Squeezing Metrology: a unified framework

Lorenzo Maccone and Alberto Riccardi

Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy

Quantum metrology theory has up to now focused on the resolution gains obtainable thanks to the entanglement among N probes. Typically, a quadratic gain in resolution is achievable, going from the $1/\sqrt{N}$ of the central limit theorem to the $1/N$ of the Heisenberg bound. Here we focus instead on quantum squeezing and provide a unified framework for metrology with squeezing, showing that, similarly, one can generally attain a quadratic gain when comparing the resolution achievable by a squeezed probe to the best N-probe classical strategy achievable with the same energy. Namely, here we give a quantification of the Heisenberg squeezing bound for arbitrary estimation strategies that employ squeezing. Our theory recovers known results (e.g. in quantum optics and spin squeezing), but it uses the general theory of squeezing and holds for arbitrary quantum systems.

1 Introduction

In quantum metrology one studies the resolution gains that can be attained when using quantum effects in the estimation strategy. The usual setting considers an estimation strategy where a parameter φ is encoded onto a probe state through a unitary transformation $U_\varphi = e^{iH\varphi}$, where H is the probe Hamiltonian. This is a very general setting that encompasses most estimations. While squeezing has been used in quantum metrology for specific systems [1–14], there is no general theory of squeezing-based metrology that holds for arbitrary measurements and systems. In the general case, quantum metrology theory [15–26] focuses on entanglement: it analyzes the situation in which the estimation is repeated N times and shows that there is a quadratic improvement in resolution whenever the probes are entangled. Without entanglement, one can only achieve the standard-quantum-limit resolution $\Delta\varphi \propto 1/\sqrt{N}$ of the central limit theorem, and the error decreases to $\Delta\varphi \propto 1/N$ of the Heisenberg bound using entangled probes [15, 19, 20, 22], Fig. 1a.

In this paper we focus on squeezing and consider the case in which a single probe is squeezed with respect to the observable A that is measured to estimate the parameter φ. We are **not** claiming to have discovered that squeezing is useful for quantum metrology (there is plenty of evidence for that in the literature [1–14]). The main result of this paper is a unified framework that describes all previous (and presumably future) metrology protocols that use squeezing in *any* quantum system and *any* observable, as it is based on the elegant general theory of squeezing for arbitrary systems [27]. As shown through various examples, the previously known results [1–14] can be recovered as specific instances of our theory. Squeezing the probe requires an amount of energy $E = \langle s|H|s\rangle$, where $|s\rangle$ is the squeezed state of the probe. As in entanglement-based quantum metrology, a quadratic resolution gain is obtained also in this case, if one compares the resolution attainable with a squeezed probe to the resolution obtainable with N classical probes (i.e. prepared in a coherent state) of total energy E, Fig. 1b. By classical probe we intend a probe prepared in a minimum uncertainty coherent state, as is customary. We employ N classical probes with the same energy as the squeezed probe because we need a dimensionless parameter (N) to measure the precision enhancement. We compare the quantum and classical strategies with same average energy because, without any energy restriction, one could achieve arbitrary precision by using arbitrarily high squeezing. In essence, our result can be summarized as follows: whereas one needs N classical probes to decrease the error from $\Delta\varphi$ to $\Delta\varphi/\sqrt{N}$, a single squeezed probe that uses the same energy can decrease the error to $\Delta\varphi/N$. We show that this

Lorenzo Maccone: maccone@unipv.it
Alberto Riccardi: alberto.riccardi01@universitadipavia.it

resolution increase is the optimal one (no further enhancement is possible), so we can call it the Heisenberg squeezing bound. We will be neglecting multiplicative constants of order one and consider only the scaling in N: it is impossible to give a general theory of the multiplicative constants because these depend on the specific implementations (as is clear from the examples below).

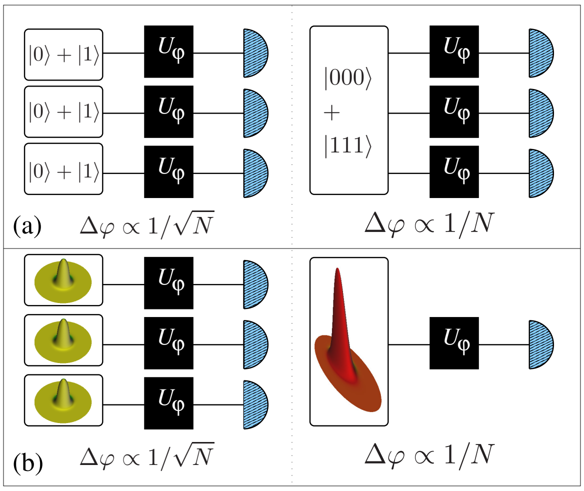

Figure 1: Conventional quantum metrology vs. squeezed metrology. (a) Typically one considers entanglement among probes as a means of achieving a quadratic enhancement in precision: N entangled probes allow a precision gain from $\Delta\varphi/\sqrt{N}$ (left) to $\Delta\varphi/N$ (right) [15]. (b) Here we consider the role of squeezing: if a probe in a classical (coherent) state allows a precision estimate of $\Delta\varphi$, one can achieve $\Delta\varphi/\sqrt{N}$ by using N such probes (left). Devoting the same energy of the N coherent probes that optimally use the energy resources to squeezing a single probe, one can decrease the error to $\Delta\varphi/N$ (right), a quadratic gain.

