

Amplification of quadratic Hamiltonians

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Speeding up the dynamics of a quantum system is of paramount importance for quantum technologies. However, in finite dimensions and without full knowledge of the details of the system, it is easily shown to be *impossible*. In contrast we show that continuous variable systems described by a certain class of quadratic Hamiltonians can be sped up without such detailed knowledge. We call the resultant procedure *Hamiltonian amplification* (HA). The HA method relies on the application of local squeezing operations allowing for amplifying even unknown or noisy couplings and frequencies by acting on individual modes. Furthermore, we show how to combine HA with dynamical decoupling to achieve amplified Hamiltonians that are free from environmental noise. Finally, we illustrate a significant reduction in gate times of cavity resonator qubits as one potential use of HA.

1 Introduction

Strong interactions between the components of a quantum device are crucial for maintaining the quantum effects relevant for quantum information processing. For instance, the generation of entanglement between system components [1, 2], the implementation of multi-qubit gates [3, 4, 5], and, in general, coherent interactions used in quantum metrology and quantum sensors [6, 7], all rely on sufficiently strong couplings. Furthermore, the speed at which a quantum system can evolve towards a desired target is inherently limited by the strength of the underlying Hamiltonian which governs the evolution [8, 9]. A generic framework describing

System dimension	Dynamical Decoupling $H_0 \rightarrow 0$	Hamiltonian Amplification $H_0 \rightarrow \lambda H_0, \lambda > 1$
Finite	✓	✗
Infinite	✗	✓

Figure 1: The paper introduces Hamiltonian Amplification (HA) that allows for amplifying an unknown Hamiltonian H_0 by an amount $\lambda > 1$. While HA is impossible in finite dimensional systems, we show that some infinite dimensional system (Eq. (2)) can be amplified through rapidly applied local parametric controls. In contrast, for some infinite dimensional systems the opposite, i.e., Dynamical Decoupling (DD) that achieves averaging out Hamiltonians, is impossible [11], while for finite dimensional systems DD can always succeed [12]. However, for infinite dimensional systems DD can be used to suppress certain interactions. Thus, we show that HA can be combined with DD to enhance desired evolutions and simultaneously fight decoherence.

how such couplings/processes can be amplified is therefore highly desirable. As schematically summarized in Fig. 1, while *Dynamical Decoupling* (DD) achieves the opposite by suppressing unknown couplings/processes through rapidly applied controls [10, 11, 12], here we propose an amplification procedure referred to as *Hamiltonian amplification* (HA). In particular, we show that through rapidly applying parametric controls, even unknown parameters of an important class of quadratic Hamiltonians (see. Eq. (2)) can generically be enhanced. Consequently, by combining HA and (partial) DD, system parameters can selectively be amplified and unwanted interactions, for instance with the environment, are simultaneously suppressed, thereby opening up a path for the manipulation of quantum systems in ways that were commonly thought to be impossible.

Given a time independent Hamiltonian H_0 , our

goal is to turn the natural evolution $\exp(-iH_0t)$ into an accelerated evolution $\exp(-i\lambda H_0t)$, $\lambda > 1$, by adding a suitable control Hamiltonian $H_c(t)$ that induces a speed-up of the system's evolution. If we know H_0 and have full control over the system, this can be trivially achieved by choosing $H_c(t) = H_c = (\lambda - 1)H_0$. But the realistic case of interest arises if some parameters of the Hamiltonian are unknown or if we have only partial control of the system. If the speed-up can be achieved under these conditions, a physical effect observed at a time t can then be observed at a shorter time t/λ without knowing the details of the system. For example, a particularly important goal would be amplification of the (possibly uncertain) interaction strength between qubits by local controls only.

While in finite dimensions such amplification is generally *not possible* with either unknown parameters or partial control (as a consequence of norm preservation [13]) we show, surprisingly, that *both* limitations can be overcome in some infinite-dimensional systems by parametrically driving the system components.

Unlike other protocols recently developed for superconducting circuits and opto-mechanics [14, 15, 16] that suffer from also increasing the interactions with the environment [17], HA provides here a general framework that goes beyond a particular implementation, and combined with DD can avoid this drawback.

2 Amplifying quadratic Hamiltonians

Consider a quantum system described by a Hamiltonian H_0 that is driven by means of an external, possibly time dependent, control Hamiltonian $H_c(t)$ so that the total Hamiltonian governing the system's evolution reads $H(t) = H_0 + H_c(t)$. Analogous to DD [12], in order to introduce HA we separate the total evolution into that of the controller alone $U_c(t)$ and its action on H_0 . The total evolution is then given by $U(t) = U_c(t)\tilde{U}(t)$ where the evolution in the frame rotating with $U_c(t)$ reads $\tilde{U}(t) = \exp(-it\bar{H}(t))$ where $\bar{H}(t)$ is given by the Magnus expansion [12, 18]. The goal of HA is to find controls so that $\bar{H}(t) = \lambda H_0$, $\lambda > 1$, for all t . As proven in Appendix A, this is impossible for finite dimensional systems. Note, however, that DD, i.e., $\lambda = 0$ for traceless H_0 , can always be achieved when the

system is finite dimensional [10]. The situation changes when continuous variable systems, such as quantum harmonic oscillators, living in an infinite dimensional space, are considered. Here DD *does not* always work [11], while, as we show below, HA is possible.

