Quantum Algorithms for Graph Connectivity and Formula Evaluation

Stacey Jeffery[1] and Shelby Kimmel[2]

[1]QuSoft and CWI, Amsterdam, the Netherlands
[2]Middlebury College, Middlebury, VT, USA

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$O(\sqrt{N})$ N $O(|V|^{3/2})$ V

1 Introduction

Stacey Jeffery: jeffery@cwi.nl, SJ completed parts of this work while at the Institute for Quantum Information and Matter (IQIM), Caltech

Shelby Kimmel: shelby.kimmel@gmail.com, SK completed parts of this work while at the Joint Center for Quantum Information and Computer Science (QuICS), University of Maryland

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NAND $O(\sqrt{N})$ N NAND $\Theta(N^{.753})$ $\Omega(N^{.51})$ NAND *k-fault trees*, $O(2^k)$, $\Omega((\log \frac{\log N}{k})^k)$, *superpolynomial* k.

Contributions.

- G $G \cup \{\{s,t\}\}$ —
  - –
  - –
- N $O(\sqrt{N})$
  - –
  - –

Open Problems.

Organization:

2 Preliminaries

2.1 Graph Theory

Definition 1 (Unit *st*-flow)**.** *Let G be an undirected weighted graph with $s, t \in V(G)$, and s and t connected. Then a* unit *st*-flow *on G is a function $\theta : \overrightarrow{E}(G) \to \mathbb{R}$ such that:*

1. *For all $(u, v, \lambda) \in \overrightarrow{E}(G)$, $\theta(u, v, \lambda) = -\theta(v, u, \lambda)$;*
2. *$\sum_{v,\lambda:(s,v,\lambda)\in\overrightarrow{E}} \theta(s, v, \lambda) = \sum_{v,\lambda:(v,t,\lambda)\in\overrightarrow{E}} \theta(v, t, \lambda) = 1$; and*
3. *for all $u \in V(G) \setminus \{s, t\}$, $\sum_{v,\lambda:(u,v,\lambda)\in\overrightarrow{E}} \theta(u, v, \lambda) = 0$.*

Definition 2 (Unit Flow Energy)**.** *Given a unit st-flow θ on a graph G, the* unit flow energy *is*

$$J(\theta) = \sum_{(\{u,v\},\lambda)\in E(G)} \theta(u,v,\lambda)^2. \tag{1}$$

Definition 3 (Effective resistance)**.** *Let G be a graph with $s,t \in V(G)$. If s and t are connected in G, the* effective resistance *is $R_{s,t}(G) = \min_\theta J(\theta)$, where θ runs over all unit st-flows. If s and t are not connected, $R_{s,t}(G) = \infty$.*

$R_{s,t}(G)$ s t s t G, $R_{s,t}(G)|E(G)|$ *commute time* s t s t s G e G s t, $R_{s,t}(G)$

network G *weight* $c : E(G) \to \mathbb{R}^+$.

Definition 4 (Effective Resistance with weights)**.** *Let $\mathcal{N} = (G,c)$ be a network with $s,t \in V(G)$. The* effective resistance of $\mathcal{N}$ *is $R_{s,t}(\mathcal{N}) = \min_\theta \sum_{(\{u,v\},\lambda)\in E(G)} \frac{\theta(u,v,\lambda)^2}{c(\{u,v\},\lambda)}$, where θ runs over all unit st-flows.*

u $(\{u,v\},\lambda)$ $c(\{u,v\},\lambda)$. s t $R_{s,t}(\mathcal{N}) \sum_{e\in E(G)} c(e)$. $\mathcal{N}$ e $1/c(e)$, $R_{s,t}(\mathcal{N})$ G $e = (\{s,t\},\lambda)$ $c(e)$, $R_{s,t}(G) = 1/c(e)$. $R_{s,t}$,

conductance e $c(e)$,

Claim 5. *Let two networks $\mathcal{N}_1 = (G_1,c_1)$ and $\mathcal{N}_2 = (G_2,c_2)$ each have nodes s and t. If we create a new graph G by identifying the s nodes and the t nodes (i.e. connecting the graphs in parallel) and define $c : E(G) \to \mathbb{R}^+$ by $c(e) = c_1(e)$ if $e \in E(G_1)$ and $c(e) = c_1(e)$ if $e \in E(G_2)$, then*

$$R_{s,t}(G,c) = \left(\frac{1}{R_{s,t}(G_1,c_1)} + \frac{1}{R_{s,t}(G_2,c_2)}\right)^{-1}. \tag{2}$$