The paper outline: we prove the general theory of squeezing metrology, giving the yields of the possible strategies that employ squeezing; we then illustrate it with some prototypical cases: quadrature squeezing, interferometric phase estimation, and spin squeezing. We conclude with a step-by-step procedure to obtain new squeezing based metrology protocols.

2 Results

Two observables are relevant in an estimation problem: the observable A that is measured and whose outcomes are used to estimate the parameter φ and the Hamiltonian H that encodes the parameter onto the probes, namely the generator of translations of φ through $U_\varphi = e^{iH\varphi}$. It is then natural to consider as "classical" the strategy that uses the coherent states for these two observables [3, 27]. Then the "quantum" strategies are the ones where we squeeze A and anti-squeeze H. Squeezed states for two observables A and H are the eigenstates of the operator $L(\lambda) \equiv \lambda A + iH$ [27], where $\lambda \in \mathbb{C}$ is a squeezing parameter: the states are squeezed in A and antisqueezed in H for $|\lambda| > 1$ and squeezed in H and antisqueezed in A for $|\lambda| < 1$ [1]. All states with $|\lambda| = 1$ are coherent states. [The converse is not true in Perelomov's definition of coherent states [3], where $|\lambda| \neq 1$ for some. Here we use Trifonov's notation [27]: all have $|\lambda| = 1$.] In the following, among all eigenstates with $|\lambda| = 1$, we will consider the minimum uncertainty ones: these are the ones that are customarily considered the "classical" states. These definitions of squeezed and coherent arise from the Schrödinger uncertainty relation

$$\Delta A^2 \Delta H^2 \geqslant |\frac{1}{2}\langle [A, H] \rangle|^2 \quad (1)$$
$$+ |\frac{1}{2}\langle \{A, H\} \rangle - \langle A \rangle \langle H \rangle|^2, \quad (2)$$

where ΔX^2 is the variance of X, $\langle X \rangle$ its expectation value, and $[\,,]$ and $\{\,,\}$ are the commutator and anticommutator. For the eigenstates of $L(\lambda)$, [27]

$$\Delta A^2 = \frac{|\langle C \rangle|}{2Re\lambda},\ \Delta H^2 = |\lambda|^2 \frac{|\langle C \rangle|}{2Re\lambda},$$
$$\Delta AH = -\frac{|\langle C \rangle| Im\lambda}{2Re\lambda}, \quad (3)$$

where $C \equiv [A, H]$ and $\Delta AH \equiv \frac{1}{2}\langle \{A, H\} \rangle - \langle A \rangle \langle H \rangle$ (and additional dimensionful constant may be present if A and H have different units). By considering a real λ, we can restrict to the Heisenberg-Robertson inequality [28] $\Delta A \Delta H \geqslant \frac{1}{2}|\langle C \rangle|$ and squeezed states: the best metrological advantage is obtained in this case (this can be easily shown by repeating the derivation below with complex λ), so we will consider only real positive λ.

When using the outcomes of A to estimate the parameter φ, error propagation implies $\Delta\varphi = \Delta A / |\frac{\partial}{\partial\varphi}\langle A \rangle|$. We can evaluate the derivative recalling that the state of the probe is evolved by

[1] In the case in which $i[A, B]$ is not strictly positive or negative, as for spin systems, additional squeezed states may exist [27].

the unitary $U_\varphi = e^{iH\varphi}$, so that

$$\frac{\partial}{\partial\varphi}\langle A\rangle = \frac{\partial}{\partial\varphi}\langle 0|e^{-iH\varphi}Ae^{iH\varphi}|0\rangle = -i\langle[H,A]\rangle \quad (4)$$

where $|0\rangle$ is the initial state of the probe. So

$$\Delta\varphi = \Delta A/|\langle C\rangle| = 1/\sqrt{2\lambda|\langle C\rangle|}\ , \quad (5)$$

where we used (3). This equation is also valid for coherent states where $\lambda = 1$. When repeating the estimation procedure N times, the central limit theorem implies that the rmse reduces by $\sqrt{N}$ to $\Delta\varphi(N) \equiv \Delta\varphi/\sqrt{N}$.

We now show that a squeezed state with the same energy as N minimum-uncertainty coherent states can allow a quadratic gain when comparing $\Delta\varphi_{sq}$ to $\Delta\varphi_{cl}(N) = \Delta\varphi_{cl}/\sqrt{N}$ (i.e. what can be achieved classically by repeating N times), where the suffix sq and cl indicates that the respective quantities are calculated on squeezed or minimum-uncertainty coherent states. Namely, $\Delta\varphi_{sq}/\Delta\varphi_{cl} \sim 1/N$ which is a quadratic gain over $\Delta\varphi_{cl}(N)/\Delta\varphi_{cl} = 1/\sqrt{N}$. Here N is the number of coherent probes that can be produced with the squeezed state's energy, namely

$$N = (\langle H\rangle_{sq} - E_0)/(\langle H\rangle_{cl} - E_0)\ , \quad (6)$$

where E_0 is the ground state energy. Clearly we are interested in comparing the best estimation strategies, since one can always get a worse strategy by simply wasting energy resources.