In order to introduce the concept, we start by considering the evolution of a quantum harmonic oscillator with frequency ω described by the Hamiltonian $H_0 = \frac{\omega}{2}(x^2 + p^2)$ where x and p are the canonical position and momentum operators, respectively. We begin by considering bang-bang type controls that correspond to delta-function like pulses implementing, at time intervals $\Delta t = \frac{t}{2n}$, the squeezing operations $S^{(\pm)} = \exp[\pm i\frac{r}{2}(xp + px)]$ where r is the squeezing parameter. The canonical operators transform under squeezing according to $S^{(\pm)\dagger}xS^{(\pm)} = \exp(\mp r)x$ and $S^{(\pm)\dagger}pS^{(\pm)} = \exp(\pm r)p$. Thus, if we alternate between $S^{(+)}$ and $S^{(-)}$ the modified dynamics is approximately given by $\left(S^{(-)\dagger}e^{-i\frac{H_0t}{2n}}S^{(-)}S^{(+)\dagger}e^{-i\frac{H_0t}{2n}}S^{(+)}\right)^n \approx e^{-i\cosh(2r)H_0t}$, which becomes exact in the limit of rapid squeezing, i.e., $\Delta t \rightarrow 0$, $n \rightarrow \infty$ with t fixed. Consequently the operations $V = \{S^{(\pm)}\}$ amplify the harmonic oscillator by a factor $\lambda = \cosh(2r)$. The physics behind the amplification may be readily understood: squeezing in the x -direction maps the operator x onto $e^{-r}x$, while squeezing in p -direction maps x onto $e^r x$. In the case of very fast alternation, the operator is effectively averaged into $\cosh(r)x$ and thus amplified by a factor $\cosh(r)$, and the same applies to any other phase-space quadrature. For the operators x^2 and p^2 , the same reasoning leads to an amplification by $\cosh(2r)$. We now generalize this idea to more complex continuous-variable systems.

Consider again bang-bang controls that instantaneously implement a set of (Gaussian) unitary operations V in a Suzuki-Trotter type sequence $\Lambda_{t/n} = \prod_{v \in V} v^\dagger \exp(-iH_0 \frac{t}{|V|n})v$, where H_0 is a quadratic Hamiltonian [19, 20] of a system of continuous variables (see below for details) and we denote by $|V|$ the number of elements in V . Since quadratic Hamiltonians can be represented by matrices, the map $\Lambda_{t/n}$ converges and in the limit we have $\lim_{n \rightarrow \infty} \Lambda_{t/n}^n = \exp(-itM(H_0))$ where the dynamics is governed by the average

[12],

$$M(H_0) = \frac{1}{|V|} \sum_{v \in V} v^\dagger H_0 v. \quad (1)$$

A most natural class of continuous-variable systems to consider is described by quadratic Hamiltonians of the form

$$H_0 = \sum_{i,j} (\omega_{i,j}^{(x)} x_i x_j + \omega_{i,j}^{(p)} p_i p_j), \quad (2)$$

which typically serve as a model for a wide range of quantum-optical and opto-mechanical systems [21] consisting of N linearly interacting quantum harmonic oscillators, where $\omega_{i,j}^{(x)}$ and $\omega_{i,j}^{(p)}$ determine the frequencies ($i = j$) and coupling constants ($i \neq j$). Now, assume that the dynamics can be modified through $V = \{\mathcal{S}^{(\pm)}\}$ where $\mathcal{S}^{(\pm)} = \prod_{i=1}^N S_i^{(\pm)}$, is a product of local squeezing operations $S_i^{(\pm)}$ in x_i and p_i direction of the i th oscillator. We find $M(H_0) = \cosh(2r)H_0$ for all Hamiltonians of the form (2), concluding that through local squeezing operations any quadratic Hamiltonian (2) can be amplified by a factor $\cosh(2r)$ without knowledge of the frequencies and coupling constants present in H_0 . Thus, contrary to the case of finite-dimensional systems, coupling strengths and frequencies can be amplified through local unitary operations. We remark here that for two linearly interacting harmonic oscillators the coupling can also be enhanced by acting on a single oscillator alone. This could be of particular importance for amplifying the entanglement creation between a light mode and a mechanical oscillator in an optomechanical system [2, 22].

The quadratic Hamiltonian (2) differs from a generic quadratic Hamiltonian $H = \frac{1}{2} \sum_{i,j} A_{i,j} R_i R_j$ where A is a real and symmetric $2N \times 2N$ matrix and the canonical operators constitute the vector $R = (x_1, p_1, \dots, x_N, p_N)$ [20], only by single-mode and two-mode squeezing terms $\propto x_i p_j$, which we collect to form the Hamiltonian H_1 . Such terms are left invariant by squeezing the oscillator in x and p directions and thus, a generic quadratic Hamiltonian $H = H_0 + H_1$ is transformed under amplification through $\{\mathcal{S}^{(\pm)}\}$ according to $M(H) = \cosh(2r)H_0 + H_1$. We see that if the actual system under consideration deviates from the quadratic Hamiltonian (2) by the presence of two-mode squeezing terms,

such terms become negligible when the system is sufficiently amplified. We further remark that two-mode squeezing terms can be amplified by locally squeezing around different angles. However, we did not find a procedure yet that amplifies generic quadratic Hamiltonians.

The HA procedure above assumed that the squeezing operations were implemented instantaneously and alternating infinitely fast. The performance of the protocol is now considered under more realistic conditions. The error ϵ that is induced for finite n , i.e. for a finite waiting time Δt , can be upper-bounded by using well-known bounds for the Trotter sequence [23] and exploiting the symplectic representation of quadratic Hamiltonians [19, 20]. That is, the time evolution $U = \exp(-iH_0 t)$ of a quadratic Hamiltonian H_0 is represented by a symplectic transformation $\exp(-tA\Omega) \in \text{Sp}(2N, \mathbb{R})$ where Ω is the symplectic form. Using the symplectic representation of $\Lambda_{t/n}$ and $M(H_0)$, with further details found in Appendix B, we can upper-bound the error by

$$\epsilon \leq \frac{t\Delta t \omega_{\max}^2 N^2}{4} |\sinh(4r)| e^{t\omega_{\max} N} \sqrt{\frac{\cosh(4r)}{2}}, \quad (3)$$

where the error was evaluated using the Hilbert-Schmidt norm difference of the symplectic matrices and $\omega_{\max} = \max_{i,j} \{|\omega_{i,j}^{(x)}|, |\omega_{i,j}^{(p)}|\}$ is the highest frequency/coupling constant present in the amplified Hamiltonian¹. While HA is about transforming Hamiltonians rather than preparing states, in Appendix D we also investigate an example of the ‘‘Gaussian’’ fidelity error [27] to speed up the preparation of Gaussian states using HA.

