_1, g_2 \in G$ if for any GUR $\mathcal{T}$ there exists a region R_1 , containing g_1 , for which there is no region R_2 , containing g_2 , that is homologous to R_1 through $\mathcal{T}$, with $g_1 \bowtie_{\mathcal{T}} g_2$.*

In simple words, the evolution given by the global rule $\mathcal{V}$ allows for discrimination of the two systems $g_1, g_2 \in G$ if there exists a test that gives different probabilities depending on whether it is applied at g_1 or at g_2 , or if the tests that one can run on R_1 cannot be performed on any region R_2 containing g_2 , or viceversa.

Now, the notion of discrimination of systems g_1 and g_2 with respect to a third system e is given as follows.

Definition 45 (Relative discrimination). *The evolution $\mathcal{V}$ discriminates two systems $g_1, g_2 \in G$ relatively to a reference system e , with $e \in G$, if for every $\mathcal{T}$ there exists a region R_1 , containing $\{g_1, e\}$, for which there is no region R_2 , containing $\{g_2, e\}$, homologous to R_1 through $\mathcal{T}$ with $e \bowtie_{\mathcal{T}} e$ and $g_1 \bowtie_{\mathcal{T}} g_2$.*

We can now require the evolution described by a global rule to be homogeneous by a precise statement.

Principle 1 (Homogeneity). *The global rule $\mathcal{V}$ discriminates any two systems $g_1 \neq g_2 \in G$, relatively to an arbitrary reference system $e \in G$, but not absolutely.*

Before providing a convenient restatement of the principle, we analyse some of its aspects and consequences. In the first place, the principle says that if we do not choose a reference element $e \in G$, every two systems g_1 and g_2 cannot be discriminated. If we take the negation of definition 44, we obtain a straightforward consequence of homogeneity.

Lemma 58. Let $\mathcal{V}$ satisfy the homogeneity principle 1. For every pair $g_1, g_2 \in G$, there exists $\mathcal{T}$ such that for every choice of region $R_1 \ni g_1$, there exists an operationally equivalent region $R_2 \ni g_2$ through $\mathcal{T}_i \in \llbracket A_{R_1} \rightarrow A_{R_2} \rrbracket$ such that also $N_{R_1}^+$ and $N_{R_2}^+$ are operationally equivalent through $\mathcal{T}_o \in \llbracket A_{R_1'} \rightarrow A_{R_2'} \rrbracket$, and for all schemes $(\{\mathcal{A}_k^{(i)}\}_{k=1}^m, \{\mathcal{B}_l^{(i)}\}_{l=1}^n)$ representing the same tests for $i = 1, 2$, one has $\mathcal{B}_l^{(1)} \mathcal{V} \mathcal{A}_k^{(1)} = \mathcal{T} \mathcal{B}_l^{(2)} \mathcal{V} \mathcal{A}_k^{(2)} \mathcal{T}^{-1}$.

Proof. The statement follows negating definition 44. $\square$

We can now draw a first non trivial consequence from the statement of principle 1.

Lemma 59. For a homogeneous GUR $(G, A, \mathcal{V}^\dagger)$, all local systems A_g for $g \in G$ are operationally equivalent.

Proof. Let us consider two elements $g_1, g_2 \in G$. As a consequence of the homogeneity principle 1, for the special choice $R_1 = \{g_1\}$, there exists $R_2 \ni g_2$ such that $R_2 \bowtie \{g_1\}$ for a suitable $\mathcal{T}$. Now, by lemma 54, it must be $|R_2| = |\{g_1\}| = 1$, i.e. $R_2 = \{g_2\}$. Thus, we have

$$A_{g_2} \cong A_{g_1}. \quad \square$$

Corollary 15. If the region R_2 containing g_2 is homologous to $\{g_1\}$ through $\mathcal{T}$, then $R_2 = \{g_2\}$, and

$$\mathcal{T}^{-1} \llbracket A_{g_1}^{(G)} C \rightarrow A_{g_1}^{(G)} C \rrbracket \mathcal{T} = \llbracket A_{g_2}^{(G)} C \rightarrow A_{g_2}^{(G)} C \rrbracket. \quad (102)$$

Proof. By the proof of lemma 59 one has that $R_2 = g_2$. The thesis then follows from lemma 57. $\square$

The second result that we prove is a technical lemma that will play a crucial role in the following results.

Lemma 60. Let $(G, A, \mathcal{V}^\dagger)$ satisfy the homogeneity requirement. Then for every region R one has $A_R = \bigotimes_{x \in R} A_{g_x} \cong A_0^{\otimes |R|}$, where $A_0 \cong A_g$ for every $g \in G$.

Proof. By lemma 59 we know that for every $g \in G$ one has $A_g \cong A_0$. The remaining part of the statement thus follows straightforwardly. $\square$

Lemma 61. Let $\mathcal{V}$ be non trivial and satisfy the homogeneity requirement, and $R_2 \bowtie R_1$. Then $|R_1| = |R_2|$.

Proof. This follows from lemmas 54 and 60, since R_1 and R_2 are operationally equivalent by definition. $\square$

Let now g_1 and g_2 be an arbitrary pair of systems in G , and $\mathcal{V}$ a homogeneous GUR. By the homogeneity principle 1, there exists $\mathcal{T}$ such that for every region $R_1 \ni g_1$ there is a region $R_2 \ni g_2$ homologous through $\mathcal{T}$ to R_1 . The set of these transformations is denoted as $T_{\mathcal{V}}$, i.e.

$$T_{\mathcal{V}} := \{ \mathcal{T} \mid \exists g_1, g_2 \in G : \forall R_1 \ni g_1 \exists R_2 \ni g_2, R_2 \bowtie_{\mathcal{T}} R_1 \}.$$

Lemma 62. Let $\mathcal{V}$ satisfy homogeneity, and let $\mathcal{T} \in T_{\mathcal{V}}$. Then there exists a permutation $\pi : G \rightarrow G$ such that if $R_2 \bowtie_{\mathcal{T}} R_1$ it is

$$R_2 = \pi(R_1) \quad (103)$$

Proof. By definition, if $\mathcal{T} \in T_{\mathcal{V}}$ there exist $g_1, g_2 \in G$ such that for every $S_1 \ni g_1$ there is $S_2 \ni g_2$ such that $S_2 \bowtie_{\mathcal{T}} S_1$. Let now $R_2 \bowtie_{\mathcal{T}} R_1$, and $f_1 \in R_1$. Consider the finite region $V_1 := \{f_1, g_1\}$. Then there is a region $V_2 \ni g_2$ such that $V_2 \bowtie_{\mathcal{T}} V_1$, and by lemma 61, one has $|V_2| = |V_1| = 2$. Consequently, it must be $V_2 = \{f_2, g_2\}$. Moreover, since we know that $g_2 \bowtie_{\mathcal{T}} g_1$, by definitions 41, 42, and 43, it is $f_2 \bowtie_{\mathcal{T}} f_1$. Thus, for every $f_1 \in R_1$ there is a unique $f_2 \in R_2$ such that $f_2 \bowtie_{\mathcal{T}} f_1$. We then define $\pi : G \rightarrow G$ by setting $\pi(f_1) := f_2$. By lemma 55 the map π is both injective and surjective, and thus it is a permutation of G . $\square$

In the following, given $\mathcal{T} \in T_{\mathcal{V}}$, we will denote $\mathcal{T} = \mathcal{T}_\pi$ where π is the permutation of G in lemma 62.

Lemma 63. Let $(G, A, \mathcal{V}^\dagger)$ be a homogeneous GUR, and $\pi \in \Pi_{\mathcal{V}}$. Then the GUR $\mathcal{T}_\pi$ is unique.

Proof. Let $\mathcal{T}_\pi, \mathcal{S}_\pi \in T_{\mathcal{V}}$ correspond to the same permutation π . Let $a \in \llbracket \bar{A}_G \bar{C} \rrbracket_{Q\mathbb{R}}$. Then by lemma 28 one has $a = \mathcal{A}^\dagger e_{G'}$, with $\mathcal{A} \in \llbracket A_G C \rightarrow A_G C \rrbracket_{Q\mathbb{R}}$. Thus

$$\begin{aligned} (\mathcal{S}_\pi \mathcal{T}_\pi^{-1})^\dagger a &= [(\mathcal{S}_\pi \mathcal{T}_\pi^{-1})^\dagger \otimes \mathcal{I}_C^\dagger] \mathcal{A}^\dagger e_{G'} \\ &= (\mathcal{T}_\pi^{-1\dagger} \mathcal{S}_\pi^\dagger \otimes \mathcal{I}_C^\dagger) \mathcal{A}^\dagger (\mathcal{T}_\pi^{-1\dagger} \mathcal{S}_\pi^\dagger \otimes \mathcal{I}_C^\dagger) e_{G'}, \end{aligned}$$

where $G' = G \cup h_0$ and $A_{h_0} \cong C$. Now, by definition

$$\begin{aligned} (\mathcal{S}_\pi^\dagger \otimes \mathcal{S}_C^\dagger) \mathcal{A}^\dagger (\mathcal{S}_\pi^{-1\dagger} \otimes \mathcal{S}_C^\dagger) \\ = (\mathcal{S}_\pi^\dagger \otimes \mathcal{S}_C^\dagger) \mathcal{A}^\dagger (\mathcal{S}_\pi^{-1\dagger} \otimes \mathcal{S}_C^\dagger) \\ = (\mathcal{S}_i^\dagger \otimes \mathcal{S}_C^\dagger) \mathcal{A}^\dagger (\mathcal{S}_i^{-1\dagger} \otimes \mathcal{S}_C^\dagger), \end{aligned}$$

which implies

$$(\mathcal{S}_\pi \mathcal{S}_\pi^{-1})^\dagger a = a.$$

Since the latter holds for every C and every a , we conclude that $\mathcal{S}_\pi = \mathcal{T}_\pi$. $\square$

Corollary 16. *The set $T_\mathcal{V} = \{\mathcal{T}_\pi\}$ is a representation of a group $\Pi_\mathcal{V} = \{\pi \mid \mathcal{T}_\pi \in T_\mathcal{V}\}$ of permutations of G .*

We can summarise the above results as follows. For every pair g_1, g_2 there exists a reversible map $\pi : G \rightarrow G$ such that $\pi(g_1) = g_2$ and a reversible transformation $\mathcal{T}_\pi$ of $[[A_G]]_Q$ such that $\mathcal{T}_\pi^{-1} \cdot \mathcal{T}_\pi$ is an automorphism of $[[A_G \rightarrow A_G]]_Q$ and for every $R \subset G$ and every system C

$$\begin{aligned} [[A_R^{(G)} C \rightarrow A_R^{(G)} C]]_{Q*} \\ = \mathcal{T}_\pi^{-1} [[A_{\pi(R)}^{(G)} C \rightarrow A_{\pi(R)}^{(G)} C]]_{Q*} \mathcal{T}_\pi, \end{aligned}$$

and $\mathcal{T}_\pi$ leaves the GUR $\mathcal{V}$ invariant, i.e.

$$\mathcal{T}_\pi^{-1} \mathcal{V} \mathcal{T}_\pi = \mathcal{V}.$$

The transformations π clearly form a group $\Pi_\mathcal{V}$ whose action $\Pi_\mathcal{V} \times G \rightarrow G$ is *transitive*.

We now proceed considering the second part of the homogeneity principle, i.e. the statement that discrimination of every pair $g_1 \neq g_2$ is possible relative to some third element $e \in G$. The main consequence is that the action $\Pi_\mathcal{V} \times G \rightarrow G$ is *free*.

Lemma 64. *Let $\mathcal{V}$ satisfy the homogeneity principle 1. If $e \neq \pi \in \Pi_\mathcal{V}$, there is no element g of G such that $\pi(g) = g$.*

Proof. Let $\mathcal{T}_\pi \in T_\mathcal{V}$. Suppose that there is $g \in G$ such that $\pi(g) = g$. Then for every region $R_1 \ni g$, the region $R_2 := \pi(R_1)$ contains g and one has $R_2 \bowtie_{\mathcal{T}} R_1$, with $g \bowtie_{\mathcal{T}} g$. This is in contradiction with definition 45, i.e. with the homogeneity principle 1. $\square$

Corollary 17. *Let $\Pi_\mathcal{V}$ be the group of permutations of G such that $T_\mathcal{V}$ is a representation of $\Pi_\mathcal{V}$. The action of $\Pi_\mathcal{V}$ on G is regular.*

We can now summarise all the results that we drew from homogeneity as follows: for every $g_1, g_2 \in G$ there exists a permutation π of G such that $\pi(g_1) = g_2$, and a GUR $\mathcal{T}_\pi$ of $[[\bar{A}_G]]_{Q\mathbb{R}}$ such that for every $R \in \mathcal{R}^{(G')}$ and C

$$\begin{aligned} [[A_R^{(G)} C \rightarrow A_R^{(G)} C]]_{Q*} \\ = \mathcal{T}_\pi^{-1} [[A_{\pi(R)}^{(G)} C \rightarrow A_{\pi(R)}^{(G)} C]]_{Q*} \mathcal{T}_\pi, \end{aligned} \quad (104)$$

and $\mathcal{T}_\pi$ leaves the evolution invariant, i.e.

$$\mathcal{T}_\pi^{-1} \mathcal{V} \mathcal{T}_\pi = \mathcal{V}.$$

The transformations π form a group $\Pi_\mathcal{V}$ that acts regularly (i.e. transitively and freely) on G .

Now, it is clear that, chosen $e \in G$, the elements of G can be identified with the elements of $\Pi_\mathcal{V}$, as the action of $\Pi_\mathcal{V}$ on G is regular. In other terms, G is a principal homogeneous space for $\Pi_\mathcal{V}$, and there is a bijection $\Pi_\mathcal{V} \leftrightarrow G$. However, the homogeneity principle implies an even stronger result: the set G can be seen as the set of vertices of a special Cayley graph of $\Pi_\mathcal{V}$. The following arguments, largely inspired to Refs. [25, 46], make the mentioned result rigorous.

As a consequence of homogeneity, one has

$$\pi(g) \rightarrow \pi(g') \Leftrightarrow g \rightarrow g'. \quad (105)$$

In other words, $N_{\pi(g)}^\pm = \pi(N_g^\pm)$ for $\pi \in \Pi_\mathcal{V}$, as can be checked starting from the definition of $N_g^\pm$. The proof is provided in Appendix D.

Let now $g \in G$ and $\pi \in \Pi_\mathcal{V}$ such that $\pi(e) = g$. One can then label the elements $g' \in N_g^\pm$ by $\pi^{-1}(g') \in N_e^\pm = \{h_i^{\pm 1}\}_{i=1}^k$. Notice that for the moment we have no reason to claim that $k < \infty$. We will associate all the pairs $(g, g') \in E$ with the *color* h_i if $g = \pi(e)$ and $g' = \pi(h_i)$ for some $\pi \in \Pi_\mathcal{V}$ ². This construction enriches the graph $\Gamma(G, E)$, which becomes a vertex-transitive coloured directed graph, with colours corresponding to the labels $h_i \in N_e^+$

Given two elements $g, g' \in G$, a *path* p from g to g' is an array of elements $p = (g_1, g_2, \dots, g_k) \in$

²In principle, the colour of an edge (g, g') might be ambiguously defined if there existed two permutations $\pi, \sigma \in \Pi_\mathcal{V}$ such that $\pi^{-1}(g) = \sigma^{-1}(g) = e$ and $\pi^{-1}(g'), \sigma^{-1}(g') \in N_e^\pm$, but $\pi^{-1}(g') \neq \sigma^{-1}(g')$. However, due to lemma 64, this is never the case, since it would imply that $\sigma\pi^{-1}(g') \neq g'$, and thus $\sigma\pi^{-1} \neq e$, but $[\sigma\pi^{-1}](g) = \sigma(e) = g$.

$G^{\times k}$ such that, setting $g_0 := g$ and $g_{k+1} := g'$, one has

$$\forall 0 \leq i \leq k \quad (g_i \rightsquigarrow g_{i+1}) \vee (g_{i+1} \rightsquigarrow g_i).$$

The *length* of a path p , denoted $\ell(p)$ is $k+1$ if $p \in G^{\times k}$. In the following we will restrict attention to the case where $\Gamma(G, E)$ is *connected*, i.e. for every two elements $g, g' \in G$ there exists a path from g to g' . The set of paths from g to g' will be denoted by $\mathbf{p}_{g,g'}^+$. We can now introduce the graph metric of $\Gamma(G, E)$ as follows. Let $g, g' \in G$. Then we define

$$d_\Gamma(g, g') := \min_{p \in \mathbf{p}_{g,g'}^+} \ell(p)$$

We remark that $d_\Gamma(g, g') = d_\Gamma[\pi(g), \pi(g')]$ for all $\pi \in \Pi_\gamma$, since $g_1, \dots, g_k$ is a path from g to g' if and only if $\pi(g_1), \dots, \pi(g_k)$ is a path from $\pi(g)$ to $\pi(g')$ due to condition (105).

We now use the alphabet $N_e^+ \cup N_e^-$ to form arbitrary words, obtaining a free group F : composition corresponds to word juxtaposition, with the empty word λ representing the identity, and the formal rule $h_i h_i^{-1} = h_i^{-1} h_i = \lambda$. An element $w = h_{i_1}^{p_1} h_{i_2}^{p_2} \dots h_{i_m}^{p_m}$ of F —with $p_j \in \{-1, 1\}$ —thus corresponds to a path on the graph, where the symbol $h_i^{-1} \in N_e^-$ denotes a backwards step along the arrow h_i . For every $h_{i_1}^{p_1} h_{i_2}^{p_2} \dots h_{i_m}^{p_m} = w \in F$, one has $w^{-1} = h_{i_m}^{-p_m} \dots h_{i_2}^{-p_2} h_{i_1}^{-p_1}$.