For arbitrary settings, not just the setup here, the best estimation strategies are the ones that satisfy the quantum measurement bound of [29] with equality. This bound implies that

$$\Delta\varphi \geqslant \max\Big[\frac{\kappa}{\nu(\langle H\rangle - E_0)}, \frac{\gamma}{\sqrt{\nu}\Delta H}\Big]\ , \quad (7)$$

where κ and γ are numerical constants of order 1 (see [29]) and ν is the number of times the estimation is repeated (here we will consider $\nu = 1$). The first term in the max expresses a quantum speed limit [30] and the second arises from the time-energy uncertainty [31] (our choice $\nu = 1$ follows from these results [29]). Eq. (7) implies that any energy beyond the standard deviation will be wasted for the estimation: if $\langle H\rangle - E_0 > \zeta\Delta H$ (with $\zeta \equiv \kappa/\gamma$), then the error $\Delta\varphi$ is dominated by ΔH: "too much energy" strategies. To avoid wastes, we should choose $\langle H\rangle - E_0 \simeq \zeta\Delta H$ ("good strategies"), where the "$\simeq$" sign emphasizes that only the order of magnitude of the two terms is important since Eq. (7) is not a tight bound [29]. Eq. (7) also implies that estimation strategies that have $\langle H\rangle - E_0 < \zeta\Delta H$ ("too little energy" strategy) have error $\Delta\varphi$ dominated by the energy: they cannot achieve the error of Eq. (5), but are limited to $\Delta\varphi \sim \kappa/(\langle H\rangle - E_o)$.

The "good strategy" energy requirement $\langle H\rangle - E_0 \simeq \zeta\Delta H$ can always be enforced when squeezing is used, since the ratio between energy and its standard deviation can be tuned through the squeezing parameter λ: one can assign an arbitrary fraction of the energy budget to squeezing the probe. Instead, for classical strategies, the requirement of having a coherent state may be inconsistent with good strategies: for example spin coherent states have "too much energy". Otherwise typically the "good strategy" requirement can be satisfied only for a specific value of the energy, as for the harmonic oscillator. Indeed, Eq. (3) fixes ΔH^2 to $|\langle C\rangle|/2$, and fixing it also to $\langle H - E_0\rangle^2$ may be possible for coherent states only for a given energy. For classical strategies with too little energy, using (6) will underestimate N since some of the coherent state energy is not used for estimation, see Eq. (7). For these strategies, one should use $N = (\langle H\rangle_{sq} - E_0)/\Delta H_{cl}$ in place of (6) to count only for the energy that is actually employed for the estimation in classical strategies. The focus on "good strategies" is not a limitation of our paper, because these are the strategies that are optimal: the ones that do not waste energy.

Consider the "good strategies" first. In this case, replacing $\langle H\rangle - E_0 = \zeta\Delta H$ into (6) we find

$$N \simeq \frac{\Delta H_{sq}}{\Delta H_{cl}} = \sqrt{\lambda\,|\langle C\rangle_{sq}/\langle C\rangle_{cl}|}\ , \quad (8)$$

where we used (3) in the last equality. Using (5), we find

$$\frac{\Delta\varphi_{sq}}{\Delta\varphi_{cl}} = \sqrt{|\langle C\rangle_{cl}/\lambda\langle C\rangle_{sq}|} = \frac{1}{N}\ , \quad (9)$$

a quadratic gain over the classical strategy of repeating N times the classical estimation: $\Delta\varphi_{cl}(N)/\Delta\varphi_{cl} = 1/\sqrt{N}$. In other words, when comparing the squeezed and classical strategy *of equal energy*, we find $\Delta\varphi_{sq}/\Delta\varphi_{cl}(N) = 1/\sqrt{N}$, the main result of our paper. Since we obtained the quadratic gain when comparing the best estimation strategies, this is optimal: one cannot achieve a larger gain (unless one employs suboptimal classical strategies). This justifies our claim that (9) represents the Heisenberg squeezing bound.

We emphasize that in this derivation we have neglected multiplicative and additive constant factors of order one: as a general theory containing them is impossible (they depend on the specific implementations). In principle, in the abstract scenario we analyzed here, one may consider a *single* coherent state that is not minimum-uncertainty (namely a state with equal spreads $\Delta A = \Delta H$ which are larger than their minimum value) and devote all the energy to it. In this case, that state would already achieve the optimal precision without any need for squeezing. However, this strategy does not exist in all the examples we analyzed because those coherent states have always a bounded variance and the "good strategy" condition cannot be met. Moreover, these large-uncertainty coherent states would not be recognized as a "classical" strategy.