Instead of using instantaneous bang-bang operations, we now turn to amplification using smooth pulses with a finite amplitude. A time-dependent squeezing parameter $r(t)$ is considered such that the Hamiltonian of the controller reads $H_c(t) = r(t) \sum_{j=1}^N (x_j p_j + p_j x_j)$. We take a periodic controller $U_c(t + 2n\Delta t) = U_c(t)$ for $n \in \mathbb{N}$ so that the time evolution at times $t = n2\Delta t$ in the

¹While the Hilbert-Schmidt norm difference ϵ of symplectic matrices is typically used as a distance measure for (quadratic) continuous variable control systems (see e.g., [24, 25]), we remark here that it is not an operational fidelity measure. However, in the spirit of [26], ϵ can be made operational if one imposes energy constraints on the system Hamiltonian H by, for instance, considering only Gaussian states for which the expectation of H is upper bounded by a predefined value.

presence of H_0 is given by the unitary transformation $U(t) = \exp(-2in\Delta t\bar{H})$. Instead of modifying the dynamics with instantaneous unitary operations $\mathcal{S}^{(\pm)}$ applied in cycles of duration $2\Delta t$, we now change the dynamics smoothly, such that for small Δt the dynamics is effectively governed by the 1st order term of the Magnus expansion $\bar{H}^{(0)} = \frac{1}{2\Delta t} \int_0^{2\Delta t} U_c^\dagger(t) H_0 U_c(t) dt$, which is the continuous equivalent to the map (1). The higher-order terms of the Magnus expansion introduce errors which can be neglected if Δt is sufficiently small. As an example, with details found in Appendix C, the pulse $r(t) = \frac{\pi K}{2\Delta t} \cos(\frac{\pi t}{\Delta t})$ yields $\bar{H}^{(0)} = \lambda H_0$ with $\lambda = I_0(K)$ being the modified Bessel function of the first kind. Here, the amount of amplification is chosen by the constant K , which, together with the frequency of the pulse $\propto \Delta t^{-1}$, determines the pulse amplitude. We remark that for $\Delta t \rightarrow 0$, which yields perfect amplification, the pulse amplitude as well as its frequency becomes infinitely large, thus approaching an infinitely fast bang-bang type control. However, we further show in Appendix C that a large class of pulses exist that amplify the 1st order and simultaneously suppress the 2nd order of the Magnus expansion. Such pulses can be more effective in achieving HA, which was confirmed by numerical simulations. Furthermore, a detailed analysis in Appendix D of the efficiency of imperfect controls upon amplification indicates that errors in the amplification pulses enter for sufficiently small Δt linearly in the amplification process.

3 Amplification and Decoupling

While the system parameters are amplified, possibly unwanted couplings with the environment are amplified too. However, we now show how HA can be combined with some form of DD so that interactions with the environment are suppressed while system parameters are amplified, thereby achieving amplification that is free from environmental noise.

As shown in [11], it is not possible to average generic quadratic Hamiltonians to zero (or to multiples of the identity) through rapidly applied unitary operations (i.e., DD in the sense introduced before). However, parts of the Hamiltonian, such as interactions quadratic in the quadrature operators, can be suppressed [11, 28].

In general, in the limit of infinitely fast operations the dynamics is governed by a map $M^{(DD)}$ of the form (1), where the decoupling set $V^{(DD)}$ consists of unitary transformations that allow for suppressing the unwanted interactions. Consider a continuous-variable system S that interacts with another continuous-variable system E in a linear way described by the total Hamiltonian $H = H_S + H_E + H_{SE}$ where H_S , H_E are the Hamiltonians of S and E , respectively, and $H_{SE} = \sum_{i \in S, j \in E} (\omega_{i,j}^{(x)} x_i x_j + \omega_{i,j}^{(p)} p_i p_j)$ describes the interaction, assuming that all Hamiltonians are of the form (2). The dynamics of system S can always be decoupled from E using the operations $V_S^{(DD)} = \{\mathcal{R}_S(0), \mathcal{R}_S(\pi)\}$ where $\mathcal{R}_S(\phi) = \prod_{i \in S} \exp(-i\frac{\phi}{2}(x_i^2 + p_i^2))$ is a product of local rotations of the system oscillators [11, 28]. The operation $\mathcal{R}_S(\pi)$ inverts the sign in front of H_{SE} so that the interaction between S and E is suppressed by acting on S alone, i.e., $M_S^{(DD)}(H) = H_S + H_E$. In contrast to finite-dimensional systems, the dynamics of S remains invariant under the decoupling map $M_S^{(DD)}$, which is the attractive feature of why DD and HA can be combined.