The action $M : F \times G \rightarrow G$ of words $w = h_{i_1}^{p_1} \dots h_{i_{m-1}}^{p_{m-1}} h_{i_m}^{p_m} \in F$ on elements $g \in G$ can be defined as follows

$$\begin{aligned} M(w, g) &= M(h_{i_1}^{p_1} h_{i_2}^{p_2} \dots h_{i_m}^{p_m}, g) \\ &:= M[h_{i_2}^{p_2} \dots h_{i_m}^{p_m}, M(h_{i_1}^{p_1}, g)], \end{aligned}$$

where the action of $h_i \in S$ on the elements $g \in G$ is defined as $M(h_i, g) = g'$ if there exists $\pi \in \Pi_\gamma$ such that $\pi^{-1}(g) = e$ and $\pi^{-1}(g') = h_i$, while $M(h_i^{-1}, g) = g'$ if there exists $\pi \in \Pi_\gamma$ such that $\pi^{-1}(g) = h_i$ and $\pi^{-1}(g') = e$ ³. In the following we will use the shortcut

$$gw := M(w, g).$$

³The action is well defined. Suppose indeed that there are two elements f, f' both satisfying the definition of $M(h_i, g)$. This means that there exist π, σ such that $\pi^{-1}(g) = \sigma^{-1}(g) = e$, while $\pi^{-1}(f) = \sigma^{-1}(f') = h_i$. However, according to lemma 64, the first condition implies that $\sigma = \pi$. By the same lemma, the second condition then implies that $f = f'$. A similar argument shows that $M(h_i^{-1}, g)$ is well defined.

This notation is particularly handy, since

$$g(h_{i_1}^{p_1} h_{i_2}^{p_2} \dots h_{i_m}^{p_m}) = (gh_{i_1}^{p_1}) h_{i_2}^{p_2} \dots h_{i_m}^{p_m}.$$

For every $w \in F$, and for every $g \in G$ and for every permutation $\pi \in \Pi_\gamma$, we now show that $\pi(gw) = \pi(g)w$. The first step consists in proving the result for $w = h_i^{\pm 1}$. Let $g' = gh_i$, namely by definition $\sigma^{-1}(g) = e$ and $\sigma^{-1}(g') = h_i$ for some (unique) $\sigma \in \pi_\gamma$. By Eq. (105) it is $\pi(g) \rightsquigarrow \pi(g')$. Moreover, if we define $f := \pi(g)$ and $f' := \pi(g')$ we have

$$\begin{aligned} h_i &= \sigma^{-1}(g') = \sigma^{-1}\pi^{-1}(f'), \\ e &= \sigma^{-1}(g) = \sigma^{-1}\pi^{-1}(f), \end{aligned}$$

namely $f' = fh_i$. This implies

$$\pi(gh_i) = \pi(g') = \pi(g)h_i. \quad (106)$$

Similarly, if $g' = gh_i^{-1}$ by definition $\sigma^{-1}(g) = h_i$ and $\sigma^{-1}(g') = e$ for some (unique) $\sigma \in \pi_\gamma$. Thus, setting again $f := \pi(g)$ and $f' := \pi(g')$, we have

$$\begin{aligned} h_i &= \sigma^{-1}(g) = \sigma^{-1}\pi^{-1}(f), \\ e &= \sigma^{-1}(g') = \sigma^{-1}\pi^{-1}(f'), \end{aligned}$$

namely $f' = fh_i^{-1}$, and

$$\pi(gh_i^{-1}) = \pi(g') = \pi(g)h_i^{-1}. \quad (107)$$

Notice that the last result implies that for every pair $f, g \in G$, given $\pi \in \Pi_\gamma$ such that $\pi(f) = g$, there is a bijection $N_f^\pm \leftrightarrow N_g^\pm$, given by

$$N_{\pi(g)}^\pm = \pi(N_f^\pm). \quad (108)$$

One can now prove that $\pi(fw) = \pi(f)w$ by induction on the length $l(w)$ of the word w . Indeed, we know that it is true for $l(w) = 1$. Suppose now that for $l(w) = n - 1$ one has $\pi(fw) = \pi(f)w$, and consider w' with $l(w') = n$. Then $w' = wh_i^p$ with $l(w) = n - 1$ and $p = \pm 1$. We have

$$\begin{aligned} \pi(fw') &= \pi(fwh_i^p) = \pi[(fw)h_i^p] \\ &= \pi(fw)h_i^p = \pi(f)wh_i^p = \pi(f)w', \end{aligned}$$

where the induction hypothesis is used in the fourth equality.

Let us now suppose that for some $f \in G$ and some word $w \in F$ one has $fw = f$. Then for every $f' \in G$ one can take π such that $\pi(f) = f'$, thus obtaining

$$f'w = \pi(f)w = \pi(fw) = \pi(f) = f'.$$

Thus, if a path $w \in F$ is closed starting from $f \in G$, then it is closed also starting from any other $g \in G$.

We can now easily see that the subset R of F corresponding to words r such that $gr = g$ for all $g \in G$ is a normal subgroup. Indeed, R is a subgroup because the juxtaposition of two words $s, s' \in R$ is again a word $ss' \in R$, and for every word $s \in R$ also $s^{-1} \in R$. To prove that R is normal in F we just show that it coincides with its normal closure, i.e. for every $w \in F$ and every $r \in R$, we have $wrw^{-1} \in R$. Indeed, defining for arbitrary g the element $g' := gw$, we have $g'w^{-1} = g$, and thus $gwrw^{-1} = g'r w^{-1} = g'w^{-1} = g$, namely $wrw^{-1} \in R$.

We thus identified a normal subgroup R containing all the words r corresponding to closed paths. If one takes the quotient F/R , one obtains a group whose elements are equivalence classes of words in F . If we label an arbitrary element of G by e , it is clear that the elements of G are in one-to-one correspondence with the vertices of G , since for every $g \in G$ there is one and only one class in F/R whose elements lead from e to g . We can then write $g = w$ for every $w \in F$ such that w represents a path leading from e to g . Notice that the elements of F/R , i.e. equivalence classes of words $[w]_R$, correspond to elements of G , i.e. for every $g \in G$ there is a unique class $[w]_R$ such that $g = [w]_R$. The criterion for class membership of words is very simple: $w \in g$ iff w connects the vertex $e \in G$ to the vertex $g \in G$.

As a consequence of the above arguments, we can now show that for every $\pi \in \Pi_{\mathcal{V}}$ and for every $g = [w]_R \in G$, one has $\pi(g) = [sw]_R$ for a fixed word $s \in F$. Indeed, consider $g_0 = [w_0]_R$. Let $[w']_R = \pi(g_0)$. Then $w' = w'_0 w_0^{-1} w_0 = sw_0$, with $s := w'_0 w_0^{-1}$, and $\pi(g_0) = [sw_0]_R$. Let now $f \in G$. We can always find $t \in F$ so that $f = [w_0 t]_R$, e.g. by setting $t := w_0^{-1} z$, $f = [z]_R$. Then we have $\pi(f) = \pi([w_0 t]_R) = \pi(g_0 t) = \pi(g_0) t = [sw_0 t]_R = [sz]_R$, where we use the definition $\pi(k)r := [xr]_R$ with $\pi(k) = [x]_R$. This definition clearly makes sense, since when $[a]_R = [b]_R$ and $[r]_R = [u]_R$, also $[ar]_R = [bu]_R$.

In technical terms, the graph $\Gamma(G, E) = \Gamma(G, S)$ is the Cayley graph of the group $G = F/R$. Homogeneity thus implies that the set G is a group G that can be presented as $G = \langle S | R \rangle$, where S is the set of *generators* of G and R is the group of *relators*. In the following, if $h_i = h_i^{-1}$

we will draw an undirected edge to represent h_i . The presentation can be chosen by arbitrarily dividing S into $S_+ \subseteq S$ and $S_- := S_+^{-1}$ in such a way that $S_+ \cup S_- = S$. The above arbitrariness is inherent the very notion of group presentation and corresponding Cayley graph, and is exploited in the literature, in particular in the definition of isotropy [25, 57].

For convenience of the reader we remind the definition of *Cayley graph*. Given a group G and a set S_+ of generators of the group, the Cayley graph $\Gamma(G, S_+)$ is defined as the colored directed graph having vertex set G , edge set $\{(g, gh) \mid g \in G, h \in S_+\}$, and a color assigned to each generator $h \in S_+$. Notice that a Cayley graph is *regular*—i.e. each vertex has the same degree—and *vertex-transitive*—i.e. all vertices are equivalent, in the sense that the graph automorphism group acts transitively upon its vertices. The Cayley graphs of a group G are in one to one correspondence with its presentations, with $\Gamma(G, S_+)$ corresponding to the presentation $\langle S_+ | R \rangle$.

7 Locality

If a GUR were to represent a physical law, we would like that, in order to determine it, it is sufficient to test it in a finite region and for a finite amount of time. This possibility is granted by homogeneity in conjunction with a second principle—*locality*—to which the present section is devoted. The locality requirement can be phrased in simple words by stipulating that in order to calculate the effects of a physical law, i) it is possible to adopt a reductionist procedure, decomposing an arbitrary system into elementary parts and then calculating its evolution by evolving every part separately, considering only pairwise interactions between parts, and ii) if the reduction is operated, only finite systems will be involved in the calculation for every elementary part.

In the present section we introduce the precise statement of the locality principle, and analyse its main consequences. In order to make this notion independent of homogeneity, we formulate it for general admissible GURs. In particular, this implies that the mathematical statement of locality alone will not allow for determination of a local GUR by observation of a finite region.

Before giving a formal definition, we provide a

few heuristic steps leading to the final formulation. Let us start considering an admissible GUR $(G, A, \mathcal{V}^\dagger)$, with uniformly bounded neighbourhood, namely such that there exists $k < \infty$ so that for every $g \in G$ it is $|N_g^\pm| \leq k$. We remark that, since $|N_g^\pm| \leq k$, it is $|P_g^\pm| \leq k^2$. This also implies that the transformations $\mathcal{S}'_g$ are transformations of finitely many systems.

In the remainder, we will often omit the labels 0, 1 on the wires in diagrams, and adopt the convention that the upper half is labelled 0 and the lower one 1.

Now, reminding Eq. (92), given $a \in \llbracket \bar{A}_R^{(G)} \mathbb{C} \rrbracket_{Q\mathbb{R}}$, one has

$$\begin{aligned} [(\mathcal{V}^\dagger a)_{(N_R^-, 1)\mathbb{C}}] &= [\{(\mathcal{V}_R^\dagger \otimes \mathcal{I}_C^\dagger) a\}_{(N_R^-, 1)\mathbb{C}}] \\ &= [\{(\mathcal{S}'_{N_R}^\dagger \otimes \mathcal{I}_C^\dagger) a\}_{(N_R^-, 1)\mathbb{C}}]. \end{aligned}$$

Thus, we have

$$\begin{array}{c} \text{Diagram (109):} \\ \text{Left side: } N_R^- \text{ wire entering a box } \mathcal{V}_R, \text{ with } R \text{ wire exiting.} \\ \text{Right side: } \text{A box } \mathcal{S}'_{N_R} \text{ with } N_R^- \text{ wire entering from the bottom.} \\ \text{Top of } \mathcal{S}'_{N_R}: \text{A box } \psi \text{ with } P_R^- \text{ wire entering from the top.} \\ \text{Outputs of } \mathcal{S}'_{N_R}: \text{ } R \text{ wire exiting top, } P_R^- \setminus R \text{ wire exiting middle, and } N_R^- \text{ wire exiting bottom.} \\ \text{The } P_R^- \setminus R \text{ and } N_R^- \text{ wires then enter a box } \mathbb{C} \text{ with two outputs labeled } e. \end{array}$$

Notice that, by admissibility of the GUR $\mathcal{V}$, and thanks to the no-restriction hypothesis,

$$\mathcal{V}_R \in \llbracket A_{N_R^-} \rightarrow A_R \rrbracket_1.$$

Moreover, the GUR $\mathcal{V}$ can be equivalently defined starting from its action on $\llbracket \bar{A}_G \rrbracket_{L\mathbb{R}}$, by setting $\mathcal{V}^\dagger[a_R] := [(\mathcal{V}_R^\dagger a)_{N_R^-}]$. Indeed, following this definition, one can prove that $\mathcal{V}$ is right-invertible, by the argument discussed in Appendix E. We remark that the construction of the inverse provides a new GUR $\mathcal{W}$.

As a consequence, we have that $\mathcal{W}^\dagger \mathcal{V}^\dagger = \mathcal{V}^\dagger \mathcal{W}^\dagger = \mathcal{I}_G^\dagger$ for some GUR $\mathcal{W}$, and thus $\mathcal{W} = \mathcal{V}^{-1}$. We want to stress here that in proving this result we only use properties (88), (89), (94) and (97) of $\mathcal{S}'_g$, derived in section 5.4, along with uniform boundedness of the neighbourhoods for the GUR $\mathcal{V}$ (see Appendix E for the details).

Let us now abstract our focus from the situation where a GUR $(G, A, \mathcal{V}^\dagger)$ is given. Instead, we will suppose that, given the pair (G, A) , a finite neighbourhood $N_g^+ \subseteq G$ is defined for every $g \in G$, and for $f \in G$ we set

$$N_f^- := \{h \in G \mid f \in N_h^+\}.$$

For $X \in \mathbb{R}^{(G)}$ we then define $N_X^\pm := \bigcup_{g \in X} N_g^\pm$, and the sets $P_g^\pm$ are then defined exactly as for a GUR. Finally, we suppose that a family of maps $\mathcal{S}'_g$ is defined, with the properties (88) and (89), and an analogue of (94) and (97).

Definition 46 (Local rule). *A k -local rule $(N^+, A, \mathcal{S}')$ on G is i) a map $N^+ : G \rightarrow \mathbb{R}^{(G)}$ that associates $g \in G$ with a finite region $N_g^+ \in \mathbb{R}^{(G)}$ such that $|N_g^+|, |N_g^-| \leq k$ for every $g \in G$; ii) a map $\mathcal{S}' : g \mapsto \mathcal{S}'_g$, where $\mathcal{S}'_g$ are maps in $\llbracket A_{(N_g^+, 0)} A_{(g, 1)} \rightarrow A_{(N_g^+, 0)} A_{(g, 1)} \rrbracket_1$ with the following properties for every $f, g \in G$ and $R, S \in \mathbb{R}^{(G)}$:*

1. $\mathcal{S}'_g \mathcal{S}'_g = \mathcal{I}_{A_{N_g^+} A_g}$;
2. $\mathcal{S}'_g \mathcal{S}'_f = \mathcal{S}'_f \mathcal{S}'_g$;
3. for every $\mathcal{C} \in \llbracket A_{(R, 0) \cup (S, 1)} \mathbb{C} \rightarrow A_{(R, 0) \cup (S, 1)} \mathbb{C} \rrbracket$

$$\begin{array}{c} \text{Diagram (110):} \\ \text{Top row: } P_R^- \setminus N_S^+ \text{ wire entering a box } \mathcal{S}'_{N_R^- \cup S}. \\ \text{Middle row: } N_S^+ \text{ wire entering a box } \mathcal{C}. \\ \text{Bottom row: } N_R^- \text{ wire entering a box } \mathcal{S}'_{N_R^- \cup S}. \\ \text{Left side: } S \setminus N_R^- \text{ wire entering a box } \mathcal{S}'_{N_R^- \cup S}. \\ \text{Right side: } P_R^- \setminus N_S^+ \text{ wire exiting a box } \mathcal{S}'_{N_R^- \cup S}. \\ \text{Central box } \mathcal{C}: \text{ } R \text{ wire entering top, } R \text{ wire exiting top, } C \text{ wire entering middle, } C \text{ wire exiting middle, } S \text{ wire entering bottom, } S \text{ wire exiting bottom.} \\ \text{Bottom row: } S \setminus N_R^- \text{ wire exiting a box } \mathcal{S}'_{N_R^- \cup S}. \end{array}$$

$$= \begin{array}{c} \text{Simplified diagram:} \\ \text{Top: } P_R^- \setminus N_S^+ \text{ wire entering a box } \mathcal{C}'. \\ \text{Middle: } N_S^+ \text{ wire entering a box } \mathcal{C}'. \\ \text{Bottom: } N_R^- \text{ wire entering a box } \mathcal{C}'. \\ \text{Left: } S \setminus N_R^- \text{ wire entering a box } \mathcal{C}'. \\ \text{Right: } P_R^- \setminus N_S^+ \text{ wire exiting a box } \mathcal{C}'. \end{array}, \quad (110)$$

where $\mathcal{S}'_X := \prod_{g \in X} \mathcal{S}'_g$ for every $X \subseteq G$.

In the following, we will often omit the subscript X in $\mathcal{S}'_X$ in the diagrams, since X corresponds to the label of the lower half wires. Moreover, in order to make the equations lighter, from now on we will always write or draw transformations and effects that do not involve explicitly the external system $\mathbb{C}$, however it will always be assumed that the conditions hold also locally on extended systems, unless explicitly stated otherwise.