Consider now bad estimation strategies. For classical strategies with too much energy, the above results still hold if one uses the appropriate N discussed above. For strategies with too little energy, Eq. (5) does not apply and we need to amend the above derivation, but we still obtain the same result: consider first the case of a suboptimal classical estimation strategy, where $\Delta\varphi_{cl} \sim \kappa/(\langle H\rangle_{cl} - E_o)$, so

$$\frac{\Delta\varphi_{sq}}{\Delta\varphi_{cl}} = \frac{\langle H\rangle_{cl} - E_0}{\kappa\sqrt{2\lambda|\langle C\rangle_{sq}|}} = \frac{1}{2\kappa}\frac{1}{N} \,, \qquad (10)$$

where we used the fact that $N = \Delta H_{sq}/(\langle H\rangle_{cl} - E_0)$ in this case. Again we find a quadratic gain (apart from an inconsequential numerical constant of order one). We obtain analogous results when comparing two bad estimation strategies or a bad squeezed strategy with a good classical strategy.

3 Discussion

(i) The above proof elucidates why squeezing is beneficial: only a squeezed estimation strategy can attain the equality $\langle H\rangle - E_0 = \zeta\Delta H$ for all energies. A coherent state typically attains this only for a specific energy, so it is suboptimal when the procedure is repeated N times: the $\langle H\rangle$ term in (7) scales linearly with $\nu = N$, whereas the ΔH term scales as $\sqrt{\nu} = \sqrt{N}$. Hence, given an energy budget, repeating the measurement is never advantageous and a single-shot *classical* strategy that optimally uses all resources may exist only for a specific energy. Then, only for this value of the energy, the single-shot classical strategy performs as well as the squeezed one, otherwise squeezing the probe is always better. (ii) The squeezing parameter λ is not present in the final result of (9). Nonetheless, the results obtained hold only for $\lambda > 1$, corresponding to a reduction in the fluctuations of the observable A that is measured, as expected. Indeed, for $\lambda < 1$, corresponding to an increase in fluctuations of A, Eq. (8) typically implies that $N < 1$ since the optimal squeezed strategy's energy is less than the classical one, and the quadratic gain would favor the classical strategies in that case. Intuitively, this can be seen by considering the minimum uncertainty states for which $\Delta A\Delta H = |\langle C\rangle|/2$. Then

$$\Delta\varphi = \Delta A/|\frac{\partial}{\partial\varphi}\langle A\rangle| = \Delta A/\|\langle C\rangle| = 1/(2\Delta H) \,. \qquad (11)$$

Then, for $\lambda > 1$ we get better accuracy since ΔH is increased, whereas the opposite happens for $\lambda < 1$. (iii) The theory above refers to the estimation of φ from the measurement of an observable A, but it applies also to POVM measurements. Indeed a POVM can be extended to a projective POVM through the Naimark dilation theorem [32] without changing the measurement outcome statistics. One can then assign arbitrary "eigenvalues" to each element of this POVM to obtain an observable A to which the above theory applies. (iv) As for entanglement-based quantum metrology [22, 33, 34], the presence of noise complicates the situation enormously and will be analyzed elsewhere.

4 Examples

We now show some simple examples that refer to different (finite and infinite-dimensional) Hilbert spaces *and* to different observables.

4.1 Position measurements

Consider the situation where we want to estimate the position X. The generator of translation of position is the momentum P, so we choose $H = P$. This Hamiltonian is not lower bounded so the average energy is infinite for any state: we need to introduce an energy cutoff, e.g. by supposing that states have negligible negative momenta components. [Alternatively, one could also consider $H = |P|$ as the Hamiltonian.] We consider a position-momentum squeezed state [1], whose wavefunction is a Gaussian [2]. Introducing (arbitrary) constants m (mass) and ω (angular frequency), we can write X and P in terms of creation and annihilation operators through $a = X\sqrt{m\omega/2\hbar} + iP/\sqrt{2\hbar m\omega}$. The squeezed states are eigenstates of $\mu a + \nu a^\dagger$ with $\mu = (\lambda+1)/\sqrt{4\lambda}$ and $\nu = (\lambda-1)/\sqrt{4\lambda}$. As before, the coherent state is obtained for $\lambda = 1$, yielding the "standard" coherent states. The rmse are [1] $\Delta X = \sqrt{\hbar/(2\lambda m\omega)}$ and $\Delta P = \sqrt{m\omega\hbar\lambda/2}$. Then, assuming zero-energy ground state, and using the least energetic squeezed and coherent states $\langle P\rangle \simeq \Delta P$ (the ones with smallest energy that still have negligible negative energy components), we have

$$N = \frac{\langle P\rangle_{sq}}{\langle P\rangle_{cl}} = \frac{\Delta P_{sq}}{\Delta P_{cl}} \simeq \sqrt{\lambda} \Rightarrow \frac{\Delta X_{sq}}{\Delta X_{cl}} = \frac{1}{\sqrt{\lambda}} \simeq \frac{1}{N}\ , \tag{12}$$

a quadratic improvement over the classical strategy of repeating N times the optimal coherent state measurement, which gives $\Delta X_{cl}(N)/\Delta X_{cl} = 1/\sqrt{N}$.