Let $M_S^{(HA)}$ be the map that amplifies system S by a factor $\cosh(2r)$ through the set of operation $V_S^{(HA)} = \{\mathcal{S}_S^{(\pm)}\}$ where $\mathcal{S}_S^{(\pm)} = \prod_{i \in S} \mathcal{S}_i^{(\pm)}$ so that $M_S^{(HA)}(H_S) = \cosh(2r)H_S$, noting that any linear interaction with E is amplified too, i.e., $M_S^{(HA)}(H_{SE}) = \cosh(r)H_{SE}$. However, through concatenating both maps $\mathcal{G}_S \equiv M_S^{(HA)} \circ M_S^{(DD)}$ we see that $\mathcal{G}_S(H) = \cosh(2r)H_S + H_E$. Thus, system S becomes simultaneously amplified and decoupled from the environment E . The map $\mathcal{G}_S(\cdot) = \frac{1}{|\mathcal{G}_S|} \sum_{g \in \mathcal{G}_S} g^\dagger(\cdot)g$ that achieves amplification and decoupling is determined by the set of operations $\mathcal{G}_S = \{\mathcal{R}_S(\pi)\mathcal{S}_S^{(+)}, \mathcal{S}_S^{(+)}, \mathcal{R}_S(\pi)\mathcal{S}_S^{(-)}, \mathcal{S}_S^{(-)}\}$ applied only on S . Thus, through alternating between squeezing and rotating the system oscillators, the interaction with another system is suppressed and the system of interest is amplified, noting that DD can also be achieved using smooth pulses [11].

4 Amplification of qubit interactions

The proposed HA only works in an infinite-dimensional setting. However, finite-dimensional systems, such as qubits, can be coupled through the interaction with a quantum harmonic oscilla-

tor such that an effective qubit-qubit interaction is obtained [29, 30, 31, 32, 33, 34]. If the original couplings between the qubits and the oscillator can be amplified, so can the effective coupling strength between the qubits. For instance, for circuit QED systems [29, 30, 31] it was recently shown that the interaction strength between a charge qubit and a resonator mode is enhanced in the regime where the resonator mode is strongly squeezed [14]. However, strongly squeezing the resonator mode is experimentally challenging. In contrast, using HA to amplify the qubit-resonator coupling has the advantage that operating in the strong squeezing regime is not required. Instead, the qubit-resonator coupling is enhanced by rapidly alternating between squeezing the resonator mode along different directions.

The interaction between a resonator mode and two charge qubits can be described by a Jaynes-Cummings type Hamiltonian $H_0 = H_r + H_q + H_{\text{int}}$ where $H_r = \omega_r a^\dagger a$ and $H_q = \sum_j \frac{\omega_j}{2} \sigma_z^{(j)}$ are the free Hamiltonians of the resonator mode and the qubits $j = 1, 2$, respectively, and $H_{\text{int}} = \sum_j g_j (\sigma_+^{(j)} a + \sigma_-^{(j)} a^\dagger)$ describes the interaction. If one assumes that the oscillator is initially prepared close to the vacuum state and the qubits are strongly detuned from the resonator frequency, $\Delta_j = |\omega_j - \omega_r| \gg g_j$, the oscillator degrees of freedom can be eliminated so that between the qubits an effective exchange interaction $V = \Omega (\sigma_-^{(1)} \sigma_+^{(2)} + \sigma_+^{(1)} \sigma_-^{(2)})$ is induced. For simplicity, we assume that $\omega = \omega_j$ and $g = g_j$ for $j = 1, 2$, so that the interaction strength is given by $\Omega = \frac{g^2}{\Delta}$. For $t = \frac{\pi}{2\Omega}$ such a Hamiltonian implements a SWAP gate. Since the bosonic annihilation operator transforms under the squeezing operation $S^{(\pm)}$ according to $S^{(\pm)\dagger} a S^{(\pm)} = a \cosh(r) \mp a^\dagger \sinh(r)$ we find for the Jaynes Cummings Hamiltonian $M(H_0) = H_q + \cosh(2r)H_r + \cosh(r)H_{\text{int}}$. Thus, in the limit of squeezing the oscillator infinitely fast, the qubit-qubit interaction is determined by the amplified interaction strength

$$\Omega_{\text{amp}}(r) = \frac{\cosh^2(r)g^2}{|\omega - \cosh(2r)\omega_r|}, \quad (4)$$

so that the time to implement a SWAP gate is reduced to $t_{\text{swap}}(r) = \frac{\pi}{2\Omega_{\text{amp}}(r)}$. However, the effective qubit-qubit interaction strength cannot be arbitrarily enhanced, since for large squeezing parameters $r \gg 1$ the amplified strength Ω_{amp}

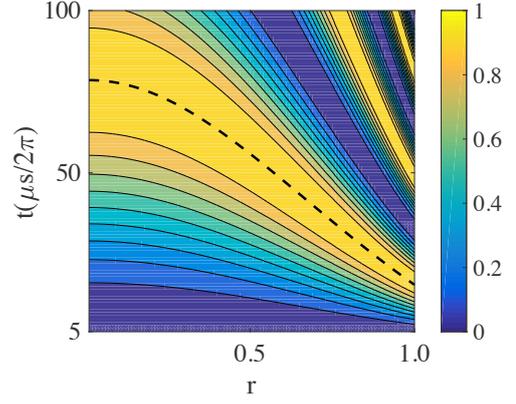


Figure 2: Probability of swapping the states of two qubits (colormap) that interact via a quantum harmonic oscillator as a function of the amount r the oscillator is rapidly squeezed and the total evolution time t . The time to swap the state (dashed black curve) is determined by the amplified frequency (4) with parameters $\omega_r/2\pi = 2.5$ GHz, $\omega/2\pi = 15$ GHz, $g/2\pi = 50$ MHz, taken in the range of a typical circuit QED setup [31].

tends to $\frac{g^2}{2\omega_r}$, as a consequence of the effect of the squeezing on the oscillator frequency as well as on the couplings². However, numerical investigations indicate that for finite squeezing a significant speed-up can be achieved.