Remark 8. Notice that for a transformation $\mathcal{C}$ of the form $\mathcal{C}_0 \otimes \mathcal{I}_{(R \setminus R', 0) \cup (S \setminus S', 1)}$, due to the definition of $\mathcal{S}'_{N_R^- \cup S}$ and the property 2 of $\mathcal{S}'_g$, one

has that

$$\begin{array}{c}
\frac{P_R^- \setminus N_S^+}{\frac{N_S^+}{C} \quad \mathcal{C}' \quad \frac{N_S^+}{C} \quad N_R^-} = \frac{P_R^- \setminus N_S^+}{\frac{N_S^+ \setminus N_{S'}^+}{C} \quad \mathcal{C}'_0 \quad \frac{N_S^+ \setminus N_{S'}^+}{C} \quad N_{R'}^-} \\
\frac{[P_R^- \cup N_S^+][P_{R'}^- \cup N_{S'}^+]}{= \frac{P_{R'}^- \setminus N_{S'}^+}{N_{S'}^+} \quad \mathcal{S}' \quad \frac{N_{R'}^- \cup S'}{R'} \quad \mathcal{C}_0 \quad \frac{N_{R'}^- \cup S'}{R'} \quad \mathcal{S}' \quad \frac{P_{R'}^- \setminus N_{S'}^+}{N_{S'}^+} \quad N_{R'}^-} \\
\frac{[N_R^- \cup S][N_{R'}^- \cup S']}{S' \setminus N_{R'}^-}
\end{array}$$

The first property that we prove is the following.

Lemma 65. *Given a local rule $(N^+, A, \mathcal{S}')$ on G , the following identities hold:*

1. For every $S \in R^{(G)}$, and every $\mathcal{A} \in \llbracket A_S C \rightarrow A_S C \rrbracket$,

$$\begin{array}{c}
\frac{(N_S^+, 0)}{(S, 1)} \quad \mathcal{S}'_{N_S^-} \quad \frac{(N_S^+, 0)}{(S, 1)} \quad \mathcal{A} \quad \frac{(N_S^+, 0)}{(S, 1)} \quad \mathcal{S}'_{N_S^-} \quad \frac{(N_S^+, 0)}{(S, 1)} \\
= \frac{(N_S^+, 0)}{(S, 1)} \quad \mathcal{A}^+ \quad \frac{(N_S^+, 0)}{(S, 1)}
\end{array} \quad (111)$$

2. for every $R \in R^{(G)}$ and every $\mathcal{B} \in \llbracket A_R C \rightarrow A_R C \rrbracket$

$$\begin{array}{c}
\frac{(P_R^-, 0)}{(N_R^-, 1)} \quad \mathcal{S}'_{N_R^-} \quad \frac{(P_R^- \setminus R, 0)}{(R, 0)} \quad \mathcal{B} \quad \frac{(P_R^-, 0)}{(N_R^-, 1)} \quad \mathcal{S}'_{N_R^-} \quad \frac{(P_R^-, 0)}{(N_R^-, 1)} \\
= \frac{(P_R^-, 0)}{(N_R^-, 1)} \quad \mathcal{B}^- \quad \frac{(P_R^-, 0)}{(N_R^-, 1)}
\end{array} \quad (112)$$

Proof. The equalities follow taking $S = \emptyset$ and $R = \emptyset$ in Eq. (110), respectively. $\square$

Remark 9. Given a local rule $(N^+, A, \mathcal{S}')$ we define the forward and backward evolution of local transformations $\mathcal{F} \in \llbracket A_R C \rightarrow A_R C \rrbracket$ for a finite region $R \in R^{(G)}$ through the expressions (111) and (112), as follows

$$[\mathbf{V}_L^\pm(\mathcal{F})]_{N_R^\pm C} := [\mathcal{F}^\pm]_{N_R^\pm C}. \quad (113)$$

Now, once we established the notion of a local rule and its application to local transformations, we can define the property of a global rule of being reducible to a local rule.

Definition 47 (Reduction to a local rule). We say that the GUR $(G, A, \mathcal{V}^\dagger)$ is reducible to a local rule if there exists a k -local rule $(N^+, A, \mathcal{S}')$ for some $k < \infty$ such that for every $R \in R^{(G)}$ and every $\mathcal{F} \in \llbracket A_R C \rightarrow A_R C \rrbracket$

$$\begin{aligned}
(\mathcal{V} \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{V}^{-1} \otimes \mathcal{I}_C) &= \mathbf{V}_L^-(\mathcal{F}) \otimes \mathcal{I}_{G \setminus N_R^-}, \\
(\mathcal{V}^{-1} \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{V} \otimes \mathcal{I}_C) &= \mathbf{V}_L^+(\mathcal{F}) \otimes \mathcal{I}_{G \setminus N_R^+}.
\end{aligned} \quad (114)$$

We now prove the two main results of this section, which relate local rules and GURs in a one-to-one correspondence.

Theorem 9. Every local rule $(N^+, A, \mathcal{S}')$ on G identifies a GUR $(G', A, \mathcal{W}^\dagger)$, with $G' = G \times \{0, 1\}$, and $\mathcal{W}^{-1} = \mathcal{I}_G \mathcal{W} \mathcal{I}_G$.

Proof. First of all, one can define the linear maps $\mathcal{W}_C^\dagger$ on $\llbracket \bar{A}_G \bar{C} \rrbracket_{L\mathbb{R}}$ as follows. For $a_{ZC} \in \llbracket \bar{A}_G \bar{C} \rrbracket_{L\mathbb{R}}$, with $Z = (R, 0) \cup (S, 1)$, we set

$$\mathcal{W}^\dagger[a_{ZC}] := [a_{(N_R^-, 0) \cup (N_S^+, 1)C}^-],$$

where

$$\begin{array}{c}
\frac{\bar{N}_R^+}{\bar{N}_R^- \cup S} \quad e \quad \frac{\bar{S} \setminus N_R^-}{e} \quad \frac{\bar{N}_R^-}{C} \quad a^- \quad \frac{\bar{N}_S^+}{C} \\
:= \frac{\bar{N}_R^+}{\bar{N}_R^- \cup S} \quad e \quad \frac{\bar{S} \setminus N_R^-}{S \setminus N_R^-} \quad \mathcal{S} \quad \frac{\bar{N}_R^-}{N_R^-} \quad \mathcal{S}' \quad \frac{\bar{N}_S^+}{N_S^+} \quad a \quad \frac{\bar{N}_R^-}{S} \quad e
\end{array}$$

We omitted the subscript for $\mathcal{S}$ and $\mathcal{S}'$, as the systems on which the transformations act are

piece of information besides knowing $\mathcal{S}'_e$. Actually, determining the cellular automaton with local tests also requires knowledge of the regions N_R^+ for every $R \in \mathcal{R}^{(G)}$ —or, in other words, knowledge of the structure of the graph of influence relations. This information is indeed necessary in order to know how to correctly calculate the transformation $V_L^\pm(\mathcal{A}_R)$ for a given region R . Indeed, it might happen that knowledge of finitely many closed paths from the elements $g \in R$ is not sufficient to reconstruct all the new closed paths that appear as the size of the region R increases. This means that one needs infinitely many rules to identify systems in $N_{g_1}^\pm \cap N_{g_2}^\pm$. However, this is not the case if closed paths in the graph of influence relations can be decomposed into elementary closed paths having a uniformly bounded size, let us say by a constant length l . In this case, knowledge of local rules and of the local structure of the graph up to some finite distance l is sufficient to calculate the action of $\mathcal{V}$ in an arbitrary finite region. This is the reason for the following definition.

Definition 48 (Decomposability into bounded regions). *We say that a global update rule $(G, A, \mathcal{V}^\dagger)$ is decomposable into bounded regions if the closed paths of the influence graph of $\mathcal{V}$ can be decomposed into elementary closed paths of uniformly bounded length.*

We are finally in position to formulate the locality principle, which can be thought of as the requirement that finite, local information is needed in order to reconstruct the update rule.

Principle 2 (Locality). *The GUR $(G, A, \mathcal{V}^\dagger)$ is reducible to a local rule and decomposable into bounded regions.*

8 Cellular Automata

The principles of homogeneity and locality single out the special class of global rules that we will call *cellular automata*.

Definition 49. *Let $(G, A, \mathcal{V}^\dagger)$ be a global update rule obeying the principles of homogeneity and locality. We say that $(G, A, \mathcal{V}^\dagger)$ is a cellular automaton.*

Notice that the influence graph of a cellular automaton is the Cayley graph of a *finitely presented* group, i.e. a group that is finitely generated and can be presented through finitely many

relators. This fact has the very important consequence that every cellular automaton defines a Cayley graph that is quasi-isometric to some smooth Riemannian manifold of dimension $d \leq 4$ (see [47], sec. IV pag. 90).

An important remark is in order. Thanks to homogeneity, one can easily show that the local rule $\mathcal{S}'_g$ for a cellular automaton on the Cayley graph $\Gamma(G, S_+)$ has the property that $\mathcal{S}'_g = \mathcal{S}'_{e_G}$ for every $g \in G$. Thus, cellular automata can be identified by their Cayley graph $\Gamma(G, S_+)$ and by a reversible transformation $\mathcal{S}'_{e_G} \in \llbracket A_{N_{e_G}^+} A_{e_G} \rightarrow A_{N_{e_G}^+} A_{e_G} \rrbracket_1$.

8.1 Results

We now prove two important theorems regarding cellular automata. The first one shows that, in order to obey the properties of homogeneity and locality, a cellular automaton cannot be in the quasi-local algebra.

Theorem 11. *Let $|G| = \infty$. The homogeneous and local cellular automaton $\mathcal{V}$ is an element of $\llbracket A_G \rightarrow A_G \rrbracket_{Q1}$ if and only if it is trivial: $\mathcal{V} = \mathcal{I}_G$.*

Proof. Let $\mathcal{V}$ be in the quasi-local algebra, and $\mathcal{V}_{nR_n}$ be a sequence of local transformations in the class of $\mathcal{V}$. Then for $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$ one has

$$\|\mathcal{V}_{nR_n} - \mathcal{V}\|_{\sup} \leq \varepsilon.$$

Moreover, for $\pi \in \Pi_{\mathcal{V}}$, one has $\mathcal{T}_\pi \mathcal{V} \mathcal{T}_\pi^{-1} = \mathcal{V}$. Then

$$\|\mathcal{V}_{nR_n} - \mathcal{T}_\pi \mathcal{V} \mathcal{T}_\pi^{-1}\|_{\sup} \leq \varepsilon.$$

Since $\mathcal{T}_\pi$ is a GUR, it is isometric and thus

$$\begin{aligned} \|\mathcal{T}_\pi^{-1} \mathcal{V}_{nR_n} \mathcal{T}_\pi - \mathcal{V}\|_{\sup} \\ = \|\mathcal{V}_{nR_n} - \mathcal{T}_\pi \mathcal{V} \mathcal{T}_\pi^{-1}\|_{\sup} \leq \varepsilon. \end{aligned} \quad (115)$$

Now, this implies that

$$\|\mathcal{V}_{nR_n} - \mathcal{T}_\pi^{-1} \mathcal{V}_{nR_n} \mathcal{T}_\pi\|_{\sup} \leq 2\varepsilon.$$

It is easy to prove that, if $|G| = \infty$, for every $R \in \mathcal{R}^{(G)}$ there must exist $\pi \in \Pi_{\mathcal{V}}$ so that $R \cap \pi(R) = \emptyset$. Suitably choosing π , condition (115) then takes the form

$$\begin{aligned} \|\mathcal{W} \otimes \mathcal{I} - \mathcal{I} \otimes \mathcal{W}\|_{\sup} \\ = \|\mathcal{W} \otimes \mathcal{I} - \mathcal{W} \otimes \mathcal{W} + \mathcal{W} \otimes \mathcal{W} - \mathcal{I} \otimes \mathcal{W}\|_{\sup} \\ \leq 2\|\mathcal{W} - \mathcal{I}\|_{\sup} \leq \varepsilon. \end{aligned}$$

Being ε arbitrary, we must conclude that $\mathcal{W} = \mathcal{I}$, namely $\mathcal{V}_{nR_n} = 1_\emptyset$. $\square$

The second result pertains the following types of theory.

1. Theories with local discriminability [8, 9, 15, 58], or more generally every theory where the algebra of transformations $\mathcal{A} \in \llbracket A_1 \dots A_n \rightarrow A_1 \dots A_n \rrbracket$ is generated by local transformations, i.e. transformations of the form

$$\mathcal{A}_i \in \mathcal{I}_{\bar{A}_i} \otimes \llbracket A_i \rightarrow A_i \rrbracket,$$

where $\bar{A}_i$ is the composite system made of all $A_1 \dots A_n$ except from A_i . This class includes, among others, classical information theory and quantum theory.

2. Theories where the algebra of transformations $\mathcal{A} \in \llbracket A_1 \dots A_n \rightarrow A_1 \dots A_n \rrbracket$ is generated by bipartite transformations, i.e. transformations of the form

$$\mathcal{A}_{i,j} \in \mathcal{I}_{\bar{A}_i \bar{A}_j} \otimes \llbracket A_i A_j \rightarrow A_i A_j \rrbracket,$$

where $\bar{A}_i \bar{A}_j$ is the composite system made of all $A_1 \dots A_n$ except from A_i and A_j . This case includes the Fermionic theory as well as real quantum theory.

In the above cases, for groups G with suitable properties that we will immediately specify, the admissible local rules for a cellular automaton can be sought considering finite-dimensional systems. In the remainder we will then restrict to the families of theories introduced above.

Before proving this result, we need some preliminary definition and lemma.

Definition 50 (Quotient group). *Let G, H be two groups, and suppose that there exists a group homomorphism $\varphi : G \rightarrow H$. Then H is a quotient of G . If H is finite, it is called a finite quotient of G .*

The reason why we are interested in finite quotients is that, under suitable assumptions that will be made rigorous shortly, the same local rule defines an automaton on G and an automaton on any of its finite quotients H . This result makes it much easier to provide unitarity conditions in all those cases where the finite quotients of the

group G satisfy the mentioned requirements. We now explain the hypotheses in detail, and finally prove the main theorem.

We start remarking that if the group G has a finite presentation

$$G = \langle a_1, \dots, a_n | r_1, \dots, r_k \rangle,$$

it is straightforward to verify that any quotient H of G has a finite presentation of the form

$$H = \langle \varphi(a_1), \dots, \varphi(a_n) | \varphi(r_1), \dots, \varphi(r_k), b_1, \dots, b_j \rangle,$$

where the new relations b_k do not belong to the group R , conjugate closure of $\{\varphi(r_1), \dots, \varphi(r_k)\}$. When H is a quotient of G , referring to the presentation above, we will denote by R' the conjugate closure of the group generated by $\{\varphi(r_l)\}_{l=1}^k \cup \{b_m\}_{m=1}^j$. Moreover, the neighbourhoods of f in $\Gamma(H, \phi(S_+))$ will be denoted by $N_f^\pm$ for $f \in H$. We now state two lemmas that we use in the following. The proofs are provided in Appendix G

Lemma 66. *Let N^+ be a neighbourhood system corresponding to the Cayley graph $\Gamma(G, S_+)$ of a finitely generated group G , and let $H = \phi(G)$ be a finite quotient of G such that for every $h_a, h_b, h_c, h_d \in S_+$ one has $\phi(h_a h_b^{-1}) \in R'$, if and only if $h_a h_b^{-1} \in R$ and $\phi(h_a h_b^{-1} h_c h_d^{-1}) \in R'$ if and only if $h_a h_b^{-1} h_c h_d^{-1} \in R$.*

1. $N_{\phi(g)}^\pm = \phi(N_g^\pm)$, and $|N_{\phi(g)}^\pm| = |N_g^\pm|$.
2. $P_{\phi(g)}^\pm = \phi(P_g^\pm)$, and $|P_{\phi(g)}^\pm| = |P_g^\pm|$.
3. Given $f_1, f_2 \in H$, one can choose $g_i \in \phi^{-1}(f_i)$ such that $N_{f_1}^\pm \cap N_{f_2}^\pm = \phi(N_{g_1}^\pm \cap N_{g_2}^\pm)$ and $|N_{f_1}^\pm \cap N_{f_2}^\pm| = |N_{g_1}^\pm \cap N_{g_2}^\pm|$.
4. Given $f_1, f_2 \in H$, one can choose $g_i \in \phi^{-1}(f_i)$ such that $P_{f_2}^\mp \cap N_{f_1}^\pm = \phi(P_{g_2}^\mp \cap N_{g_1}^\pm)$ and $|P_{f_2}^\mp \cap N_{f_1}^\pm| = |\phi(P_{g_2}^\mp \cap N_{g_1}^\pm)|$.
5. If $g_2 \notin N_{g_1}^\pm$ and $\phi(g_2) \in N_{\phi(g_1)}^\pm$, it is $P_{g_1}^\pm \cap N_{g_2}^\mp = \emptyset$.

Lemma 67. *In addition to the hypotheses of lemma 66, let $\phi(h_a h_b^{-1} h_c h_d^{-1} h_e h_f^{-1}) \in R'$ iff $h_a h_b^{-1} h_c h_d^{-1} h_e h_f^{-1} \in R$ for every $h_a, h_b, h_c, h_d, h_e, h_f \in S_+$. Then*

1. Given $f_1, f_2 \in H$ with $f_2 \in P_{f_1}^\pm$, one has $P_{f_1}^\pm \cap P_{f_2}^\pm = \phi(P_{g_1}^\pm \cap P_{g_2}^\pm)$, for some $g_i \in \phi^{-1}(f_i)$, with $g_2 \in P_{g_1}^\pm$, and $|P_{f_1}^\pm \cap P_{f_2}^\pm| = |P_{g_1}^\pm \cap P_{g_2}^\pm|$.

2. Let $g_2 \notin P_{g_1}^\pm$ and $\phi(g_2) \in P_{\phi(g_1)}^\pm$. Then $N_{g_1}^\pm \cap N_{g_2}^\pm = P_{g_1}^\pm \cap P_{g_2}^\pm = \emptyset$.

Corollary 18. Under the hypotheses of lemma 66, the homomorphism ϕ is invertible on $N_g^\pm$ and $P_g^\pm$ for every $g \in G$.