4.2 Optical interferometry

As a second example, consider interferometric phase ϕ measurements [35]. In this case $H = a^\dagger a \equiv \hat{N}$ and A is whatever observable (or POVM) is measured. Since the phase is periodic, the rmse is not an appropriate uncertainty measure [36] unless the uncertainty is small compared to 2π. One must resort to Susskind-Glogower uncertainty relations (SGUR) [37]

$$\Delta\hat{N}\Delta\hat{C} \geqslant \frac{1}{2}\langle\hat{S}\rangle\ ,\ \Delta\hat{N}\Delta\hat{S} \geqslant \frac{1}{2}\langle\hat{C}\rangle\ , \tag{13}$$

where $\hat{C} = (E_+ + E_-)/2$ and $\hat{S} = i(E_+ - E_-)/2$ are the "cosine" and "sine" operators with $E_\pm = \sum_n |n\rangle\langle n \mp 1|$ (in the Fock basis). Both $\hat{S}$ and $\hat{C}$ can be used as the observable A of the theory given above. Indeed, the SGUR can be used because $\hat{S} = \frac{\partial}{\partial\phi}\hat{C} = i[H,\hat{C}]$ and $\hat{C} = \frac{\partial}{\partial\phi}\hat{S} = i[H,\hat{S}]$. [37]. Hence Eq. (5) can be replaced by $\Delta\phi = \Delta C/|\langle S\rangle| = \Delta S/|\langle C\rangle|$ and we can write SGUR as $\Delta\hat{N}\Delta\phi \geqslant \frac{1}{2}$ (meaningful if $\Delta\phi \ll 2\pi$, with an appropriate choice of boundaries far from the average ϕ). This implies that the minimum uncertainty squeezed and coherent states for SGUR can be employed in the theory presented here to get a quadratic improvement for phase sensing. (The proof is basically identical to the one for X and P presented below, since ϕ and $\hat{N}$ can be considered as conjugate variables in the above regime, where $\langle\phi\rangle$ is basically null at the boundaries.) These states were determined in [38], but unfortunately they seem to have no physical relevance, although they can be approximated in particular regimes, e.g. [39, 40]. The usual coherent states $|\alpha\rangle$, eigenstates of the annihilation a, are not minimum uncertainty states for SGUR [38], although they approximate them for large average photon number [37].

The prototypical squeezed-light interferometric measurement is the one proposed by Caves [41]. While the original proposal is not optimal [22], one with a modified detection strategy is: it has a phase uncertainty $\Delta\phi_{sq} \simeq 1/\langle a^\dagger a\rangle$ at the optimal working point [42]. In contrast, coherent states $|\alpha\rangle$ can only achieve the standard quantum limit $\Delta\phi_{cl} \simeq 1/\sqrt{\langle a^\dagger a\rangle}$, so the classical strategy that optimally employs the energy resources, i.e. the one for which $\langle a^\dagger a\rangle \simeq \Delta H$, is the one that employs coherent states with $\langle a^\dagger a\rangle = 1$ since the coherent state's Poissonian statistic implies that $\langle a^\dagger a\rangle = \Delta H^2$. Then a quadratic gain over *this* strategy (when repeated N times) follows: the energy of the repeated strategy is $\langle a^\dagger a\rangle_N = N$, so that $\Delta\phi_{sq}/\Delta\phi_{cl}(N) = \sqrt{N}/\langle a^\dagger a\rangle = 1/\sqrt{N}$ or, when comparing with the single-shot optimal classical strategy, $\Delta\phi_{sq}/\Delta\phi_{cl} = 1/\langle a^\dagger a\rangle = 1/N$.

As a further example of phase estimation that is optimal only in certain regimes, consider quadrature squeezing for phase estimation (quadrature squeezing is different from SGUR squeezing). Namely, estimate phase shifts ϕ generated by $H = a^\dagger a$, by measuring the quadrature $P = i(a^\dagger - a)/\sqrt{2}$. Then $\Delta\phi = \Delta P/|\langle X\rangle|$, with $X = (a + a^\dagger)/\sqrt{2}$ since $\frac{\partial}{\partial\phi}P = i[a^\dagger a, P] = -X$. For a quadrature squeezed displaced state we find $\langle X\rangle = \sqrt{2}Re(\alpha)$, and $\Delta P = e^{-\xi}/\sqrt{2}$, where α and ξ are the displacement and squeezing parameters and where we choose real ξ (the only interesting case here). Since $\langle H\rangle = |\alpha|^2 + \sinh^2|\xi|$ [44], it is clear that the good strategies (the ones which do not waste energy) are the ones where