Based on the full Jaynes-Cummings Hamiltonian, we numerically studied excitation transfer from one qubit to another, using bang-bang amplification. The qubits were initially prepared in a product state consisting of one up and one down eigenstate of σ_z and the colormap of Fig. 2 shows the probability for swapping the qubit states as a function of the evolution time t and the squeezing parameter r . The parameters can be found in the caption of Fig. 2. We see that the oscillations are accelerated by increasing the amount of squeezing. According to $t_{\text{swap}}(r)$ shown as the black dashed line, the time to swap the qubit states is significantly reduced. Unfortunately, in order to achieve such an improvement for the chosen parameters $\Delta t \approx 1$ ps. However, the better performance for the case when bounded smooth con-

²We remark here that in the regime in which the resonator mode is rapidly squeezed along x and p such that the dynamics is effectively governed by $M(H_0)$, the initial state of the resonator mode is not changed. Thus, as long as the condition $|\omega - \cosh(2r)\omega_r| \gg \cosh(r)g$ is satisfied to eliminate the resonator mode in the presence of HA, then independently of how strong the resonator is squeezed an effective two qubit interaction determined by (4) is obtained.

trols are used suggests that more sophisticated shaped pulses could lead to an efficient amplification of qubit interactions. Moreover, by combining HA with DD, obtained by additionally applying $\mathcal{R}(\pi)$ on the oscillator and σ_z on the qubits such that the Jaynes-Cummings interactions stays invariant while interactions with the environment are suppressed, gates can be implemented faster and at the same time decoherence is reduced.

5 Conclusions

As indicated in Fig. 1, every (traceless) Hamiltonian describing a finite-dimensional quantum system can be averaged to zero using a decoupling sequence. However, in finite dimensions the Hamiltonian cannot be amplified. In contrast, some infinite-dimensional systems cannot be averaged out, whereas, as we have shown, some infinite-dimensional systems can be amplified. This observation has formed the basis of *Hamiltonian Amplification* introduced here and proven to be applicable to a broad class of continuous-variable systems through local squeezing operations even without full knowledge of, or full control over, the parameters present in the Hamiltonian. We showed how Hamiltonian Amplification can be combined with Dynamical Decoupling, which allows for simultaneously amplifying the system parameters and suppressing unwanted interactions. We further illustrated Hamiltonian Amplification in a hybrid system to speed up the implementation of quantum logic gates, thereby showing the broad scope of the proposed scheme in manipulating time scales of quantum systems. By amplifying/suppressing the relevant coupling and frequency components, we believe that the scheme opens up new prospects for a wide range of areas [7, 2, 22, 35, 36, 37, 38]. We further note that the simplest form of HA was recently demonstrated in a trapped-ion experiment by amplifying the position operator through alternating between squeezing and anti-squeezing, indicating that HA is feasible with current technology [39].

There is certainly room for more sophisticated pulse sequences, possibly obtained from well-known methods used in dynamical decoupling [40, 41, 42, 43] and optimal control theory [44], that would allow for even more efficient and robust amplification. Furthermore, in this

work Hamiltonian Amplification was introduced for quadratic Hamiltonians but the observation that $M^{(\text{HA})}(x^n + p^n) = \cosh(nr)(x^n + p^n)$ with n being an integer suggest that some non-linear terms can be amplified too, which will be investigated in future work.

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Appendix

A No amplification of bounded Hamiltonians

The proof that finite dimensional quantum systems cannot be amplified follows directly from results obtained in [13]. We consider a finite dimensional quantum system whose total Hamiltonian is given by $H(t) = H_0 + H_c(t)$, where H_0 is the Hamiltonian that we want to amplify by a coherent controller described by the Hamiltonian $H_c(t)$. The total evolution $U(t) = U_c(t)\tilde{U}(t)$ can be separated into the evolution of the controller alone $U_c(t) = \mathcal{T} \exp(-i \int_0^t H_c(t') dt')$ and its action on H_0 given by $\tilde{U}(t) = \mathcal{T} \exp(-i \int_0^t U_c^\dagger(t') H_0 U_c(t') dt') = \exp(-it\bar{H}(t))$ where $\bar{H}(t)$ is given by the Magnus expansion. Since there always exist a set of unitary transformations $\{W(s)\}$ that allows to express $\bar{H}(t)$ as $\bar{H}(t) = \frac{1}{t} \int_0^t W^\dagger(t') U_c^\dagger(t') H_0 U_c(t') W(t') dt'$, we have that for any unitarily invariant norm $\|\bar{H}(t)\| \leq \|H_0\|$ for all t , which implies that Hamiltonian Amplification is not possible for finite dimensional systems. Note, however, that the proof becomes meaningless when infinite dimensional systems described by unbounded operators are considered, as then the norms are not defined.

B Derivation of the error bound

In order to bound the error for obtaining an evolution that is generated by the amplified Hamiltonian $M(H_0) = \frac{1}{2}(\mathcal{S}^{(+)\dagger} H_0 \mathcal{S}^{(+)} + \mathcal{S}^{(-)\dagger} H_0 \mathcal{S}^{(-)}) = \cosh(2r)H_0$, valid for any quadratic Hamiltonian of the form given in Eq. (2) of the main body of the manuscript, we use the symplectic representation of quadratic Hamiltonians. That is, the unitary time evolution $U = e^{-iH_0 t}$ is represented by a symplectic matrix $e^{-A_0 \Omega t} \in \text{Sp}(2N, \mathbb{R})$ where $A_0 \in \mathbb{R}^{2N \times 2N}$ is a symmetric matrix and Ω is the symplectic form. If we denote by $A^{(\pm)}$ the to $\mathcal{S}^{(\pm)\dagger} H_0 \mathcal{S}^{(\pm)}$ corresponding real and symmetric matrices such that $\frac{1}{2}(A^{(+)} + A^{(-)}) = \cosh(2r)A_0$, the error $\epsilon = \left\| e^{-\cosh(2r)A_0 \Omega t} - e^{-\frac{t}{2n} A^{(+)} \Omega} e^{-\frac{t}{2n} A^{(-)} \Omega} \right\|$ for obtaining an amplified A_0 is upper bounded by [23],