In order to make the hypotheses of lemmas 66 and 67 clearer, in Fig. 2 we show two examples of finite quotients of the group $\mathbb{Z}^2$, the first one satisfying the hypotheses of lemma 66 but not those of lemma 67, and the second one satisfying both.

Figure 2: (a) The Cayley graph of $\mathbb{Z}^2$ presented as $\langle a, b | aba^{-1}b^{-1} \rangle$; (b) the Cayley graph of the finite quotient of $\mathbb{Z}^2$ $\langle \phi(a), \phi(b) | \phi(aba^{-1}b^{-1}), \phi(a)^6, \phi(b)^5 \rangle$, satisfying the hypotheses of lemma 66 but not those of 67; (c) the Cayley graph of the finite quotient of $\mathbb{Z}^2$ $\langle \phi(a), \phi(b) | \phi(aba^{-1}b^{-1}), \phi(a)^8, \phi(b)^7 \rangle$, satisfying the hypotheses of both lemmas 66 and 67. Notice that we implicitly assume that S_+ contains also a^{-1} and b^{-1} , but we do not express the presentation accordingly, to keep the formulas readable. In principle there should be two extra generators c, d (and $\phi(c), \phi(d)$ in the finite quotients), along with the relations ac and bd (and $\phi(ac), \phi(bd)$, accordingly).

We can now prove the following results, which are the core of the final theorem at the end of the section.

Lemma 68. For theories of type 1, under the hypotheses of lemma 66, given a homogeneous

local rule $(N^+, A, \mathcal{S}')$ with neighbourhood system N^+ such that $|N_{e_G}^+| = |S_+|$, the local rule $(N'^+, A, \tilde{\mathcal{S}}')$, with $\tilde{\mathcal{S}}'_{e_H} := \mathcal{S}'_{e_G}$ is well-defined on the Cayley graph $\Gamma(H, \phi(S_+))$, with $N'_{\phi(g)} = \phi(N_g^+)$. Viceversa, given a local rule $(N'^+, A, \tilde{\mathcal{S}}')$ whose neighbourhood system N'^+ corresponds to a Cayley graph $\Gamma(H, \phi(S_+))$, the local rule $(N^+, A, \mathcal{S}')$, with $\mathcal{S}'_e := \tilde{\mathcal{S}}'_e$ is well-defined on the Cayley graph $\Gamma(G, S_+)$, with $\phi(N_g^+) = N'_{\phi(g)}$.

Proof. We remark that, by definition of the maps $\mathcal{S}'_g$ (see Definition 46) a given choice of $\mathcal{S}'_{e_H}$ is in principle compatible with every Cayley graph with $|N'_{e_H}| = |N_{e_G}^+|$, i.e. the same number of generators (we remind that $\mathcal{S}'_g = \mathcal{S}'_{e_H} \in \llbracket A_{N_{e_G}^+} A_{e_G} \rightarrow A_{N_{e_G}^+} A_{e_G} \rrbracket_1$). By lemma 66, the cardinality of N'_{e_H} is the same as that of $N_{e_G}^+$. Then $\tilde{\mathcal{S}}'_{e_H} = \mathcal{S}'_{e_G}$ is well defined on $\Gamma(H, \phi(S_+))$ iff it is well defined on $\Gamma(G, S_+)$. What remains to be proved is that the map $\tilde{\mathcal{S}}'_{e_H} = \mathcal{S}'_{e_G}$ identifies a local rule on the system of neighbourhoods N'^+ —namely that Eq. (110) as well as items 1 and 2 in definition 46 hold—iff it does on the system of neighbourhoods N^+ . Item 1 is satisfied by $\tilde{\mathcal{S}}' = \mathcal{S}'$ by definition in both cases. As to item 2, it is trivial to verify that the only important feature in order to decide whether it holds in any neighbourhood system N'^+ is the cardinality of $N'_{e_H} \cap N'_f$ (or $N_{e_G}^+ \cap N_g^+$) for every $f \in P'_{e_H}$ (or $g \in P_{e_G}^+$), since this identifies the systems where both $\tilde{\mathcal{S}}'_{e_H} = \mathcal{S}'_{e_G}$ and $\tilde{\mathcal{S}}'_f = \mathcal{S}'_g$ act non-trivially, namely those where the products $\tilde{\mathcal{S}}'_{e_H} \tilde{\mathcal{S}}'_f$ or $\mathcal{S}'_{e_G} \mathcal{S}'_g$ might not be commutative. Lemma 66 thus ensures that item 2 is satisfied by $\tilde{\mathcal{S}}'$ iff it is by $\mathcal{S}'$. Let us then focus on Eq. (110). For theories of type 1, it is sufficient to verify that Eq. (110) holds for $|R| = 1$ and $|S| = 0$ or $|R| = 0$ and $|S| = 1$. Both conditions easily follow if we consider lemma 66. $\square$

Lemma 69. For theories of type 2, under the hypotheses of lemma 67, given a homogeneous local rule $(N^+, A, \mathcal{S}')$ with neighbourhood system N^+ such that $|N_{e_G}^+| = |S_+|$, the local rule $(N'^+, A, \tilde{\mathcal{S}}')$, with $\tilde{\mathcal{S}}'_{e_H} := \mathcal{S}'_{e_G}$ is well-defined on the Cayley graph $\Gamma(H, \phi(S_+))$, with $N'_{\phi(g)} = \phi(N_g^+)$. Viceversa, given a local rule $(N'^+, A, \tilde{\mathcal{S}}')$ whose neighbourhood system N'^+ corresponds to a Cayley graph $\Gamma(H, \phi(S_+))$, the local rule $(N^+, A, \mathcal{S}')$, with $\mathcal{S}'_e := \tilde{\mathcal{S}}'_e$ is

well-defined on the Cayley graph $\Gamma(G, S_+)$, with $\phi(N_g^+) = N_{\phi(g)}^+$

Proof. The first part of the proof proceeds exactly as that of lemma 68. The non trivial part of the proof regards the equivalence of Eq. (110) for both $\mathcal{S}'$ on G and $\mathcal{S}'$ on H . Suppose that $\mathcal{S}'$ defines a local rule for the neighbourhood system N_g^+ . In theories of type 2, it is sufficient to verify that Eq. (110) holds for regions with $|R| = 2$, $|S| = 0$, or $|R| = 1$, $|S| = 1$, or $|R| = 0$, $|S| = 2$. Let then $\tilde{R}, S \subseteq \Gamma(H, \phi(S_+))$, with $R = \phi(\tilde{R})$ and $S = \phi(\tilde{S})$. In the first case we have $R = \{f_1, f_2\}$, with $\tilde{R} = \{g_1, g_2\}$ and $f_i = \phi(g_i)$, while $S = \emptyset$. The relevant informations in order to verify Eq. (110) are the cardinalities i) $|N_{f_1}^- \cap N_{f_2}^-|$ and ii) $|P_{f_1}^- \cap P_{f_2}^-|$. If $f_2 \in P_{f_1}^-$, by lemma 67 we know that we can choose g_1 and g_2 so that $|N_{f_1}^- \cap N_{f_2}^-| = |N_{g_1}^- \cap N_{g_2}^-|$ and $|P_{f_1}^- \cap P_{f_2}^-| = |P_{g_1}^- \cap P_{g_2}^-|$. Thus, since Eq. (110) holds for $\mathcal{C} \in \llbracket A_{\tilde{R}}^{(G)} \rightarrow A_{\tilde{R}}^{(G)} \rrbracket$, and

$$A_{\tilde{R}}^{(G)} \cong A_{\tilde{R}}^{(H)}, \quad \tilde{\mathcal{S}}' = \mathcal{S}',$$

we conclude that Eq. (110) holds also for $\mathcal{C} \in \llbracket A_R^{(H)} \rightarrow A_R^{(H)} \rrbracket$. On the other hand, if $f_2 \notin P_{f_1}^-$, we can calculate $\tilde{\mathcal{S}}'_{N_{f_1}, f_2} \mathcal{C} \tilde{\mathcal{S}}'_{N_{f_1}, f_2}$ in two steps. First we apply $\tilde{\mathcal{S}}'_{N_{f_1}} = \mathcal{S}'_{N_{g_1}^-}$, treating the system f_2 as an external system. Reminding that $f_2 \notin P_{f_1}^-$, it is $N_{f_1}^- \cap N_{f_2}^- = \emptyset$. By Eq. (110), that holds for $\mathcal{S}'_{N_{g_1}^-}$ we then have

$$\begin{aligned} & \begin{array}{c} \text{---} (P_{f_2}^- \setminus [P_{f_1}^- \cup f_2], 0) \text{---} \\ \text{---} (f_2, 0) \text{---} \end{array} \\ & \begin{array}{c} \text{---} (P_{f_1}^-, 0) \text{---} \end{array} \begin{array}{c} \text{---} (f_1, 0) \text{---} \end{array} \mathcal{C} \begin{array}{c} \text{---} (f_1, 0) \text{---} \end{array} \begin{array}{c} \text{---} (P_{f_1}^-, 0) \text{---} \end{array} \\ & \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \\ & \text{---} (P_{f_1}^- \cup P_{f_2}^- \setminus f_2, 0) \text{---} \\ & = \begin{array}{c} \text{---} (f_2, 0) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \mathcal{C}' \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_2}^-, 1) \text{---} \end{array} \end{aligned}$$

Then, we apply $\tilde{\mathcal{S}}'_{N_{f_2}^-}$, obtaining

$$\begin{aligned} & \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (f_2, 0) \text{---} \end{array} \mathcal{C}' \begin{array}{c} \text{---} (f_2, 0) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \\ & \begin{array}{c} \text{---} (N_{f_2}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (P_{f_2}^- \setminus f_2, 0) \text{---} \end{array} \begin{array}{c} \text{---} (P_{f_1}^-, 0) \text{---} \end{array} \\ & \text{---} (P_{f_2}^- \setminus [P_{f_1}^- \cup f_2], 0) \text{---} \\ & = \begin{array}{c} \text{---} (P_{f_1}^- \cup P_{f_2}^-, 0) \text{---} \end{array} \\ & = \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_1}^-, 1) \text{---} \end{array} \mathcal{C}'' \begin{array}{c} \text{---} (N_{f_2}^-, 1) \text{---} \end{array} \begin{array}{c} \text{---} (N_{f_2}^-, 1) \text{---} \end{array}, \quad (116) \end{aligned}$$

where on the l.h.s. we reversed the ordering of the wires for the sake of simplicity of the diagram. Overall, considering that $\tilde{\mathcal{S}}'_{N_{f_1}^- \cup N_{f_2}^-} = \tilde{\mathcal{S}}'_{N_{f_1}^-} \tilde{\mathcal{S}}'_{N_{f_2}^-} = \tilde{\mathcal{S}}'_{N_{f_2}^-} \tilde{\mathcal{S}}'_{N_{f_1}^-}$, we then proved Eq. (110) for the local rule $\tilde{\mathcal{S}}'_f$ on the neighbourhood scheme given by the graph $\Gamma(H, \phi(S_+))$. In the second case, let $R = f_1 = \phi(g_1)$, $S = f_2 = \phi(g_2)$. The informations that matter are whether one can choose g_1, g_2 so that $f_2 \in N_{f_1}^-$ iff $g_2 \in N_{g_1}^-$, and $|P_{f_1}^- \cap N_{f_2}^+| = |P_{g_1}^- \cap N_{g_2}^+|$. Again by lemma 66, we know that the answer is positive. Then, also for $|R| = |S| = 1$, since Eq. (110) holds for $\mathcal{C} \in \llbracket A_{(\tilde{R}, 0) \cup (\tilde{S}, 1)}^{(G)} \rightarrow A_{(\tilde{R}, 0) \cup (\tilde{S}, 1)}^{(G)} \rrbracket$, and

$$A_{(\tilde{R}, 0) \cup (\tilde{S}, 1)}^{(G)} \cong A_{(R, 0) \cup (S, 1)}^{(H)}, \quad \tilde{\mathcal{S}}' = \mathcal{S}',$$

we conclude that Eq. (110) holds also for $\mathcal{C} \in \llbracket A_{(R, 0) \cup (S, 1)}^{(H)} \rightarrow A_{(R, 0) \cup (S, 1)}^{(H)} \rrbracket$. The argument for $|R| = 0$, $|S| = 2$ is the same as for $|R| = 2$, $|S| = 0$. For the converse, one can follow the same argument as for the direct statement. The only differences are that in the case $|R| = 2$ and $|S| = 0$ one might have $g_2 \notin P_{g_1}^-$ while $f_2 = \phi(g_2) \in \phi(P_{g_1}^-) = P_{f_2}^-$, and in the case $|R| = |S| = 1$ it might happen that $g_2 \notin N_{g_1}^-$ while $f_2 \in N_{f_1}^-$. By lemma 67, the first case can occur only if $N_{g_1}^+ \cap N_{g_2}^+ = P_{g_1}^- \cap P_{g_2}^- = \emptyset$, while by lemma 66 the second case occurs only if $P_{g_1}^- \cap N_{g_2}^+ = \emptyset$. In both cases, the different structure of the neighbourhoods between G and H does not preclude the derivation of Eq. (110) in the case of G from the same identity for the case of H , as one can immediately realise by direct inspection of Eq. (110). $\square$

Theorem 12 (Wrapping lemma). *For a theory of type 1, under the hypotheses of lemma 66, the local rule $(N^+, A, \mathcal{S}')$ defines a cellular automaton $(G, A, \mathcal{V})$ on the Cayley graph $\Gamma(G, S_+)$ if and only if the same local rule defines a cellular automaton $(H, A, \mathcal{W})$ on the Cayley graph $\Gamma(H, \phi(S_+))$. The same is true of a theory of type 2, under the hypotheses of lemma 67.*

Proof. The thesis now easily follows from lemmas 68 and 69, and theorem 10. $\square$

The above result allows us, at least in theories of the two types considered so far, and for CAs on Cayley graphs of groups with finite quotients satisfying the hypotheses of lemmas 66 or 67, to reduce the classification of CAs on graphs of infinite groups to that of CAs on finite groups, which is in principle a much easier task. Moreover, once a suitable finite quotient is found, one can simulate the evolution of a finite region and for a finite number of steps considering the same evolution on the finite *wrapped* graph instead of having to consider the actual infinite graph.

9 Examples

9.1 Classical case

Classical systems can be classified in terms of the number d of pure states ρ_y , $y = 1, \dots, d$ which coincides with the dimension of the state space $d = \dim[\mathbb{d}]_{\mathbb{R}}$. All the pure states of a classical system are jointly discriminable by atomic effects a_x , $x = 1, \dots, d$, i.e. $(a_x | \rho_y) = \delta_{xy}$. For the sake of simplicity, we will restrict attention to classical *computation*, i.e. the sub-theory of classical theory where there is only one kind of non-trivial elementary system, the *bit* having $d = 2$, and all other systems are composite systems of n bits, having dimension $d = 2^n$. Every effect in $[\bar{A}_G]_Q$ for an infinite system $|G| = \infty$ can be decomposed as a series of effects corresponding to finite, arbitrarily large bit strings b_R , with

$$b_R := [a_R], \quad a = a_{b_{r_1}} \otimes a_{b_{r_2}} \otimes \dots \otimes a_{b_{r_{|R|}}},$$

with $R = \{r_1, r_2, \dots, r_{|R|}\}$. In this case, let b_R denote a local effect. A global rule $\mathcal{V}$ can be specified by the input/output relation $\mathcal{V}^\dagger b_R = b'_{R'}$. However, in the extensive literature about classical cellular automata—see e.g. [59] for reference,

though we cannot even try to provide an exhaustive account of the relevant bibliography—it is customary to express $\mathcal{V}$ through its local rule, which is specified by its action on local states. The only point we want to discuss here is the apparent clash of our general wrapping lemma with results in the literature. In particular, in Ref. [31] a counterexample to the classical version of the wrapping lemma is provided, referring to [48]. The local rule for the counterexample is the following. Let $G = \mathbb{Z}_r$ with $r \notin 3\mathbb{Z}$. The rule defined by

$$b'_j = b_{j \ominus_r 1} \oplus b_j \oplus b_{j \oplus_r 1} \quad (117)$$

What we claim here is that, according to our notion of neighbourhood based on definition 36, the above automaton has a neighbourhood system where N_i^+ is actually much larger than $\{j \ominus_r 1, j, j \oplus_r 1\}$. Indeed, if one considers the action of $\mathcal{V}$ on the algebra of transformations instead of its action on states, one can easily realise that $N_i^+ = G$ for every i . However, by checking the conditions for no-signalling, it is easy to realise that the rule in Eq. (117) only signals to sites $i \ominus 1$, i , and $i \oplus 1$. Thus, while no-signalling is necessary for no causal influence, the converse is not true.

We remark that the definition of causal influence that we give, based on transformations, is inspired by the quantum definition, which indeed involves the generators of the local algebra, and thus the Kraus representatives of transformations rather than the mere set of local states. In this respect, the notion of neighbourhood used in the literature on CA is different from the quantum one, and we point to this relevant difference as the responsible for the apparent incompatibility discussed in Ref. [31]. The subject was extensively studied in Ref. [60] and later in Ref. [61], where the authors maintain the notion of neighbourhood for classical CA based on the signalling condition, and prove a bound on the mismatch between the classical neighbourhood of a given CA and the neighbourhood of the quantum version of the same CA.

9.2 Quantum case

Quantum cellular automata represent now a widely studied field of research. After the very general analysis of Refs. [31, 32, 33, 61], there are various results in the literature about quantum

cellular automata. Here we cite the very general classification of Ref. [62, 63], while for extensive reviews we refer to Refs. [64, 65].