α_{sq} and α_{cl} are real. For these,

$$\frac{\Delta\phi_{sq}}{\Delta\phi_{cl}} = \frac{\Delta P_{sq}}{\Delta P_{cl}} \frac{|\langle X\rangle_{cl}|}{|\langle X\rangle_{sq}|} = \frac{e^{-\xi}|\alpha_{cl}|}{|\alpha_{sq}|}\ , \tag{14}$$

$$N = \frac{\langle H\rangle_{sq}}{\langle H\rangle_{cl}} = \frac{\sinh^2|\xi|+\alpha_{sq}^2}{\alpha_{cl}^2} \simeq \tfrac{1}{2}e^{2|\xi|} + \alpha_{sq}^2, \tag{15}$$

where the last equality holds for large squeezing $|\xi| \gg 1$ and for the optimal classical strategy that, again, is for $\langle a^\dagger a\rangle = \Delta H = \Delta H^2 = \alpha_{cl}^2 = 1$. Since $x^2 + y^2 \geqslant 2xy$ for any real x, y, we find

$$\frac{\Delta\phi_{sq}}{\Delta\phi_{cl}} = \frac{1}{e^{\xi}|\alpha_{sq}|} \geqslant \frac{2}{e^{2|\xi|}/2 + \alpha_{sq}^2} = \frac{2}{N}, \tag{16}$$

if we choose $\xi > 0$ (i.e. P-direction squeezing). The inequality (16) becomes an equality for $\alpha_{sq}^2 = \frac{1}{2}e^{2|\xi|}$, namely if we devote half of the energy to squeezing and half to displacing, which is known to be the best way to allocate the energy [43]. So, in the limit of large P-direction squeezing, we have optimality also in this case. Many other optical phase estimation strategies based on squeezing are known, e.g. [10, 13, 45, 46].

4.3 Spin squeezing

As a final example consider spin squeezing, with $A = J_x$ and $H = -J_y$. In this case, not all squeezed states are eigenstates of $L(\lambda)$ since C is not strictly positive or negative [27]. Thus, instead of limiting ourselves to the eigenstates of $L(\lambda = 1)$, we will employ the $su(2)$ coherent states [3] which are a larger class [27]. These are defined as $|\beta\rangle \equiv \exp(\beta J_+ - \beta^* J_-)|j;-j\rangle$, with $J_\pm \equiv J_x \pm iJ_y$, $\beta \in \mathbb{C}$, and $|j;-j\rangle$ the lowest-weight eigenvector of J_z. The latter state is the only eigenstate of $L(\lambda = 1) = J_x - iJ_y \equiv J_-$ (this is the reason for the minus sign in the definition of H, without which we would obtain the equivalent coherent state class originating from $|j;+j\rangle$). The states $|\beta\rangle$ have $\Delta J_x^2 = j(1 - \sin^2\theta\cos^2\phi)/2$, $\Delta H^2 = \Delta J_y^2 = j(1 - \sin^2\theta\sin^2\phi)/2$, $\langle -J_y\rangle - E_0 = j(1 + \sin\theta\sin\phi)$, and $|\langle[J_x, J_y]\rangle| = |\langle J_z\rangle| = j|\cos\theta|$, where θ and ϕ are defined as $\beta = -e^{i\phi}\tan(\theta/2)$ [3]. All these coherent states give rise to a "too much energy" strategy, since $\langle H\rangle_{cl} - E_0 > \Delta H$ (except for the case of $|\beta\rangle$ eigenstate of J_y which is obviously useless for estimation). Thus, we need to define $N = (\langle H\rangle_{sq} - E_0)/\Delta H_{cl}$ in order to count only the energy that is actually employed in the estimation in the classical strategy. The precision achievable in estimating a rotation by an angle φ around the y axis is $\Delta\varphi = \Delta J_x/|\langle J_z\rangle| = \sqrt{\frac{1-\sin^2\theta\cos^2\phi}{2j\cos^2\theta}} \geqslant \frac{1}{\sqrt{2j}}$, where the last inequality becomes an equality on the eigenstates of J_z (i.e. for $\theta = 0, \pi$) as expected. The squeezed strategy should employ squeezed spin states [6] with reduced fluctuations in J_x, namely $\Delta J_x = 1/\sqrt{2}$, $\Delta J_y = j/\sqrt{2}$ for which $\langle -J_y\rangle + j = j$. This is a "good strategy", since $\langle H\rangle_{sq} - E_0 = j \simeq \Delta H_{sq} = j/\sqrt{2}$. It achieves the Heisenberg squeezing bound precision $\Delta\varphi \simeq 1/j$ [6]. The comparison between the classical and the squeezed strategies follows:

$$N = (-\langle J_y\rangle_{sq} + j)/\Delta J_{y\,cl} = j/\sqrt{j/2} = \sqrt{2j}$$
$$\Rightarrow \Delta\varphi_{sq}/\Delta\varphi_{cl} = \sqrt{2j}/j = 2/N, \tag{17}$$

again a quadratic improvement (apart from a constant of order one) over the $\Delta\varphi_{cl}(N)/\Delta\varphi_{cl} = 1/\sqrt{N}$ precision obtained by repeating the classical strategy N times. Note that in all spin-squeezing literature N is defined differently, as the number of elementary spin-1/2 particles equivalent to a j-spin system [6]. Thus elsewhere spin squeezing is analyzed in terms of the entanglement among these particles, using the entanglement-based theory of quantum metrology [15].

5 New protocols

We now detail the step-by-step procedure that one can employ to devise a new metrology protocol based on squeezing, employing the results of our paper:

1. Identify the Hamiltonian H that generates the translations of the parameter of observable that we want to estimate. Identify the observable A that we can employ in the estimation. Note that A is in general not unique (as shown in the above examples).

2. Solve the eigenvalue equation for the non-Hermitian operator $\lambda A + iH$ to find the squeezed states of the system (the intelligent states) [27].