$$\epsilon \leq \frac{t\Delta t}{8} \|[A^{(+)}\Omega, A^{(-)}\Omega]\| e^{\frac{t}{2}(\|A^{(+)}\Omega\| + \|A^{(-)}\Omega\|)}, \quad (5)$$

where $\Delta t = \frac{t}{n}$ and $\|A\| = \sqrt{\text{tr}\{A^\dagger A\}}$ is the Hilbert-Schmidt norm. Since $\|[A^{(+)}\Omega, A^{(-)}\Omega]\| \leq 2|\sinh(4r)|N^2 \max_{i,j} |A_{i,j}|^2$, and $\|A^{(\pm)}\Omega\| \leq \sqrt{2 \cosh(4r)}N \max_{i,j} |A_{i,j}|$, with $\omega_{\max} = \max_{i,j} |A_{i,j}|$ we arrive at the desired bound (3).

C 1st and 2nd order terms of the Magnus expansion

Here we derive the pulse given in the main body of the manuscript as well as show that a large class of pulses simultaneously amplifies the 1st order and suppresses the 2nd order terms of the Magnus expansion. We recall from the main body of the paper that the first order of the Magnus expansion is

given by

$$\bar{H}^{(0)} = \frac{1}{2\Delta t} \int_0^{2\Delta t} H(t) dt, \quad (6)$$

where

$$\begin{aligned} H(t) &= U_c^\dagger(t) H_0 U_c(t) \\ &= \sum_{i,j} \left(\omega_{i,j}^{(x)} x_i x_j e^{-2R(t)} + \omega_{i,j}^{(p)} p_i p_j e^{2R(t)} \right), \end{aligned} \quad (7)$$

with $R(t) = \int_0^t r(t') dt'$ being the integrated pulses. Thus, in order to amplify the 1st order of the Magnus expansion we need to find a pulse $r(t)$ that yields $\bar{H}^{(0)} = \lambda H_0$ where $\lambda = \frac{1}{2\Delta t} \int_0^{2\Delta t} \exp(\pm 2R(t)) dt > 1$. Since $\int_0^{2\Delta t} \exp(\pm K \sin(\pi t/\Delta t)) dt = 2\Delta t I_0(K)$ where $I_0(K)$ is the modified Bessel function of the first kind, we find that the pulse

$$r(t) = \frac{\pi K}{2\Delta t} \cos\left(\frac{\pi t}{\Delta t}\right), \quad (8)$$

amplifies the 1st order of the Magnus expansion by an amount $\lambda = I_0(K)$.

We now turn to the question whether there exist pulses that amplify the 1st order and simultaneously suppress the 2nd order of the Magnus expansion. The 2nd order terms of the Magnus expansion reads

$$\begin{aligned} \bar{H}^{(1)} &= -\frac{i}{4\Delta t} \int_0^{2\Delta t} dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)] \\ &= -\frac{i}{2\Delta t} \sum_{i,j,i',j'} \omega_{i,j}^{(x)} \omega_{i',j'}^{(p)} [x_i x_j, p_{i'} p_{j'}] \int_0^{2\Delta t} dt_1 \int_0^{t_1} dt_2 \sinh(2(R(t_2) - R(t_1))). \end{aligned} \quad (9)$$

Thus, the integrated pulse $R(t)$ that simultaneously amplifies $\bar{H}^{(0)}$ and suppresses $\bar{H}^{(1)}$ has to simultaneously satisfy the two conditions

$$I_1 \equiv \int_0^T \sinh(u(t)) dt = 0, \quad (10)$$

$$I_2 \equiv \int_0^T dt_1 \int_0^{t_1} dt_2 \sinh(u(t_1) - u(t_2)) = 0, \quad (11)$$

where from now on we use the short hand notation $T = 2\Delta t$ and $u(t) = 2R(t)$.

Theorem 1. *A function $u(x)$ such that $u(-x) = u(x + T)$ satisfies conditions (11). In particular,*

$$u(t) = \sum_{n=0}^{\infty} \left(a_n \sin\left(\frac{(2n+1)\pi t}{T}\right) + b_n \cos\left(\frac{2n\pi t}{T}\right) \right). \quad (12)$$

Proof. Introducing new variables $t = (t_2 + t_1 - T)/2$, $\tau = (t_2 - t_1 + T)/2$, the integral (11) can be written as

$$I_2 = 2 \int_0^{T/2} g(\tau) d\tau \int_{-\tau}^{\tau} dt \sinh[u(t - \tau + T) - u(t + \tau)]. \quad (13)$$

If the expression under the integral (13) is an odd function of t (for all τ), then $I_2 = 0$. This leads to the condition

$$u(-t - \tau + T) - u(-t + \tau) = -u(t - \tau + T) + u(t + \tau), \quad \forall \tau. \quad (14)$$

A solution to the equation (14) satisfies Eq. (11). Using new variables $x = t - \tau$ and $y = t + \tau$, Eq. (14) reads

$$u(-y + T) - u(y) = u(-x) - u(x + T), \quad \forall x, y. \quad (15)$$

As in the method of the separation of variables for PDEs, Eq. (15) is broken into two separate equations:

$$u(-y + T) - u(y) = \lambda, \quad \forall y, \quad (16)$$

$$u(-x) - u(x + T) = \lambda, \quad \forall x, \quad (17)$$

where λ is a separation constant. Substituting $y \rightarrow -x$ into Eq. (16) leads to

$$u(-x) - u(x + T) = -\lambda, \quad u(-x) - u(x + T) = \lambda, \quad \forall x.$$