9.3 Fermionic case

For the presentation of fermionic theory as an OPT we refer to [36, 37]. The Fermionic systems are fully specified by the algebra $\mathcal{A}$ generated by local field operators ψ_i . In fact, $\mathcal{A}$ is the algebra of field operator polynomials. This algebra is $\mathbb{Z}_2$ -graded, with two modules $\mathcal{A}_E$ and $\mathcal{A}_O$, given by the span of even and odd polynomials, respectively. No combination of even and odd polynomials is allowed. As in all quantum theories, $\mathcal{A}$ provides a simplified representation in terms of Kraus operators of the algebra of transformations, which is not graded, but rather reducible to a direct sum. Linear combinations of transformations indeed do not correspond to linear combinations of the corresponding Kraus operators. The local systems for a site g of a cellular automaton in this case are registers made of one or more local Fermionic modes. Local field operators are then labelled $\psi_{i,g}$, where $g \in G$ and $i \in J_g = \{1, \dots, n_g\}$, $|J_g|$ representing the size of the register at site g . Since the local algebra $\mathcal{A}_g$ is finitely generated, the locality requirement implies that $\mathcal{V}$ is given by a local rule, which in turn is completely specified by its action on the local generators $\psi_{i,g}$ for every $g \in G$. Thus, one has

$$\begin{aligned} \mathcal{V}(\psi_{i,f}) &= \sum_{\substack{\mathbf{g} \in (N_f^+)^{\times k} \\ \mathbf{j} \in J_{\mathbf{g}} \\ \mathbf{s}, \mathbf{t} \in \{0,1\}^*}} T_{i,f}^{(\mathbf{g}, \mathbf{j}, \mathbf{s}, \mathbf{t})} \psi_{j_1, g_1}^{s_{j_1}} \psi_{j_1, g_1}^{\dagger t_{j_1}} \cdots \psi_{j_k, g_k}^{s_{j_k}} \psi_{j_k, g_k}^{\dagger t_{j_k}}, \end{aligned}$$

where $J_{\mathbf{g}} := \times_{g \in N_f^+} J_g$. Most of the automata studied so far in the literature are *linear*, namely

$$\mathcal{V}(\psi_{i,f}) = \sum_{\substack{g \in N_f^+ \\ j \in J_g}} T_{i,f}^{j, g'} \psi_{j, g}.$$

Remarkable exceptions are represented by Refs. [44, 66, 45]. In particular, in Ref. [45] the full classification of Fermionic cellular automata with $|J_g| = 1$ on Cayley graphs of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}$ was carried out.

As regards Fermionic cellular automata, we would like to remark that, despite Fermionic

theory being of type 2, they satisfy the wrapping lemma in the form of lemma 68, instead of lemma 69. This is due to the fact that Fermionic transformations, like quantum transformations, admit a Kraus decomposition, and can thus be represented through Kraus operators instead of completely positive maps. It turns out that local Fermionic field operators $\psi_{j,g}$ and $\psi_{j,g}^\dagger$ provide a basis for the full algebra of Fermionic operators. Thus, in the Fermionic case, if Eq. (110) is satisfied for local transformations, it is always satisfied. Real quantum theory, which shares many features with Fermionic theory, is different in this respect, and we conjecture that a counterexample to the thesis of lemma 68 can be found in the latter case.

10 Conclusion

Summarising the content of the manuscript, we defined the composition of denumerably many systems from an OPT, starting from the space of quasi-local effects, and defining quasi-local transformations. We then define states of such system in terms of suitable functionals on the space of effects. Among these, there are some that can be interpreted as the result of local preparations, and that we deem quasi-local states.

We defined global update rules for finite or infinite systems, and proved a generalisation of a block-decomposition theorem for GURs. We then analysed those GURs that are homogeneous. The definition of homogeneity is a big chapter on its own, and deserves a careful treatment. As a consequence of the definition, we proved that for homogeneous GURs the memory array is organised as the Cayley graph of a group. We then defined locality for GURs, and studied its consequences in detail.

In the last section we used homogeneity and locality to define cellular automata, and proved the wrapping lemma under very general assumptions.

The theory developed here is of fundamental importance for an approach to physical laws as information-processing algorithms, and will be used for the construction of statistical mechanics in post-quantum theories, as well as the reconstruction of dynamical laws in space-time beyond the now established approach based on quantum walks, thus encompassing interacting quantum and post-quantum field theories.

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A Identification of the sup- and operational norm for effects

We show that, under the hypothesis of assumption (16), one has $\|a\|_{\text{op}} = \|a\|_{\text{sup}}$ for every $a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}}$. Indeed, if $\llbracket \bar{A} \rrbracket_+ = \llbracket A \rrbracket_+^*$, then, by theorem 5, one has

$$\llbracket \bar{A} \rrbracket = \{a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}} \mid 0 \leq (a|\rho), (e - a|\rho), \forall \rho \in \llbracket A \rrbracket\}. \quad (118)$$

First of all, one can rephrase the above condition as

$$\llbracket \bar{A} \rrbracket = \{a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}} \mid 0 \leq (a|\rho), (e - a|\rho), \forall \rho \in \llbracket A \rrbracket_1\}. \quad (119)$$

Indeed, since $\llbracket A \rrbracket_1 \subseteq \llbracket A \rrbracket$, the set on the l.h.s. in Eq. (118) is clearly a subset of that on the l.h.s. in Eq. (119). Moreover, since every state $\rho \in \llbracket A \rrbracket$ is proportional by a constant $\kappa \in [0, 1]$ to a deterministic one $\rho_0 \in \llbracket A \rrbracket_1$ (see remark 1), if $0 \leq (a|\sigma), (e - a|\sigma)$ for all $\sigma \in \llbracket A \rrbracket_1$, then $0 \leq (a|\rho), (e - a|\rho)$ for all $\rho \in \llbracket A \rrbracket$. Then, by the identity in Eq. (119), one can identify the set $\llbracket \bar{A} \rrbracket$ as the subset of $\llbracket \bar{A} \rrbracket_{\mathbb{R}}$ containing only those functionals a on $\llbracket A \rrbracket_{\mathbb{R}}$ such that $0 \leq (a|\rho)$ and $0 \leq 1 - (a|\rho)$ for $\rho \in \llbracket A \rrbracket_1$, and, in turn, the above condition is equivalent to

$$\llbracket \bar{A} \rrbracket = \{a \in \llbracket \bar{A} \rrbracket_{\mathbb{R}} \mid 0 \leq (a|\rho) \leq 1, \forall \rho \in \llbracket A \rrbracket\}.$$

Now, we remind that $\lambda \in J(a)$ if and only if $\lambda e \pm a \succeq 0$. Under our hypotheses, this means

that $\lambda \pm (a|\rho) \geq 0$ for every $\rho \in \llbracket A \rrbracket$. Thus, $|(a|\rho)| \leq \lambda$ for every $\rho \in \llbracket A \rrbracket$. Finally, we can conclude that $\|a\|_{\text{op}} \leq \|a\|_{\text{sup}}$. On the other hand, if $\|a\|_{\text{op}} < \|a\|_{\text{sup}}$, then there exists $\varepsilon > 0$ such that for every $\rho \in \llbracket A \rrbracket$ one has $|(a|\rho)| \leq \|a\|_{\text{sup}} - \varepsilon$. In particular, this is true for $\rho_1 \in \llbracket A \rrbracket_1$. Consequently

$$|(a|\rho)| = \kappa |(a|\rho_1)| \leq \kappa (\|a\|_{\text{sup}} - \varepsilon) \quad \forall \rho \in \llbracket A \rrbracket,$$

where $0 \leq \kappa \leq 1$ and $\rho = \kappa \rho_1$. Then $(e|\rho) = \kappa$, and

$$(\{\|a\|_{\text{sup}} - \varepsilon\}e \pm a|\rho) \geq 0, \quad \forall \rho \in \llbracket A \rrbracket.$$

Finally, this implies that $\|a\|_{\text{sup}} - \varepsilon \in J(a)$, which is absurd. Then, $\|a\|_{\text{sup}} = \|a\|_{\text{op}}$.

B Quasi-local states

We start considering arbitrary finite disjoint partitions of the system A_G . Let $B := \{B_i\}_{i=1}^{\infty}$ denote a disjoint partition of the set G into finite blocks, i.e. $0 < |B_i| < \infty$ for every $i \in \mathbb{N}$. The set of such partitions (quotiented by the irrelevant ordering of regions) is denoted by $\mathcal{B}$.

Definition 51. For a given partition $B \in \mathcal{B}$, we define the set of finite B -regions of G as

$$\mathcal{R}^{(G,B)} := \left\{ R \subseteq G \mid R = \bigcup_{j=1}^k B_{i_j}; \quad 0 \leq k < \infty \right\}, \quad (120)$$

and the set of B -regions of G as

$$\bar{\mathcal{R}}^{(G,B)} := \left\{ R \subseteq G \mid R = \bigcup_{j=1}^k B_{i_j}; \quad 0 \leq k \leq \infty \right\}, \quad (121)$$

where for $k = 0$ we define

$$\bigcup_{j=1}^0 B_{i_j} := \emptyset.$$

Given $R = \bigcup_{j=1}^k B_{i_j}$ in $\bar{\mathcal{R}}^{(G,B)}$, let $\mathbb{N}_R^{(B)} := \{i_j\}_{j=1}^k$.

Remark 10. Clearly, $\mathcal{R}^{(G,B)} \subseteq \bar{\mathcal{R}}^{(G,B)}$. It is also trivial to prove that $\bar{\mathcal{R}}^{(G,B)}$ and $\mathcal{R}^{(G,B)}$ are closed under $\cup, \cap, \setminus$.

Given a finite region $R \in \mathbf{R}^{(G)}$, for any partition $B \in \mathbf{B}$ it is possible to cover R with regions of B .

Definition 52. Given $B \in \mathbf{B}$, the cover in B of a region $R \subseteq G$, denoted $B(R)$, is the minimal B -region $C = \bigcup_{j=1}^k B_{i_j}$ such that $R \subseteq C$.

The above notion of cover is well defined, as proved by the following lemma.

Lemma 70. Let $R \subseteq G$. The cover $B(R)$ in B of R exists and is unique.

Proof. First of all, the set of B -regions $S \in \bar{\mathbf{R}}^{(G,B)}$ such that $R \subseteq S$ is not empty, since $R \subseteq G \in \bar{\mathbf{R}}^{(G,B)}$. We just need to prove that, having defined $\mathbf{C}(R, B) := \{S \in \bar{\mathbf{R}}^{(G,B)} \mid R \subseteq S\}$, there is always a unique minimal element $B(R) \in \mathbf{C}(R, B)$, i.e. such that no B -region strictly contained in $B(R)$ covers R . As to existence, let $h \in R$. Then $h \in B_{l_h}$ for some l_h . One can easily realise that $R \subseteq C := \bigcup_{h \in R} B_{l_h}$. Moreover, if we remove a given B -region B_{l_h} from C , and $h \in B_{l_h}$, then $h \notin C \setminus B_{l_h}$, thus C is a minimal cover of R in B . As to uniqueness, suppose that there are two different minimal covers of R in B , say C_1 and C_2 . In this case, it is not restrictive to suppose that there is $h \in G$ such that $h \in C_1$ and $h \notin C_2$. Let $B_i \in B$ be the set such that $h \in B_i$. One has $B_i \subseteq C_1$ and $B_i \cap C_2 = \emptyset$. Since $R \subseteq C_2$, we can conclude that $R \cap B_i = \emptyset$, and thus $R \subseteq (C_1 \setminus B_i) \in \mathbf{C}(R, B)$. Thus, C_1 cannot be minimal in $\mathbf{C}(R, B)$. $\square$

The cover $B(R)$ can also be thought of as the intersection of all covers of R in B .

Lemma 71. The cover $B(R)$ of a finite region R is finite.

Proof. Every element $g_k \in R$ belongs to some $B_{i_k} \subseteq B$, where $k \mapsto i_k$ is generally not injective. It is then clear that $R \subseteq \bigcup_{k=1}^{|R|} B_{i_k}$, and thus $B(R) \subseteq \bigcup_{k=1}^{|R|} B_{i_k}$. $\square$

Given one partition $B \in \mathbf{B}$, we can now define the set of states that differ on finitely many blocks from a given assignment of states of blocks in B . First, we define the reference state as follows.

Definition 53 (Reference local state). Given a partition $B \in \mathbf{B}$, a reference local state over B is a map

$$\rho_0 : \mathbb{N} \rightarrow \bigsqcup_{j \in \mathbb{N}} [\![A_{B_j}]\!]_1 :: j \mapsto \rho_{0j} \in [\![A_{B_j}]\!]_1.$$

A reference local state allows one to attribute a state to any finite region $R \in \mathbf{R}^{(G)}$ by the following procedure. First, let us consider the B -cover $B(R)$ of R . Then, the state of $B(R)$ is given by

$$\rho_0[B(R)] := \bigotimes_{i_j \in \mathbb{N}_{B(R)}^{(B)}} \rho_{0i_j}.$$

Finally, we discard all the extra systems in the region $B(R) \setminus R$.

$$\rho_{0|R} := \rho_0[B(R)]|_R.$$

The above construction is then formalised in the following definition.

Definition 54. The state of a finite region $R \in \mathbf{R}^{(G,B)}$ in the reference local state ρ_0 is defined as

$$\rho_{0|R} \text{---} R := \left(\rho_0[B(R)] \text{---} \frac{R}{B(R) \setminus R} \text{---} e \right),$$

$$\rho_{0|\emptyset} := 1.$$

Given a partition B and a reference local state ρ_0 , let us now define the set of primitive local states as follows.

Definition 55. The set of primitive local states is

$$\text{Pre}[\![A_G]\!]_{\mathbb{R}}^{(B)} := \{(\sigma, R) \mid \sigma \in [\![A_R]\!]_{\mathbb{R}}; R \in \mathbf{R}^{(G,B)}\}.$$

This set collects all the generalised local preparations of finite regions of B . As in the case of effects, we introduce the simplified notation $\sigma_S := (\sigma, S)$. A choice of reference local state ρ_0 , allows us to interpret a given element σ_S of $\text{Pre}[\![A_G]\!]_{\mathbb{R}}^{(B)}$ as a preparation of the region S in the state σ . Thus, the element σ_S allows one to attribute a generalised state $\tau \in [\![A_T]\!]_{\mathbb{R}}$ to any region $T \in \mathbf{R}^{(G)}$ as follows (see fig. 3)

$$\tau := \sigma_{|S \cap T} \otimes \rho_{0|(T \setminus S)}. \quad (122)$$

Definition 56. Given a reference local state ρ_0 over B we define the following equivalence relation in $\text{Pre}[\![A_G]\!]_{\mathbb{R}}^{(B)}$

$$\sigma_S \sim_{\rho_0} \tau_T \Leftrightarrow \begin{cases} \sigma = \nu \otimes \rho_{0|(S \setminus T)}, \\ \tau = \nu \otimes \rho_{0|(T \setminus S)}, \end{cases} \quad (123)$$

for some $\nu \in [\![A_{S \cap T}]\!]_{\mathbb{R}}$.

Figure 3: An illustrative example of a reference local state ρ_0 (a) for a partition of a finite set G into 11 regions, along with a primitive local state σ_R (b) with $R = B_4 \cup B_5 \cup B_7 \cup B_8$. The dots represent the elements of G .

The symbol $\sigma_{S\rho_0}$ will denote the equivalence class of $\sigma_S \in \text{Pre}[\mathbb{A}_G]_{\mathbb{R}}^{(B)}$ under the relation $\sim_{\rho_0}$. Notice that in Eq. (123), thanks to closure of $\mathbb{R}^{(G,B)}$ under set difference, one has $(S \setminus T), (T \setminus S) \in \mathbb{R}^{(G,B)}$, and thus the definition of the equivalence relation $\sim_{\rho_0}$ involves exclusively B -regions. Moreover, all the regions involved in the definition are finite.

Lemma 72. *Let $\sigma_S \in \text{Pre}[\mathbb{A}_G]_{\mathbb{R}}^{(B)}$. Then for every $R \in \mathbb{R}^{(G,B)}$ such that $R \cap S = \emptyset$ one has $(\sigma \otimes \rho_{0|R})_{S \cup R} \in \text{Pre}[\mathbb{A}_G]_{\mathbb{R}}^{(B)}$ and $(\sigma \otimes \rho_{0|R})_{S \cup R} \sim_{\rho_0} \sigma_S$.*

Proof. First of all, as $\mathbb{R}^{(G,B)}$ is closed under union, clearly $S \cup R \in \mathbb{R}^{(G,B)}$, and thus $(\sigma \otimes \rho_{0|R})_{S \cup R} \in \text{Pre}[\mathbb{A}_G]_{\mathbb{R}}^{(B)}$. Now, $S \cap (S \cup R) = S$, and thus the condition in Eq. (123) is satisfied for $T = S \cup R$, $T \setminus S = R$, $S \cap T = S$, $\nu = \sigma$, $\rho_{0|(T \setminus S)} = \rho_{0|R}$, and $\rho_{0|(S \setminus T)} = 1$. $\square$

We now provide a way to identify a canonical representative of the equivalence class $\sigma_{R\rho_0}$ defined as follows.

Definition 57. *The minimal representative $\tilde{\sigma}_{R\sigma}$ of the equivalence class $\sigma_{R\rho_0}$ is defined through*

$$R_{\sigma} := \bigcap_{S \in \mathbb{R}_{(\sigma, R)}} S, \quad \tilde{\sigma}_{R\sigma} \sim_{\rho_0} \sigma_R, \quad (124)$$

where

$$\mathbb{R}_{(\sigma, R)} := \{S \in \mathbb{R}^{(G,B)} \mid \exists \tau \in [\mathbb{A}_S]_{\mathbb{R}} : \tau_S \sim_{\rho_0} \sigma_R\}.$$

Lemma 73. *The minimal representative exists and is unique.*

For the proof, see the analogous lemma 9.