3. Use all the energy available to the estimation procedure to prepare a *single* probe in such squeezed state.

4. Let the probe evolve with the Hamiltonian H and then measure the observable A.

5. The results of our paper guarantee that the outcome will have an error on the parameter or observable to be estimated that scales as $1/N$, where N is the number of classical (coherent) probes that could be prepared using the same energy. These classical probes would give an error that scales as the standard quantum limit $1/\sqrt{N}$ because of the central limit theorem. Namely, the squeezed strategy described here has a quadratic gain.

6 Conclusions

In conclusion, we have proposed the general theory of quantum metrology that is focused on squeezing, instead of on entanglement. Our framework applies to arbitrary quantum systems, since it uses the general theory of squeezing. We have also provided various examples of applications of such theory. We believe that this theory will be of use to theoreticians to develop new protocols, rather than to experimentalists.

We acknowledge funding from Unipv, "Blue sky" project - grant n. BSR1718573 and the ATTRACT project funded by European Union's Horizon 2020 research and innovation programme under grant agreement No 777222.

References

[1] H.P. Yuen, Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions, Phys. Rev. Lett. **51**, 719 (1983) doi:10.1103/PhysRevLett.51.719.

[2] H.P. Yuen, Two-photon coherent states of the radiation field Phys. Rev. A **13**, 2226 (1976) doi:10.1103/PhysRevA.13.2226

[3] A. Perelomov, *Generalized coherent states and their applications* (Springer, 1986).

[4] M. Ozawa, Measurement breaking the standard quantum limit for free-mass position, Phys. Rev. Lett. **60**, 385 (1988) doi:10.1103/PhysRevLett.60.385

[5] D.A. Trifonov, Generalized uncertainty relations and coherent and squeezed states, JOSA A **17**, 2486 (2000) doi:10.1364/JOSAA.17.002486; C. Aragone, E. Chalbaud, and S. Salamó, On intelligent spin states, J. Math. Phys. **17**, 1963 (1976) doi:10.1063/1.522835.

[6] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, D. J. Heinzen, Spin squeezing and reduced quantum noise in spectroscopy, Phys. Rev. A **46**, R6797 (1992) doi:10.1103/PhysRevA.46.R6797.

[7] M. Kitagawa, M. Ueda, Squeezed spin states, Phys. Rev. A **47**, 5138 (1993) doi:10.1103/PhysRevA.47.5138.

[8] D. J. Wineland, J. J. Bollinger, W. M. Itano, D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, Phys. Rev. A 50, 67 (1994) doi:10.1103/PhysRevA.50.67.

[9] J. Ma, X. Wang, C. P. Sun, F. Nori, Quantum spin squeezing, Phys. Rep. **509**, 89 (2011) doi:10.1016/j.physrep.2011.08.003.

[10] A. Luis, J. Perina, SU(2) coherent states in parametric down-conversion, Phys. Rev. A **53**, 1886 (1996) doi:10.1103/PhysRevA.53.1886.

[11] C. Brif, A. Mann, Nonclassical interferometry with intelligent light, Phys. Rev. A **54**, 4505 (1996) doi:10.1103/PhysRevA.54.4505.

[12] A.I. Lvovsky, Squeezed light, in Photonics Volume 1: Fundamentals of Photonics and Physics, pg. 121 (Wiley, 2015).

[13] J. Flórez, E. Giese, D. Curic, L. Giner, R.W. Boyd, J.S. Lundeen, The phase sensitivity of a fully quantum three-mode nonlinear interferometer, New J. Phys. **20**, 123022 (2018) doi:10.1088/1367-2630/aaf3d2.

[14] B.E. Anderson, B.L. Schmittberger, P. Gupta, K.M. Jones, P.D. Lett, Optimal phase measurements with bright- and vacuum-seeded SU(1,1) interferometers, Phys. Rev. A **95**, 063843 (2017) doi:10.1103/PhysRevA.95.063843.

[15] V. Giovannetti, S. Lloyd, L. Maccone, Quantum metrology, Phys. Rev. Lett. **96**, 010401 (2006) doi:10.1103/PhysRevLett.96.010401.

[16] S. L. Braunstein, C. M. Caves, G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. **247**, 135-173 (1996) doi:10.1006/aphy.1996.0040.

[17] S. L. Braunstein, C. M. Caves, Statistical distance and the geometry of quantum states. Phys. Rev. Lett. **72**, 3439 (1994) doi:10.1103/PhysRevLett.72.3439.

[18] M. Zwierz, C. A. Pérez-Delgado, P. Kok, Ultimate limits to quantum metrology and the meaning of the Heisenberg limit, Phys. Rev. A **85**, 042112 (2012) doi:10.1103/PhysRevA.85.042112.

[19] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: beating the standard quantum limit, Science **306**, 1330 (2004) doi:10.1126/science.1104149.

[20] V. Giovannetti, S. Lloyd, L. Maccone, Advances in quantum metrology, Nature Phot. **5**, 222 (2011) doi:10.1038/nphoton.2011.35.

[21] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. **7**, 125 (2009) doi:10.1142/S0219749909004839.