Therefore, $\lambda = 0$ and we are left with the single linear equation:

$$u(-x) = u(x + T). \quad (18)$$

The first set of linearly independent solutions of Eq. (18) can be found by substituting $u_n(t) = \cos(2n\pi t/T)$, $n = 0, 1, 2, \dots$; whereas, the second set reads $u_n(t) = \sin([2n + 1]\pi t/T)$. \square

Using this theorem, we show that Eq. (10) and Eq. (11) have the following solutions:

$$u(t) = \sum_{n=0}^{\infty} a_n \cos \frac{2\pi(2n+1)t}{T}, \quad (19)$$

$$u(t) = \left(\sin \frac{2\pi t}{T} \right)^2 \sum_{n=0}^{\infty} a_n \cos \frac{2\pi(2n+1)t}{T}, \quad (20)$$

$$u(t) = \sin \left(\frac{4\pi t}{T} \right) \sum_{n=0}^{\infty} a_n \sin \frac{2\pi(2n+1)t}{T}, \quad (21)$$

$$u(t) = a_n \cos \frac{4\pi n t}{T}, \quad n = 1, 2, 3, \dots \quad (22)$$

Clearly, the function (19) obeys Theorem 1 by construction. Now let us show that the function (19) also satisfies the condition (10):

$$I_1 = \int_0^T \sinh \left(\sum_{n=0}^{\infty} a_n \cos \frac{2\pi(2n+1)t}{T} \right) dt = \frac{T}{2\pi} \int_0^{2\pi} \sinh \left(\sum_{n=0}^{\infty} a_n \cos[(2n+1)x] \right) dx.$$

Recalling that $\cos(y \pm [2n+1]\pi) = -\cos(y)$, we obtain

$$\int_0^{2\pi} \sinh \left(\sum_{n=0}^{\infty} a_n \cos[(2n+1)x] \right) dx = \left(\int_0^{\pi} + \int_{\pi}^{\pi+\pi} \right) \dots = \left(\int_0^{\pi} - \int_0^{\pi} \right) \dots = 0 \implies I_1 = 0. \quad (23)$$

In a similar fashion one can verify that the functions (20) and (21) satisfy the conditions (10) and (11). The family of solutions (22) evidently obeys Theorem 1. We show that it also satisfies (10),

$$\begin{aligned} I_1 &= \int_0^T \sinh \left(a_n \cos \frac{4\pi n t}{T} \right) dt = \frac{T}{4\pi n} \int_0^{4\pi n} \sinh(a_n \cos x) dx = \frac{2nT}{4\pi n} \int_0^{2\pi} \sinh(a_n \cos x) dx \implies \\ &\int_0^{2\pi} \sinh(a_n \cos x) dx = \left[\int_0^{\pi/2} + \int_{\pi/2}^{\pi+\pi/2} + \int_{\pi+\pi/2}^{2\pi} \right] \dots = \left[\int_0^{\pi/2} + \int_{\pi/2}^{\pi+\pi/2} + \int_{-\pi/2}^0 \right] \dots \\ &= \left[\int_{-\pi/2}^{\pi/2} + \int_{\pi-\pi/2}^{\pi+\pi/2} \right] \dots = \left[\int_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \right] \dots = 0 \implies I_1 = 0. \end{aligned} \quad (24)$$

Note that from (22) we have with $n = 1$ and $a_1 \equiv K$,

$$R(t) = \frac{K}{2} \cos \left(\frac{2\pi t}{\Delta t} \right), \quad (25)$$

which amplifies the first order of the Magnus expansion by a factor $\lambda = I_0(K)$.

D Error Analysis

Any realistic pulse achieving Hamiltonian Amplification contains some noise, influencing the efficiency of the amplification process. To study the effect of noise we begin by considering the bang-bang sequence given by $S^{(\pm)}$. A generic squeezing transformation around an angle θ is described by

$$S(\theta) = \exp\left(i\frac{r}{2}\left[\sin(\theta)(p^2 - x^2) + \cos(\theta)(xp + px)\right]\right), \quad (26)$$

noting that for $\theta = 0$ and $\theta = \pi$ we obtain $S^{(\pm)}$, respectively. We model a first possible source of errors as fluctuations in the squeezing angle θ up to some error δ . Then the squeezing in x and p direction perturbed by δ is described by $S_\delta^{(\pm)} = \exp(\pm i\frac{r}{2}[(xp + px)\cos(\delta) + (p^2 - x^2)\sin(\delta)])$. If we denote by M_δ the corresponding map formed by $S_\delta^{(\pm)}$, the quadratic Hamiltonian H_0 given by Eq. (2) in the main paper transforms according to $M_\delta(H_0) = \cosh(2r)H_0 + \delta \sinh^2(r)H_{\text{er}} + \mathcal{O}(\delta^2)$, where the first order error term is given by $H_{\text{er}} = \sum_{i,j}(\omega_{i,j}^{(x)} - \omega_{i,j}^{(p)})(x_i p_j + p_i x_j)$. Thus, for small perturbations in the squeezing angle, the amplified Hamiltonian is linearly perturbed by H_{er} . However, we see that for quadratic Hamiltonians of the form $H_0 = \sum_{i,j}\omega_{i,j}(x_i x_j + p_i p_j)$ the error term vanishes to first order in δ . A numerical example of the result is given by the soft orange curve in Fig. 3.