Definition 58. *The set of generalised local states over B upon ρ_0 is*

$$[\mathbb{A}_G]_{\rho_0 L\mathbb{R}}^{(B)} := \text{Pre}[\mathbb{A}_G]_{\mathbb{R}}^{(B)} / \sim_{\rho_0}. \quad (125)$$

Its elements will be denoted by symbols like $\rho_{S\rho_0}$.

We now define linear combinations of local states.

Definition 59. *Let $\sigma_{S\rho_0}, \tau_{T\rho_0} \in [\mathbb{A}_G]_{\rho_0 L\mathbb{R}}^{(B)}$, and $a \in \mathbb{R}$. Then we have*

$$\begin{aligned} a(\sigma_{S\rho_0}) &:= \begin{cases} (a\sigma)_{S\rho_0} & a \neq 0 \\ 0_{\emptyset\rho_0} & a = 0, \end{cases} \\ \sigma_{S\rho_0} + \tau_{T\rho_0} &:= \nu_{S \cup T\rho_0}, \\ \nu &:= \sigma \otimes \rho_{0|(T \setminus S)} + \rho_{0|(S \setminus T)} \otimes \tau. \end{aligned}$$

We will denote by $R_{\sigma+\tau}$ the region where the minimal representative of the class $\sigma_{S\rho_0} + \tau_{T\rho_0}$ differs from ρ_0 . As in the case of effects, it is not always true that $R_{\sigma+\tau} = R_\sigma \cup R_\tau$. As an example, consider $\sigma = \alpha_{i_1} \otimes \beta_{i_2}$ and $\tau = \alpha_{i_1} \otimes (\rho_{0i_2} - \beta_{i_2})$, with $\alpha_{i_1} \neq \rho_{0i_1}$, $\beta_{i_2} \neq \rho_{0i_2}$, and $R_\sigma = R_\tau = B_{i_1} \cup B_{i_2}$. Then $\sigma + \tau = \alpha_{i_1} \otimes \rho_{0i_2}$, and clearly $R_{\sigma+\tau} = B_{i_1}$, which is strictly included in $R_\sigma = R_\tau = R_\sigma \cup R_\tau$.

Definition 60. The set of local states over B upon ρ_0 is

$$[[A_G]]_{\rho_0 L}^{(B)} := \{\sigma_{S\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}^{(B)} \mid \sigma \in [[A_S]]\}.$$

An element $\sigma_{S\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ is a local state over B upon ρ_0 . A subset of special interest is that of deterministic local states over B upon ρ_0 , defined as

$$[[A_G]]_{\rho_0 L1}^{(B)} := \{\sigma_{S\rho_0} \in [[A_G]]_{\rho_0 L}^{(B)} \mid \sigma \in [[A_S]]_1\}.$$

We now characterise the vector space of generalised states as follows.

Lemma 74. The set of generalized local states in B upon ρ_0 equipped with the operations defined in Def. 59 is a real vector space. It is the space of finite real combinations of deterministic local states in B upon the same state ρ_0 :

$$[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)} = \text{Span}_{\mathbb{R}}([A_G]_{\rho_0 L1}^{(B)}), \quad (126)$$

The proof is trivial and we omit it.

We now show that quasi-local states are bounded linear functionals on $[[\bar{A}_G]]_{Q\mathbb{R}}$. We start considering the pairing of local states $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ with local effects $[[\bar{A}_G]]_{L\mathbb{R}}$, defined as follows.

Definition 61. Every $\sigma_{S\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}$ identifies a functional on $[[\bar{A}_G]]_{L\mathbb{R}}$ as follows. For $a_R \in [[\bar{A}_G]]_{L\mathbb{R}}$ let

$$(a_R | \sigma_{S\rho_0}) := \left(\begin{array}{c} \sigma_{R \cap S} \\ \rho_{0|R \setminus S} \end{array} \begin{array}{c} R \cap S \\ R \setminus S \end{array} \right) a. \quad (127)$$

We now prove that the above pairing is well-defined, namely it does not depend on the representative in the class $\sigma_{S\rho_0}$, nor on the representative in the class a_R . This is done in the next two lemmas.

Lemma 75. For every $a_R \in [[\bar{A}_G]]_{L\mathbb{R}}$ the result of Eq. (127) is independent of the choice of representative $\tau_{T\rho_0}$ in the class $\sigma_{S\rho_0}$.

Proof. By hypothesis, there exists $\nu \in [[A_{S \cap T}]]_{\mathbb{R}}$ such that

$$\sigma = \nu \otimes \rho_{0S \setminus T}, \quad \tau = \nu \otimes \rho_{0T \setminus S}.$$

Then we have

$$\begin{aligned} (a_R | \sigma_{S\rho_0}) &= (a | \nu_{|(S \cap T) \cap R} \otimes \rho_{0|[(S \setminus T) \cap R] \cup (R \setminus S)}) \\ (a_R | \tau_{T\rho_0}) &= (a | \nu_{|(S \cap T) \cap R} \otimes \rho_{0|[(T \setminus S) \cap R] \cup (R \setminus T)}). \end{aligned}$$

A straightforward set-theoretic calculation gives that $[(S \setminus T) \cap R] \cup (R \setminus S) = [(T \setminus S) \cap R] \cup (R \setminus T) = R \setminus (S \cap T)$, and thus $(a_R | \sigma_{S\rho_0}) = (a_R | \tau_{T\rho_0})$. $\square$

On similar lines, we have the following result.

Lemma 76. For every partition $B \in \mathcal{B}$, every reference local state ρ_0 over B , and every local state $\tau_{T\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$, the result of Eq. (127) is independent of the choice of representative b_S in the class a_R .

Proof. If $\tilde{c}_{R_c}$ is the minimal representative of $a_R = b_S$, by Eq. (10), one has

$$\begin{aligned} a &= \tilde{c} \otimes e_{R \setminus R_c}, \\ b &= \tilde{c} \otimes e_{S \setminus R_c}. \end{aligned}$$

Now, by the defining equation 127, we have that

$$\begin{aligned} (a_R | \tau_{T\rho_0}) &= (a | \tau_{R \cap T} \otimes \rho_{0|(R \setminus T)}) \\ &= (\tilde{c} \otimes e_{R \setminus R_c} | \tau_{R \cap T} \otimes \rho_{0|(R \setminus T)}) \\ &= (\tilde{c} | \tau_{R_c \cap T} \otimes \rho_{0|R_c \setminus T}), \end{aligned}$$

and similarly we obtain

$$(b_S | \tau_{T\rho_0}) = (\tilde{c} | \tau_{R_c \cap T} \otimes \rho_{0|(R_c \setminus T)}). \quad \square$$

The above results give us a hint that for every $B \in \mathcal{B}$ and every ρ_0 , the space $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ is a submanifold of the space $[[A_G]]_{\mathbb{R}}$. We now complete the real vector spaces $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ to Banach spaces. For this purpose, we start equipping all spaces $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ with a norm.

Definition 62. Given an element σ of the space $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$, its norm is defined as

$$\|\sigma\|_{B, \rho_0} := \|\tilde{\sigma}\|_{\text{op}}, \quad (128)$$

where $\|\cdot\|_{\text{op}}$ denotes the operational norm on $[[A_{R_\sigma}]]_{\mathbb{R}}$.

Lemma 77. Given an element σ of $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$, for τ_T in the corresponding equivalence class, one has

$$\|\sigma\|_{B,\rho_0} = \|\tau\|_{\text{op}},$$

where $\|\cdot\|_{\text{op}}$ denotes the operational norm on $[[A_T]]_{\mathbb{R}}$.

Proof. By definition of the relation $\sim_{\rho_0}$, and of minimal representative $\tilde{\sigma}_{R_\sigma}$ of σ , one has

$$\tau = \tilde{\sigma} \otimes \rho_{0(T \setminus R_\sigma)}.$$

Thus, $\|\tau\|_{\text{op}} = \|\tilde{\sigma} \otimes \rho_{0(T \setminus R_\sigma)}\|_{\text{op}}$. Now, by lemma 3

$$\|\tau\|_{\text{op}} = \|\tilde{\sigma}\|_{\text{op}} = \|\sigma\|_{B,\rho_0}. \quad \square$$

We can now show that vectors in $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ are bounded linear functionals on $[[\bar{A}_G]]_{Q\mathbb{R}}$.

Definition 63. Let $a = [a_{nR_n}] \in [[\bar{A}_G]]_{Q\mathbb{R}}$, and $\rho = \rho_{S\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}$. We define

$$(a|\rho) := \lim_{n \rightarrow \infty} (a_{nR_n} | \rho_{S\rho_0}). \quad (129)$$

With this definition, we can now prove that local states are actually bounded linear functionals on $[[\bar{A}_G]]_{Q\mathbb{R}}$.

Lemma 78. For $a = [a_{nR_n}] \in [[\bar{A}_G]]_{Q\mathbb{R}}$, and $\rho_{S\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}$, one has

$$|(a|\rho)| \leq \|a\|_{\text{sup}} \|\rho\|_{B,\rho_0}. \quad (130)$$

Proof. We start considering the case of a local effect a_R . In this case, we define

$$\begin{aligned} a_0 &:= (2\|a\|_{\text{sup}})^{-1} (\|a\|_{\text{sup}} e_R + a) \succeq 0, \\ a_1 &:= (2\|a\|_{\text{sup}})^{-1} (\|a\|_{\text{sup}} e_R - a) \succeq 0, \\ a_0 + a_1 &= e_R, \quad \|a\|_{\text{sup}}(a_0 - a_1) = a, \end{aligned}$$

we have that

$$\begin{aligned} |(a_R | \rho_{S\rho_0})| &= |(a_R | \rho_{|R \cap S} \otimes \rho_{0|R \setminus S})| \\ &= \|a\|_{\text{sup}} (\{\bar{a}_0 - \bar{a}_1\} \otimes e_{S \setminus R} | \rho \otimes \rho_{0|R \setminus S})| \\ &\leq \|a\|_{\text{sup}} \|\rho_{S\rho_0}\|_{B,\rho_0}, \end{aligned}$$

where $(\bar{a}_i | := (a_i | \mathcal{I}_{R \cap S} \otimes | \rho_{0|R \setminus S})$. Let now $a = [a_{nR_n}]$. Then we have

$$\begin{aligned} |(a_{nR_n} - a_{mR_m} | \rho_{S\rho_0})| \\ \leq \|a_{nR_n} - a_{mR_m}\|_{\text{sup}} \|\rho_{S\rho_0}\|_{B,\rho_0}, \end{aligned}$$

and thus actually $(a_{nR_n} | \rho_{S\rho_0})$ is a Cauchy sequence. Finally, since for every $n \in \mathbb{N}$ one has

$$|(a_{nR_n} | \rho_{S\rho_0})| \leq \|a_n\|_{\text{sup}} \|\rho_{S\rho_0}\|_{B,\rho_0},$$

taking the limit for $n \rightarrow \infty$ on both sides we have that

$$|(a | \rho_{S\rho_0})| \leq \|a\|_{\text{sup}} \|\rho_{S\rho_0}\|_{B,\rho_0}. \quad \square$$

We now show that the definition in Eq. (129) is independent of the sequence a_{nR_n} in the class of a .

Lemma 79. Let $a = [a_{nR_n}] = [b_{nR'_n}] \in [[\bar{A}_G]]_{Q\mathbb{R}}$, and $\rho_{S\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}}$. Then

$$\lim_{n \rightarrow \infty} (a_n | \rho_{S\rho_0}) = \lim_{n \rightarrow \infty} (b_n | \rho_{S\rho_0}). \quad (131)$$

Proof. Remind that by definition, for every ε there exists n_0 such that for $n \geq n_0$, it is $\|a_{nR_n} - b_{nR'_n}\|_{\text{sup}} \leq \varepsilon$. Now, by lemma 78, this implies that

$$|(a_{nR_n} - b_{nR'_n} | \rho_{S\rho_0})| \leq \varepsilon \|\rho_{S\rho_0}\|_{B,\rho_0},$$

and thus the thesis follows. $\square$

We finally close the vector space $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ by adding the limits of Cauchy sequences in the norm $\|\cdot\|_{B,\rho_0}$, thus obtaining a Banach space that we denote $[[A_G]]_{\rho_0 Q\mathbb{R}}^{(B)}$. We start considering Cauchy sequences

$$\sigma : \mathbb{N} \rightarrow [[A_G]]_{\rho_0 L\mathbb{R}}^{(B)} :: n \mapsto \sigma_{nR_n\rho_0} \in [[A_G]]_{\rho_0 L\mathbb{R}},$$

which make a real vector space $[[A_G]]_{\rho_0 C\mathbb{R}}^{(B)}$, containing $[[A_G]]_{\rho_0 L\mathbb{R}}^{(B)}$ as the subspace of constant sequences. We then quotient $[[A_G]]_{\rho_0 C\mathbb{R}}^{(B)}$ by the usual equivalence relation $\cong$, obtaining complete normed real vector space.

Definition 64. The space of quasi-local states in B upon ρ_0 is defined as the space $[[A_G]]_{\rho_0 Q\mathbb{R}}^{(B)} := [[A_G]]_{\rho_0 C\mathbb{R}}^{(B)} / \cong$.

Inside the new Banach space that we defined, we can single out a convex set of quasi-local states, corresponding to preparations that can be arbitrarily approximated by local protocols starting from ρ_0 , along with the cone that they span, and the convex set of deterministic states.

Definition 65. An element σ of $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ is a quasi-local state if there exists $\sigma_{n R_n \rho_0}$ in the class σ , and $n_0 \in \mathbb{N}$, such that, for $n \geq n_0$, $\sigma_{n R_n \rho_0} \in \llbracket A_G \rrbracket_{\rho_0 L}^{(B)}$. The set of quasi-local states in B upon ρ_0 is denoted by $\llbracket A_G \rrbracket_{\rho_0 Q}^{(B)}$.

A quasi-local state σ in $\llbracket A_G \rrbracket_{\rho_0 Q}^{(B)}$ is deterministic, if there exists a sequence $\sigma_{n R_n}$ in the class of σ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $\sigma_{n R_n} \in \llbracket A_G \rrbracket_{\rho_0 L^1}^{(B)}$. The set of deterministic quasi-local states is denoted by $\llbracket A_G \rrbracket_{\rho_0 Q^1}^{(B)}$.

We denote the subset of $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ of elements $\sigma = \lambda \rho$, for $\lambda \geq 0$ and $\rho \in \llbracket A_G \rrbracket_{\rho_0 Q}^{(B)}$, as $\llbracket A_G \rrbracket_{\rho_0 Q^+}^{(B)}$. We will equivalently write $\sigma \succeq 0$ for $\sigma \in \llbracket A_G \rrbracket_{\rho_0 Q^+}^{(B)}$.

Analogously to the case of effects, the elements of $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ will be denoted by $\rho = [\rho_{n R_n \rho_0}]$, with the convention that for a constant Cauchy sequence $\rho_{n R_n \rho_0} = \rho_{R \rho_0}$ we will set $\rho_{R \rho_0} := [\rho_{n R_n \rho_0}]$. We define $\|\rho\|_{B, \rho_0} := \lim_{n \rightarrow \infty} \|\rho_{n R_n \rho_0}\|_{B, \rho_0}$. One can straightforwardly check that the space $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ of quasi-local states in B upon ρ_0 is a Banach space, the sets $\llbracket A_G \rrbracket_{\rho_0 Q}^{(B)}$ and $\llbracket A_G \rrbracket_{\rho_0 Q^1}^{(B)}$ are convex subsets, and $\llbracket A_G \rrbracket_{\rho_0 Q^+}^{(B)}$ is a convex cone. The space $\llbracket A_G \rrbracket_{\rho_0 L \mathbb{R}}^{(B)}$ can be identified with the dense submanifold containing constant sequences $\sigma_{S \rho_0}$.

For every partition B and every reference local state ρ_0 , the Banach space $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ is separable. Indeed, one can choose a basis in $\llbracket A_{S_n} \rrbracket_{\mathbb{R}}$ for each region $S_n := \bigcup_{j=1}^n B_j$ in the increasing sequence $\{S_n\}_{n \in \mathbb{N}}$ of regions of B . Now, for every $\rho \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$, and for every $n \in \mathbb{N}$, given the class $\rho_{n R_n \rho_0}$ defining ρ , one has $R_n \subseteq S_m$ for sufficiently large m . Thus, linear combinations of the countable collection of bases for S_m are dense in $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$.

We now show that actually, for every pair (B, ρ_0) , the space $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ is a subspace of $\llbracket A_G \rrbracket_{\mathbb{R}}$, and $\|\cdot\|_* \equiv \|\cdot\|_{B, \rho_0}$ on $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. First of all, we extend the pairing of $\llbracket A_G \rrbracket_{Q \mathbb{R}}$ and $\llbracket A_G \rrbracket_{\rho_0 L \mathbb{R}}^{(B)}$ to $\llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$ and $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$.

Definition 66. Let $a = [a_{n R_n}] \in \llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$ and $\sigma = [\sigma_{n S_n \rho_0}] \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. Then we define

$$(a|\sigma) := \lim_{n \rightarrow \infty} (a|\sigma_{n S_n \rho_0}). \quad (132)$$

The above definition is well-posed, as shown by the following lemma, whose proof is a straightforward consequence of lemma 78.

Lemma 80. Let $a \in \llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$, and $\sigma = [\sigma_{n S_n \rho_0}] = [\sigma'_{n S'_n \rho_0}] \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. Then

$$\lim_{n \rightarrow \infty} (a|\sigma_{n S_n \rho_0}) = \lim_{n \rightarrow \infty} (a|\sigma'_{n S'_n \rho_0}). \quad (133)$$

Finally, we use the definition of sup-norm of effects to prove that vectors in the spaces $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ are bounded linear functionals on $\llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$. Also in this case the proof follows straightforwardly from lemma 78.