[22] R. Demkowicz-Dobrzański, M. Jarzyna, J. Kodyňski, Quantum Limits in Optical Interferometry, Progress in Optics, **60**, 345 (2015) doi:10.1016/bs.po.2015.02.003 also at arXiv:1405.7703 (2014).

[23] G. Toth, I. Apellaniz, Quantum metrology from a quantum information science perspective, J. Phys. A: Math Th.**47**, 424006 (2014) doi:10.1088/1751-8113/47/42/424006.

[24] M. Kacprowicz, R. Demkowicz-Dobrzanski, W. Wasilewski, K. Banaszek, I. A. Walmsley, Experimental quantum-enhanced estimation of a lossy phase shift, Nature Photonics **4**, 357 (2010) doi:10.1038/nphoton.2010.39.

[25] D.S. Simon, G. Jaeger, A.V. Sergienko, *Quantum Metrology, Imaging, and Communication* (Springer, 2017).

[26] J. C. F. Matthews, *et al.* Towards practical quantum metrology with photon counting, NPJ Quantum Inf. **2**, 16023 (2016) doi:10.1038/npjqi.2016.23.

[27] D.A. Trifonov, Generalized intelligent states and squeezing, J. Math. Phys. **35**, 2297 (1994) doi:10.1063/1.530553, also arXiv:1401.4118.

[28] H.P. Robertson, The uncertainty principle, Phys. Rev. **34**, 163 (1929) doi:10.1103/PhysRev.34.163.

[29] V.Giovannetti, S. Lloyd, L.Maccone, Quantum measurement bounds beyond the uncertainty relations, Phys. Rev. Lett. **108**, 260405 (2012) doi:10.1103/PhysRevLett.108.260405.

[30] N. Margolus, L. B. Levitin, The maximum speed of dynamical evolution, Physica D **120**, 188 (1998) doi:10.1016/S0167-2789(98)00054-2.

[31] L. Mandelstam, I. G. Tamm, The Uncertainty Relation Between Energy and Time in Non-relativistic Quantum Mechanics, *J. Phys. USSR* **9**, 249 (1945) doi:10.1007/978-3-642-74626-0_8.

[32] A. Peres, *Quantum Theory: Concepts and Methods*, (Kluwer ac. publ., Dordrecht, 1993).

[33] R. Demkowicz-Dobrzanski, J. Koldynski, M. Guta, The elusive Heisenberg limit in quantum-enhanced metrology, Nature Comm. **3**, 1063 (2012) doi:10.1038/ncomms2067.

[34] B. M. Escher, R. L. de Matos Filho, L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nature Phys. **7**, 406 (2011) doi:doi.org/10.1038/nphys1958.

[35] S.M. Barnett D.T. Pegg, On the Hermitian Optical Phase Operator, J. Mod. Opt. **36**, 7 (1988) doi:10.1080/09500348914550021.

[36] A.S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (Edizioni della Normale, Pisa 2011, reprinted from 1980).

[37] P. Carruthers M.M. Nieto, Phase and Angle Variables in Quantum Mechanics, Rev. Mod. Phys. **40**, 411 (1968) doi:10.1103/RevModPhys.40.411e .

[38] R. Jackiw, Minimum Uncertainty Product, Number-Phase Uncertainty Product, and Coherent States, J. Math. Phys. **9**, 339 (1968) doi:10.1063/1.1664585.

[39] Y. Yamamoto, N. Imoto, and S. Machida, Amplitude squeezing in a semiconductor laser using quantum nondemolition measurement and negative feedback, Phys. Rev. A **33**, 3243 (1986) doi:10.1103/PhysRevA.33.3243.

[40] A.D. Wilson-Gordon, V. Buek, P.L. Knight, Statistical and phase properties of displaced Kerr states, Phys. Rev. A **44**, 7647 (1991) doi:10.1103/PhysRevA.44.7647.

[41] C.M. Caves, Quantum-mechanical noise in an interferometer Phys. Rev. D **23**, 1693 (1981) doi:10.1103/PhysRevD.23.1693.

[42] L. Pezzé, A. Smerzi, Mach-Zehnder Interferometry at the Heisenberg Limit with Coherent and Squeezed-Vacuum Light, Phys. Rev. Lett. **100**, 073601 (2008) doi:10.1103/PhysRevLett.100.073601.

[43] S. Lloyd, private communication (2010).

[44] C.M.A. Dantas, N.G. de Almeida, B. Baseia, Statistical Properties of the Squeezed Displaced. Number States, Braz. J. Phys. **28**, 462 (1998) doi:10.1590/S0103-97331998000400021.

[45] T. Matsubara, P. Facchi, V. Giovannetti, K. Yuasa, Optimal Gaussian Metrology for Generic Multimode Interferometric Circuit, New J. Phys. **21**, 033014 (2019) doi:10.1088/1367-2630/ab0604.

[46] A. De Pasquale, P. Facchi, G. Florio, V. Giovannetti, K. Matsuoka, K. Yuasa, Two-mode bosonic quantum metrology with number fluctuations Phys. Rev. A **92**, 042115 (2015) doi:10.1103/PhysRevA.92.042115.