As another example, we consider smooth amplification pulses that contain some noise. A broad class of noise can be described by including a (possibly unknown) Hamiltonian $H_{\text{er}}(t)$ describing the effect of the noise as possible random fluctuations in the total Hamiltonian. For a periodic controller the 1st order term of the Magnus expansion is then given by $\bar{H}^{(0)} = \frac{1}{2\Delta t}(\lambda H_0 + H_{\text{noise}})$ where $H_{\text{noise}} = \int_0^{2\Delta t} dt' U_c^\dagger(t') H_{\text{er}}(t') U_c(t')$. Thus, when control is applied sufficiently fast, noise in the amplification pulses again adds linearly to the amplified Hamiltonian. In particular, the soft blue curve in Fig. 3 shows the error when the amplitude of the integrated pulse given in (25) suffers from Gaussian fluctuations about the desired time-dependent mean value (further details can be found in the caption). While for ideal bang-bang or smooth operations the amplification error tends to zero when Δt is reduced, we see that the noisy pulse and noisy bang-bang operations lead to a saturation of the amplification error at around $\epsilon \approx 10^{-2}$ (fidelity error $\epsilon_F \approx 10^{-5}$). The value at which the amplification error saturates is determined by the corresponding error term, which does not vanish when Δt is reduced.

As shown in Sec. B, while the error ϵ given by the Hilbert Schmidt distance of the amplified evolution in the symplectic representation $\exp(-\lambda A_0 \Omega t)$ and the evolution obtained for a finite spacing Δt of the squeezing operations can be upper bounded (solid grey curve in Fig. 1 (a)) using well known bounds for the Trotter sequence; a similar bound is challenging to obtain when the fidelity error ϵ_F for speeding up the preparation of Gaussian states is considered. However, it is interesting to note that both errors shown in Fig. 1 (a) and (b), respectively, indicate the same asymptotic behavior when Δt is reduced. Furthermore, the inset plot in Fig. 1 (b) shows the dependence of the fidelity error on the expectation of H_0 with respect to the initial state, here taken to be a coherent state $|\alpha\rangle$. The development of an upper bound for ϵ_F in terms of Δt capturing this behavior is left for future studies.

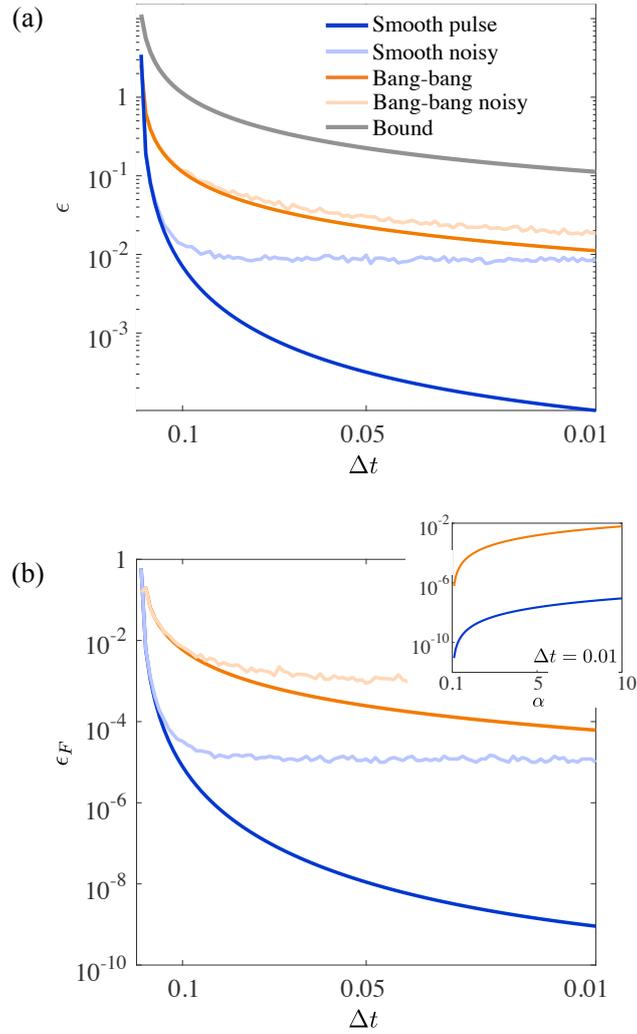


Figure 3: Numerical study of Hamiltonian Amplification of a single harmonic oscillator with frequency $\omega = 1/2$ for a fixed time $t = 1$ and amplification by a factor $\lambda = 2$. (a) error ϵ in the symplectic picture to obtain the amplified evolution $\exp(-\lambda A_0 \Omega t)$ as a function of the spacing Δt of the squeezing operation and (b) fidelity error $\epsilon_F = 1 - F$ with F being the fidelity for Gaussian states explicitly given in [27] as a function of Δt to prepare the coherent state $|\alpha \exp(-i\omega\lambda t)\rangle$ starting from an initial coherent state $|\alpha\rangle$ where $\alpha = 1$. The inset plot in (b) shows for fixed Δt the dependence of the fidelity error on α . In both case (a) and (b) the errors are evaluated for the case when the squeezing operations are applied instantaneously (orange curve) and in a smooth way (blue curve) described by the integrated pulse given in (25). The soft orange curve shows amplification through bang-bang operations for which the squeezing phase is subjected to Gaussian noise with variance $\sigma = 0.1$. The soft blue curve shows amplification through (25) containing Gaussian amplitude noise with mean zero and variance $\sigma = 0.05$. Both noisy curves were averaged over 100 trajectories. The solid grey curve represents the gate error upper bound for bang-bang operations derived in Sec A and explicitly given in the main body of manuscript. The figure shows a significant improvement in convergence in the case of smooth pulses, due to the cancelation of the second-order term in the Magnus expansion.