Lemma 81. Let $a \in \llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$ and $\rho \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. Then one has

$$|(a|\rho)| \leq \|a\|_{\text{sup}} \|\rho\|_{B, \rho_0}. \quad (134)$$

Lemma 82. The spaces $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ are closed subspaces of the extended space of states $\llbracket A_G \rrbracket_{\mathbb{R}}$. On $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ it holds that $\|\cdot\|_{B, \rho_0} \equiv \|\cdot\|_*$.

Proof. We only need to prove that $\|\cdot\|_* \equiv \|\cdot\|_{B, \rho_0}$, since we already proved that $\llbracket A_G \rrbracket_{\rho_0 L \mathbb{R}}^{(B)}$ is a linear manifold in the space of bounded linear functionals on $\llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$. First of all, by eq. (134) one clearly has $\|\rho\|_* \leq \|\rho\|_{B, \rho_0}$ for all $\rho \in \llbracket A_G \rrbracket_{Q \mathbb{R}}$. For the converse, one can use the same technique as for lemma 16, to prove that $\|\rho\|_{B, \rho_0} \leq \|\rho\|_*$ for the case of a local state $\rho \in \llbracket A_G \rrbracket_{\rho_0 L \mathbb{R}}^{(B)}$. Thus $\|\rho\|_* = \|\rho\|_{B, \rho_0}$ for local states $\rho \in \llbracket A_G \rrbracket_{\rho_0 L \mathbb{R}}^{(B)}$. It is then straightforward to conclude that equality holds for every quasi-local state $\rho \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$. $\square$

Notice that two partitions B and B' along with two local reference states ρ_0 and ρ'_0 might define the same space $\llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)} \equiv \llbracket A_G \rrbracket_{\rho'_0 Q \mathbb{R}}^{(B')}$. This is the case if for every $\sigma \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ there exists a state $\sigma' \in \llbracket A_G \rrbracket_{\rho'_0 Q \mathbb{R}}^{(B')}$ such that, $\sigma = \sigma'$, as functionals on $\llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$. We then provide the following definition.

Definition 67. We say that the two reference local states (B, ρ_0) and (B', ρ'_0) are compatible, and denote this relation by the symbol $(B, \rho_0) \sim (B', \rho'_0)$, if for all $\sigma \in \llbracket A_G \rrbracket_{\rho_0 Q \mathbb{R}}^{(B)}$ there exists a state $\sigma' \in \llbracket A_G \rrbracket_{\rho'_0 Q \mathbb{R}}^{(B')}$ such that, for every $a \in \llbracket \bar{A}_G \rrbracket_{Q \mathbb{R}}$, one has

$$(a|\sigma) = (a|\sigma').$$

The equivalence classes of pairs (B, ρ_0) under the above relation will be denoted by $[(B, \rho_0)]$.

The norm $\|\cdot\|_{B, \rho_0}$ can be extended to the space of finite linear combinations of generalised quasi-local states.

Definition 68. Let $\sigma_i \in \llbracket A_G \rrbracket_{\rho_0^{(i)} Q_{\mathbb{R}}}^{(B^{(i)})}$, for $i = 1, \dots, k$, with $[(B^{(i)}, \rho_0^{(i)})] \neq [(B^{(j)}, \rho_0^{(j)})]$. Then we define

$$\left\| \sum_{i=1}^k a_i \sigma_i \right\|_Q := \sum_{i=1}^k |a_i| \|\sigma_i\|_{B^{(i)}, \rho_0^{(i)}}.$$

The space of finite sums as from definition 68 is a real vector subspace of $\llbracket A_G \rrbracket_{\mathbb{R}}$. Moreover, as a real vector space, it is equipped with the norm $\|\cdot\|_Q$. We can define the space of Cauchy sequences in this space, and complete it by the usual procedure. This leads us to the following definition of the space of quasi-local states.

Definition 69. The space of generalised quasi-local states of A_G , denoted as $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$, is

$$\llbracket A_G \rrbracket_{Q_{\mathbb{R}}} := \bigoplus_{[B, \rho_0]} \llbracket A_G \rrbracket_{\rho_0 Q_{\mathbb{R}}}^{(B)},$$

where the direct sum denotes closure of the space of finite linear combinations in the norm $\|\cdot\|_Q$.

The norm introduced above allows us to prove that $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$ is a space of bounded linear functionals on $\llbracket \bar{A}_G \rrbracket_{Q_{\mathbb{R}}}$, namely a subspace of $\llbracket A_G \rrbracket_{\mathbb{R}}$.

Lemma 83. Every quasi-local state $\rho \in \llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$ is a continuous functional on the space of quasi-local effects $a \in \llbracket \bar{A}_G \rrbracket_{Q_{\mathbb{R}}}$, and

$$|(a|\rho)| \leq \|a\|_{\sup} \|\rho\|_Q \quad (135)$$

Proof. Let $\rho \in \llbracket A_G \rrbracket_{\mathbb{R}}$ be a finite sum of the form

$$\rho = \sum_{i=1}^k a_i \sigma_i, \quad \sigma_i \in \llbracket A_G \rrbracket_{\rho_0^{(i)} Q_{\mathbb{R}}}^{(B^{(i)})}. \quad (136)$$

Then

$$(a|\rho) := \sum_{i=1}^k a_i (a|\sigma_i). \quad (137)$$

One can easily realise that $|(a|\rho)| \leq \|a\|_{\sup} \|\rho\|_Q$. Finally, by the above inequality, also the limit of a Cauchy sequence in $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$ is a continuous linear functional on $\llbracket \bar{A}_G \rrbracket_{Q_{\mathbb{R}}}$, and satisfies

$$|(a|\rho)| \leq \|a\|_{\sup} \|\rho\|_Q. \quad \square$$

We can now prove the following important result.

Lemma 84. The space $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$ is a closed subspace of the extended space of states $\llbracket A_G \rrbracket_{\mathbb{R}}$. On $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$ it holds that $\|\cdot\|_Q \equiv \|\cdot\|_*$.

Proof. We only need to prove that $\|\cdot\|_* \equiv \|\cdot\|_Q$ on finite sums of the form of eq. (136), since we already proved that $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$ is obtained closing a linear manifold in the space of bounded linear functionals on $\llbracket \bar{A}_G \rrbracket_{Q_{\mathbb{R}}}$. The proof follows exactly the same argument as for lemma 82, using eq. (135) instead of eq. (134). $\square$

Notice that all local states allow us to define states of arbitrary finite regions $S \subseteq G$, by Eq. (122). We will now extend the definition of local state for all quasi-local states in a fixed sector $\llbracket A \rrbracket_{\rho_0 Q}^{(B)}$, as follows. First, if ρ is quasi local, then $\|\rho_n - \rho_m\|_{B, \rho_0} \leq \varepsilon$, and $\|(\rho_n - \rho_m)|_S\|_{\text{op}} \leq \|\rho_n - \rho_m\|_{B, \rho_0} \leq \varepsilon$. Thus, being the set of states $\llbracket A_S \rrbracket$ compact, $\rho_S := \lim_{n \rightarrow \infty} \rho_n|_S \in \llbracket A_S \rrbracket$.

The same is true for finite linear combinations, as well as limits of Cauchy sequences thereof. Indeed, given a finite combination of states $\rho = \sum_i a_i \rho_i$, one has

$$\rho|_S := \sum_i a_i \rho_i|_S \in \llbracket A_S \rrbracket_{\mathbb{R}}. \quad (138)$$

Finally, if $\tau : \mathbb{N} \rightarrow \llbracket A \rrbracket_{Q_{\mathbb{R}}}$, and $\|\tau_n - \tau_m\|_Q \leq \varepsilon$, then clearly $\|(\tau_n - \tau_m)|_S\|_{\text{op}} \leq \varepsilon$, and thus

$$\tau|_S := \lim_{n \rightarrow \infty} (\tau_n)|_S \in \llbracket A_S \rrbracket_{\mathbb{R}}. \quad (139)$$

In view of the above considerations, we can now define the set of states in $\llbracket A_G \rrbracket_{Q_{\mathbb{R}}}$.

Definition 70. The set $\llbracket A_G \rrbracket_Q$ of quasi-local states is the (convex) set of generalised states ρ for which, for every finite region $S \in \mathcal{R}^{(G)}$, the restriction $\rho|_S$ of ρ to S is a state in $\llbracket A_S \rrbracket$. The positive cone $\llbracket A_G \rrbracket_{Q+}$ of A_G is

$$\begin{aligned} \llbracket A_G \rrbracket_{Q+} &:= \{\sigma \in \llbracket A_G \rrbracket_{Q_{\mathbb{R}}} \mid \exists \lambda \geq 0, \rho \in \llbracket A_G \rrbracket_Q : \sigma = \lambda \rho\}. \end{aligned}$$

The set of deterministic quasi-local states of A_G , denoted as $\llbracket A_G \rrbracket_{Q1}$, is

$$\llbracket A_G \rrbracket_{Q1} := \{\rho \in \llbracket A_G \rrbracket_Q \mid (e_G|\rho) = 1\}.$$

Lemma 85. Definition 70 is compatible with definition 16

Proof. We need to prove that the definition of restriction to the finite region S provided in Eq. (33) is equivalent to that of Eq. (139). We start from the case of local states. In this case, since effects are separating for states of finite dimensional systems, one has $(a_S|\rho_T) = (a_S \otimes e_{T \setminus S}|\rho|_T \otimes \rho_{0|S \setminus T})$, and thus $\rho|_S = \rho|_{(T \cap S)} \otimes \rho_{0|S \setminus T}$, which is precisely the same as in Eq. (122). The statement in the case of a general state $\rho \in \llbracket A_G \rrbracket_Q$ is then straightforwardly proved, reminding that $|(a_S|\rho - \rho_n)| \leq \|a_S\|_{\text{sup}} \|\rho - \rho_n\|_Q$. $\square$

Corollary 19. *One has the following inclusions*

$$\llbracket A_G \rrbracket_Q \subseteq \llbracket A_G \rrbracket, \quad \llbracket A_G \rrbracket_{Q1} \subseteq \llbracket A_G \rrbracket_1. \quad (140)$$

C Proof of identity 94

In order to keep the notation lighter, we do the calculation for the case of $C \cong I$. The proof can then be adapted to the general case by suitably padding all transformations with $\mathcal{I}_C$ and extending the region $\tilde{S}$ to $\tilde{S}C$.

$$\begin{aligned} (\mathcal{V} \mathcal{A} \mathcal{V}^{-1} \otimes \mathcal{I}_{H_1})^\dagger a_{\tilde{S}} &= \mathcal{W}^{-1\dagger} (\mathcal{A} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{W}^\dagger a_{\tilde{S}} \\ &= \mathcal{W}^{-1\dagger} (\mathcal{A} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_{N_S^+ \cup N_S^+}^\dagger \mathcal{S}_{N_S^+ \cup N_S^+}^\dagger a_{\tilde{S}} \\ &= \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger \mathcal{S}_{N_S^+ \cup N_S^+ \cup R}^\dagger (\mathcal{A} \otimes \mathcal{I}_{H_1})^\dagger \\ &\quad \times \mathcal{S}_{N_S^+ \cup N_S^+}^\dagger \mathcal{S}_{N_S^+ \cup N_S^+}^\dagger a_{\tilde{S}} \\ &= \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger \mathcal{S}_{N_S^+ \cup N_S^+ \cup R}^\dagger (\mathcal{A} \otimes \mathcal{I}_{H_1})^\dagger \\ &\quad \times \mathcal{S}_{N_S^+ \cup N_S^+ \cup R}^\dagger \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger a_{\tilde{S}} \\ &= \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger \mathcal{S}_R^\dagger (\mathcal{A} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_R^\dagger \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger a_{\tilde{S}} \\ &= \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger (\mathcal{I}_{H_0} \otimes \mathcal{A})^\dagger \mathcal{S}_{P_S^+ \cup N_S^+ \cup R}^\dagger a_{\tilde{S}} \\ &= \mathcal{S}_R^\dagger (\mathcal{I}_{H_0} \otimes \mathcal{A})^\dagger \mathcal{S}_R^\dagger [a_{\tilde{S}}] \\ &= \mathcal{S}_R^\dagger \mathcal{S}_R^\dagger (\mathcal{A} \otimes \mathcal{I}_{H_0})^\dagger \mathcal{S}_R^\dagger \mathcal{S}_R^\dagger [a_{\tilde{S}}], \end{aligned}$$

where in the third identity we used Eq. (91), observing that $(\mathcal{A} \otimes \mathcal{I}_{H_1})^\dagger \mathcal{S}_{N_S^+ \cup N_S^+}^\dagger \mathcal{S}_{N_S^+ \cup N_S^+}^\dagger [a_{\tilde{S}}] \in \llbracket \bar{A}_{(N_S^+ \cup R, 0) \cup (N_S^+, 1)}^{(G)} \rrbracket_{Q\mathbb{R}}$. In the fourth identity we used the fact that $a_{\tilde{S}}$ can be padded with e_X to obtain an effect $\tilde{a}_X$ on the region $\tilde{S} \cup X$, and then $\mathcal{W}^\dagger \tilde{a}_X$ is an effect equivalent to $\mathcal{W}^\dagger a_S$. In the fifth identity we used the fact that $\mathcal{S}_X$ commutes with $\mathcal{B}_Y \otimes \mathcal{I}_{H_1}$ for $Y \cap X = \emptyset$, and in the seventh identity we used the fact that $\mathcal{S}'_X$ commutes with $\mathcal{I}_{H_0} \otimes \mathcal{B}_Y$ for $Y \cap X = \emptyset$.

D Homogeneity and causal influence

This section is devoted to the proof that, for $\pi \in \Pi_{\mathcal{V}}$,

$$\pi(g) \succ \pi(g') \Leftrightarrow g \succ g'$$

By definition, $f \succ f'$ if there exists $R \ni f$ and

$$\begin{aligned} \exists \mathcal{F} \in \llbracket A_f^{(G)} C \rightarrow A_{f'}^{(G)} C \rrbracket_{Q\mathbb{R}}, \\ (\mathcal{V} \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{V}^{-1} \otimes \mathcal{I}_C) \notin \llbracket A_{f'}^{(G)} C \rightarrow A_{f'}^{(G)} C \rrbracket_{Q\mathbb{R}}. \end{aligned} \quad (141)$$

Let now $f = \pi(g)$ for some $\pi \in \Pi_{\mathcal{V}}$. Now, by homogeneity, and in particular by Eq. (104), one has that $\mathcal{F}' \in \llbracket A_{\pi(g)}^{(G)} C \rightarrow A_{\pi(g)}^{(G)} C \rrbracket$ if and only if there exists $\mathcal{F} \in \llbracket A_g^{(G)} C \rightarrow A_g^{(G)} C \rrbracket$ such that $\mathcal{F}' = (\mathcal{T}_\pi \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{T}_\pi^{-1} \otimes \mathcal{I}_C)$. Then one has

$$\begin{aligned} (\mathcal{V} \otimes \mathcal{I}_C) \mathcal{F}' (\mathcal{V}^{-1} \otimes \mathcal{I}_C) \\ = (\mathcal{V} \mathcal{T}_\pi \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{T}_\pi^{-1} \mathcal{V}^{-1} \otimes \mathcal{I}_C) \\ = (\mathcal{T}_\pi \mathcal{V} \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{V}^{-1} \mathcal{T}_\pi^{-1} \otimes \mathcal{I}_C). \end{aligned} \quad (142)$$

As a consequence, again by eq. (104), we have that $(\mathcal{V} \otimes \mathcal{I}_C) \mathcal{F}' (\mathcal{V}^{-1} \otimes \mathcal{I}_C) \in \llbracket A_{f'}^{(G)} C \rightarrow A_{f'}^{(G)} C \rrbracket$, for $f' = \pi(g')$, if and only if $(\mathcal{V} \otimes \mathcal{I}_C) \mathcal{F} (\mathcal{V}^{-1} \otimes \mathcal{I}_C) \in \llbracket A_{g'}^{(G)} C \rightarrow A_{g'}^{(G)} C \rrbracket$. Finally, comparing with Eq. (141), this implies that $\pi(g) \succ \pi(g')$ if and only if $g \succ g'$.

E Proof of right and left invertibility of a locally defined GUR

Here we prove right and left invertibility of $\mathcal{V}$ defined starting from Eq. (109). First, notice that

Let us now define $\mathcal{W}$ by $\mathcal{W}^\dagger b_{SC} := \{(\mathcal{W}_S^\dagger \otimes \mathcal{J}_C^\dagger)b\}_{N_S^+C}$, where

$$\begin{array}{c} N_S^+ \\ \text{---} \end{array} \boxed{\mathcal{W}_S} \begin{array}{c} \text{---} S \end{array} := \begin{array}{c} N_S^+ \\ \text{---} \end{array} \boxed{\mathcal{L}'_S} \begin{array}{c} N_S^+ \\ \text{---} \end{array} \boxed{e} \begin{array}{c} \text{---} S \end{array} \begin{array}{c} \boxed{\eta} \\ \text{---} S \end{array} \quad (143)$$

Then one can calculate $\mathcal{W}^\dagger \mathcal{V}^\dagger a_{RC}$ by the following diagram

The figure consists of four circuit diagrams arranged vertically, connected by equals signs, illustrating the simplification of a quantum circuit. The first diagram shows a circuit with two large blocks labeled $\mathcal{S}'_{N_R^-}$ and a phase gate ψ . The second diagram introduces a block $\mathcal{S}'$ and a C gate. The third diagram introduces a block $\mathcal{S}$. The fourth diagram shows the final simplified circuit with a block a .

Finally, we observe that $[(a \otimes e_{P_R^- \setminus R})P_R^- \mathbb{C}] = [a_{RC}]$, then $\mathscr{W}^\dagger \mathscr{V}^\dagger = \mathscr{J}_G^\dagger$. Similarly, one can calculate $\mathscr{V}^\dagger \mathscr{W}^\dagger b_{SC}$. First we observe that $[b_{SC}] = [(b \otimes e_{P_S^+ \setminus S})P_S^+ \mathbb{C}]$. Then $\mathscr{W}^\dagger b_{SC} = \{(\mathscr{W}_{P_S^+}^\dagger(b \otimes e_{N_{P_S^+}^+ \setminus N_S^+}))\}_{N_{P_S^+ \mathbb{C}}^+}$, and

where we used Eq. (97). Now, for $\mathcal{V}^\dagger \mathcal{W}^\dagger [b_{\text{SC}}]$ we

have

The figure consists of two quantum circuit diagrams. The top diagram shows a circuit with two blocks, $\mathcal{S}'$ and $\mathcal{S}''_R$, connected by a line. The bottom diagram shows a circuit with a block $\mathcal{S}'$ and a block $\mathcal{B}''$, connected by a line. The bottom diagram is followed by an equals sign and a simplified circuit with a block b .

where Eq. (97) was used in the last step. The arguments are straightforwardly generalised to $b \in \llbracket \bar{A}_S^{(G)} \bar{C} \rrbracket_{Q\mathbb{R}}$.

F Proof of theorem 10

Proof. In the following, $G_i := (G, i)$ for $i = 0, 1$. The argument in appendix E shows that

$$\begin{array}{c} G_0 \\ \hline G_1 \end{array} \begin{array}{c} \boxed{\mathcal{W}} \\ \hline \end{array} \begin{array}{c} G_0 \\ \hline G_1 \end{array} \begin{array}{c} \text{---} e \end{array} = \begin{array}{c} G_0 \\ \hline G_1 \end{array} \begin{array}{c} \boxed{\mathcal{W}-1} \\ \hline \end{array} \begin{array}{c} G_0 \end{array},$$

where $(G, A, \mathcal{V}^{-1\dagger})$ is a GUR. Moreover, since

$$\begin{array}{c} G_0 \\ \hline \hline G_1 \end{array} \boxed{\mathcal{S}G} \begin{array}{c} G_1 \\ \hline G_0 \end{array} \boxed{e} = \begin{array}{c} G_0 \\ \hline G_1 \end{array} \boxed{e},$$

one has

namely

$$\begin{array}{c} G_1 \\ \hline G_0 \end{array} \quad = \quad \begin{array}{c} G_1 \\ \hline G_0 \end{array} \quad \begin{array}{c} \boxed{\mathcal{W}} \\ \hline \end{array} \quad \begin{array}{c} G_1 \\ \hline G_0 \end{array} \quad \begin{array}{c} \boxed{\mathcal{V}^{-1}} \\ \hline \end{array} \quad \begin{array}{c} G_0 \\ \hline \end{array} \quad \begin{array}{c} \boxed{e} \\ \hline \end{array} \quad .$$

On the other hand, one can write the latter as

$$\begin{array}{c} G_1 \\ \boxed{\mathcal{V}} \\ G_0 \end{array} \begin{array}{c} G_1 \\ e \end{array} = \begin{array}{c} G_1 \\ \boxed{\mathcal{W}} \\ G_0 \end{array} \begin{array}{c} G_1 \\ e \end{array}, \quad (144)$$

or equivalently

$$\begin{array}{c} G_0 \\ e \end{array} \begin{array}{c} G_1 \\ \boxed{\mathcal{V}} \\ G_1 \end{array} = \begin{array}{c} G_0 \\ \boxed{\mathcal{W}^{-1}} \\ G_1 \end{array} \begin{array}{c} G_0 \\ e \end{array}.$$

Now, let us consider

$$\begin{array}{c} G_0 \\ \boxed{\tilde{\mathcal{I}}_G} \\ G_1 \end{array} \begin{array}{c} G_1 \\ e \\ G_0 \end{array} \quad \begin{array}{c} G_0 \\ G_1 \\ G_2 \end{array}$$

$$:= \begin{array}{c} G_0 \\ G_1 \\ G_2 \end{array} \begin{array}{c} \boxed{\mathcal{W}^{-1}} \\ \boxed{\mathcal{I}_G} \\ \boxed{\mathcal{W}} \end{array} \begin{array}{c} G_1 \\ e \\ G_0 \end{array}$$

$$= \begin{array}{c} G_0 \\ G_2 \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{W}} \\ \boxed{\mathcal{W}} \end{array} \begin{array}{c} G_0 \\ G_2 \\ G_1 \end{array} \begin{array}{c} G_0 \\ e \end{array}$$

$$= \begin{array}{c} G_0 \\ G_2 \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{V}} \\ \boxed{\mathcal{W}} \end{array} \begin{array}{c} G_0 \\ G_2 \\ G_1 \end{array} \begin{array}{c} G_0 \\ e \end{array},$$

$$\begin{array}{c} G_1 \\ e \end{array}$$

from which one concludes that

$$\begin{array}{c} G_0 \\ \boxed{\tilde{\mathcal{I}}_G} \\ G_1 \end{array} \begin{array}{c} G_1 \\ e \\ G_0 \end{array} = \begin{array}{c} G_1 \\ e \\ G_0 \end{array} \begin{array}{c} \boxed{\mathcal{X}} \\ G_0 \end{array},$$

for some $\mathcal{X}$. Thus

$$\begin{array}{c} G_0 \\ G_2 \end{array} \begin{array}{c} \boxed{\mathcal{V}} \\ \boxed{\mathcal{W}} \end{array} \begin{array}{c} G_0 \\ G_2 \end{array} = \begin{array}{c} G_0 \\ G_2 \end{array} \begin{array}{c} \boxed{\mathcal{X}} \end{array}.$$

Finally, this implies that $\mathcal{W} = \mathcal{X} \otimes \mathcal{V}^{-1}$, and in particular, by Eq. (144), it must be $\mathcal{X} = \mathcal{V}$, namely $\mathcal{W} = \mathcal{V} \otimes \mathcal{V}^{-1}$. Consequently, we have

that

$$\begin{array}{c} C \\ G_0 \end{array} \begin{array}{c} \boxed{V_L^-(\mathcal{A}_R)} \\ G_0 \end{array} \begin{array}{c} C \\ G_1 \end{array} = \begin{array}{c} C \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{W}^{-1}} \\ G_0 \end{array} \begin{array}{c} C \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{A}_R} \\ G_0 \end{array} \begin{array}{c} C \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{W}} \\ G_1 \end{array}$$

$$= \begin{array}{c} G_0 \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{V}^{-1}} \\ G_0 \end{array} \begin{array}{c} C \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{A}_R} \\ G_0 \end{array} \begin{array}{c} C \\ G_1 \end{array} \begin{array}{c} \boxed{\mathcal{V}} \\ G_0 \end{array},$$

and

$$\begin{array}{c} G_0 \\ G_1 \\ C \end{array} \begin{array}{c} \boxed{V_L^+(\mathcal{A}_R)} \\ C \end{array} \begin{array}{c} G_0 \\ G_1 \\ C \end{array} = \begin{array}{c} G_0 \\ G_1 \\ C \end{array} \begin{array}{c} \boxed{\mathcal{W}^{-1}} \\ G_1 \end{array} \begin{array}{c} G_0 \\ G_1 \\ C \end{array} \begin{array}{c} \boxed{\mathcal{A}_R} \\ C \end{array} \begin{array}{c} G_0 \\ G_1 \\ C \end{array} \begin{array}{c} \boxed{\mathcal{W}} \\ G_1 \end{array}$$

$$= \begin{array}{c} G_1 \\ C \end{array} \begin{array}{c} \boxed{\mathcal{V}} \\ G_1 \end{array} \begin{array}{c} G_1 \\ C \end{array} \begin{array}{c} \boxed{\mathcal{A}_R} \\ C \end{array} \begin{array}{c} G_1 \\ C \end{array} \begin{array}{c} \boxed{\mathcal{V}^{-1}} \\ G_1 \end{array} \begin{array}{c} G_1 \\ C \end{array}.$$

G Proof of lemmas 66 and 67

We start with the proof of lemma 66.

Proof. (1) Notice that $N_{e_H}^{\pm} = \phi(S_{\pm}) = \phi(N_{e_G}^{\pm})$. Then $|N_{e_H}^{\pm}| \leq |N_{e_G}^{\pm}|$, with strict inequality if and only if $\phi(h_a) = \phi(h_b)$ for some $h_a \neq h_b \in N_{e_G}^{\pm}$. However, this is impossible because it would be equivalent to $\phi(h_a h_b^{-1}) \in R'$, namely $h_a = h_b$, contrarily to the hypothesis. Thus, $|N_{e_H}^{\pm}| = |N_{e_G}^{\pm}|$. Invariance of neighbourhoods under left-multiplication finally gives the thesis.

(2) Let now $f \in P_{e_H}^{\pm} = N_{N_{e_H}^{\pm}}^{\mp}$, namely there exists $h \in N_{e_H}^{\pm}$ such that $f \in N_h^{\mp}$, or equivalently $h \in N_f^{\pm}$. Then, $f \in P_{e_H}^{\pm}$ iff there exists $h \in N_{e_H}^{\pm} \cap N_f^{\pm}$. Equivalently, there exists $h \in H$ such that $h = \phi(h_a)$ and $h = f\phi(h_b)$, namely $f = \phi(h_a h_b^{-1})$. Then $P_{e_H}^{\pm} = \{\phi(h_a h_b^{-1}) \mid h_a, h_b \in N_{e_G}^{\pm}\}$. By a similar argument, we have that $P_{e_G}^{\pm} = \{h_a h_b^{-1} \mid h_a, h_b \in N_{e_G}^{\pm}\}$. Consequently, $|P_{e_H}^{\pm}| \leq |\{h_a h_b^{-1} \mid h_a, h_b \in N_{e_G}^{\pm}\}| = |P_{e_G}^{\pm}|$. Now, strict inequality in the last relation would require that $\phi(h_a h_b^{-1}) = \phi(h_a h_c^{-1})$, contrarily to the hypothesis. We conclude that $|P_{e_H}^{\pm}| = |P_{e_G}^{\pm}|$. Also in this case the thesis can be obtained by invariance of neighbourhoods under left-multiplication.

(3) Let $N_{f_1}^{\pm} \cap N_{f_2}^{\pm} = \emptyset$. Then clearly for every $g_i \in \phi^{-1}(f_i)$ it must be $N_{g_1}^{\pm} \cap N_{g_2}^{\pm} = \emptyset$. Let then now $h \in N_{f_1}^{\pm} \cap N_{f_2}^{\pm}$. This implies that $h = f_1\phi(h_a) = f_2\phi(h_b)$ for $h_a, h_b \in N_{e_G}^{\pm}$. Consequently, $f_2 = f_1\phi(h_a h_b^{-1})$. Now, either $f_1 = f_2$, and then choosing $g_1 = g_2$ the thesis trivially follows, or $\phi(h_a h_b^{-1}) \notin R'$, and then $h_a h_b^{-1} \notin R$. One can then set $g_2 = g_1 h_a h_b^{-1}$, and verify that $g_2 \in \phi^{-1}(f_2)$, as well as $k = g_1 h_a = g_2 h_b \in N_{g_1}^{\pm} \cap N_{g_2}^{\pm}$. Thus, $h = \phi(k)$, and consequently $N_{f_1}^{\pm} \cap N_{f_2}^{\pm} \subseteq \phi(N_{g_1}^{\pm} \cap N_{g_2}^{\pm})$. This implies that $|N_{f_1}^{\pm} \cap N_{f_2}^{\pm}| \leq |N_{g_1}^{\pm} \cap N_{g_2}^{\pm}|$, with a strict inequality iff $\phi(k_1) = \phi(k_2)$ for $k_1 \neq k_2 \in N_{g_1}^{\pm} \cap N_{g_2}^{\pm}$. However, from the proof of item (1) we know that if $k_1 \neq k_2 \in N_g^{\pm}$ it must be $\phi(k_1) \neq \phi(k_2)$. Thus, $|N_{f_1}^{\pm} \cap N_{f_2}^{\pm}| = |\phi(N_{g_1}^{\pm} \cap N_{g_2}^{\pm})|$, and $N_{f_1}^{\pm} \cap N_{f_2}^{\pm} = \phi(N_{g_1}^{\pm} \cap N_{g_2}^{\pm})$.

(4) Let $h \in P_{f_2}^{\mp} \cap N_{f_1}^{\pm}$. Then $h = f_2\phi(h_c^{-1}h_b) = f_1\phi(h_a)$, with $h_a, h_b, h_c \in N_{e_G}^{\pm}$. This implies that $f_2 = f_1\phi(h_a h_b^{-1}h_c)$. We can now set $g_2 := g_1 h_a h_b^{-1}h_c$ with $g_1 \in \phi^{-1}(f_1)$. In this way, one has $f_2 = \phi(g_2)$. Let us set $k := g_1 h_a = g_2 h_c^{-1}h_b \in P_{g_2}^{\mp} \cap N_{g_1}^{\pm}$. Then, $h = \phi(k)$, and $P_{f_1}^{\mp} \cap N_{f_2}^{\pm} \subseteq \phi(P_{g_2}^{\mp} \cap N_{g_1}^{\pm})$. By the same argument as for item (3), the thesis follows.

(5) Let $h \in P_{g_1}^{\pm} \cap N_{g_2}^{\mp}$. Then $h = g_1 h_a h_b^{-1}$, $h = g_2 h_c^{-1}$, for some $h_a, h_b, h_c \in N_{e_G}^{\pm}$, and thus $g_2 = g_1 h_a h_b^{-1}h_c$. On the other hand, by hypothesis, for every $h_x \in N_{e_G}^{\pm}$ one has $g_2 \neq g_1 h_x$. Collecting all the informations, we have $h_a h_b^{-1}h_c h_x^{-1} \notin R$, i.e. $\phi(h_a h_b^{-1}h_c h_x^{-1}) \notin R'$, for every $h_x \in N_{e_G}^{\pm}$. Then one has $\phi(g_2) = \phi(g_1)\phi(h_a h_b^{-1}h_c)$. By hypothesis, it is also $\phi(g_2) = \phi(g_1)\phi(h_d)$ for some $h_d \in N_{e_G}^{\pm}$. This is clearly absurd, thus it must be $P_{g_1}^{\pm} \cap N_{g_2}^{\mp} = \emptyset$. $\square$

Now, we provide the proof of lemma 67

Proof. (1) Let $f_2 \in P_{f_1}^{\pm}$, namely $f_2 = f_1\phi(h_a h_b^{-1})$ for $h_a, h_b \in N_{e_G}^{\pm}$. If $f_1 = f_2$, it is sufficient to choose $g_1 = g_2 \in \phi^{-1}(f_1)$. Let now $f_1 \neq f_2$. Then, let $g_1 \in \phi^{-1}(f_1)$, and $g_2 := g_1 h_a h_b^{-1}$. By construction, $\phi(g_2) = f_2$ and $g_2 \in P_{g_1}^{\pm}$. Now, let $h \in P_{f_1}^{\pm} \cap P_{f_2}^{\pm}$, namely $h = f_1\phi(h_f h_e^{-1}) = f_2\phi(h_c h_d^{-1}) = f_1\phi(h_a h_b^{-1}h_c h_d^{-1})$. Then we have $\phi(h_a h_b^{-1}h_c h_d^{-1}h_e h_f^{-1}) \in R'$, and necessarily $h_a h_b^{-1}h_c h_d^{-1}h_e h_f^{-1} \in R$. This implies that $g_1 h_f h_e^{-1} = g_1 h_a h_b^{-1}h_c h_d^{-1} = g_2 h_c h_d^{-1}$. Now, setting $k := g_1 h_f h_e^{-1}$, we have $k = g_2 h_c h_d^{-1} \in P_{g_1}^{\pm} \cap P_{g_2}^{\pm}$, and $h = \phi(k)$. Thus, $P_{f_1}^{\pm} \cap P_{f_2}^{\pm} \subseteq$

$\phi(P_{g_1}^{\pm} \cap P_{g_2}^{\pm})$. Finally, by the same argument as for the proof of item (2) of lemma 66, we get the thesis.

(2) The hypothesis clearly implies that $N_{g_1}^{\pm} \cap N_{g_2}^{\pm} = \emptyset$. Suppose now that $h \in P_{g_1}^{\pm} \cap P_{g_2}^{\pm}$. Then $h = g_1 h_a h_b^{-1} = g_2 h_d h_c^{-1}$ for $h_a, h_b, h_c, h_d \in N_{e_G}^{\pm}$. This implies that $g_2 = g_1 h_a h_b^{-1}h_c h_d^{-1}$, and consequently $\phi(g_2) = \phi(g_1)\phi(h_a h_b^{-1}h_c h_d^{-1})$. On the other hand, we proved that for every $h_e, h_f \in N_{e_G}^{\pm}$ one has $g_2 \neq g_1 h_f h_e^{-1}$, while by hypothesis $\phi(g_2) = \phi(g_1)\phi(h_f h_e^{-1})$ for some $h_e, h_f \in N_{e_G}^{\pm}$. Thus, while $h_a h_b^{-1}h_c h_d^{-1}h_e h_f^{-1} \notin R$, it must be $\phi(h_a h_b^{-1}h_c h_d^{-1}h_e h_f^{-1}) \notin R'$. We reached a contradiction, and thus it must be $P_{g_1}^{\pm} \cap P_{g_2}^{\pm} = \emptyset$. $\square$

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