However, if we create a new graph G by identifying the t node of G_1 with the s node of G_2, relabeling this node $v \notin \{s,t\}$ (i.e. connecting the graphs in series) and define c as before, then

$$R_{s,t}(G,c) = R_{s,t}(G_1,c_1) + R_{s,t}(G_2,c_2). \tag{3}$$

$R_{s,t}(G_1, c_1)$ $R_{s,t}(G_2, c_2)$ FALSE, TRUE, (3) OR, $0+0=0$, $0+\infty=\infty+0=\infty+\infty=\infty$. (2) AND $\frac{1}{0}=\infty$ $\frac{1}{\infty}=0$.

Definition 6 (st-cut)**.** *Given a graph G with $s,t \in V(G)$, if s and t are not connected, an st-*cut *is a function $\kappa : V(G) \to \{0,1\}$ such that $\kappa(s)=1$, $\kappa(t)=0$, and $\kappa(v)-\kappa(u)=0$ whenever $\{u,v\} \in E(G)$.*

κ $S \subset V(G)$ $s \in S$, $t \notin S$, G S, $\overline{S}$. s t G, s t.

Definition 7 (Dual Graph)**.** *Let G be a planar graph (with an implicit embedding). The* dual *graph, $G^\dagger$, is defined as follows. For every face $f \in F(G)$, $G^\dagger$ has a vertex v_f, and any two vertices are adjacent if their corresponding faces share an edge, e. We call the edge between two such vertices the* dual edge *to e, $e^\dagger$. By convention, e and $e^\dagger$ will always have the same label, so that if $e=(\{u,v\},\lambda)$, then $e^\dagger=(\{v_f,v_{f'}\},\lambda)$ for f and f' the faces of G on either side of the edge e.*

2.2 Span Programs and Quantum Query Algorithms

Definition 8 (Span Program)**.** *A span program $P=(H,U,\tau,A)$ on $\{0,1\}^N$ is made up of* **(I)** *finite-dimensional inner product spaces $H=H_1\oplus\cdots\oplus H_N$, and $\{H_{j,b}\subseteq H_j\}_{j\in[N],b\in\{0,1\}}$ such that $H_{j,0}+H_{j,1}=H_j$,* **(II)** *a vector space U,* **(III)** *a non-zero* target vector *$\tau \in U$, and* **(IV)** *a linear operator $A : H \to U$. For every string $x \in \{0,1\}^N$, we associate the subspace $H(x) := H_{1,x_1}\oplus\cdots\oplus H_{N,x_N}$, and an operator $A(x) := A\Pi_{H(x)}$, where $\Pi_{H(x)}$ is the orthogonal projector onto $H(x)$.*

Definition 9 (Positive and Negative Witness)**.** *Let P be a span program on $\{0,1\}^N$ and let x be a string $x \in \{0,1\}^N$. Then we call $|w\rangle$ a* positive witness for x in P *if $|w\rangle \in H(x)$, and $A|w\rangle=\tau$. We define the* positive witness size of x *as:*

$$w_+(x,P)=w_+(x)=\min\{\||w\rangle\|^2 : |w\rangle \in H(x), A|w\rangle=\tau\}, \tag{4}$$

if there exists a positive witness for x, and $w_+(x)=\infty$ otherwise. Let $\mathcal{L}(U,\mathbb{R})$ denote the set of linear maps from U to $\mathbb{R}$. We call a linear map $\omega \in \mathcal{L}(U,\mathbb{R})$ a negative witness for x in P *if $\omega A\Pi_{H(x)}=0$ and $\omega\tau=1$. We define the* negative witness size of x *as:*

$$w_-(x,P)=w_-(x)=\min\{\|\omega A\|^2 : \omega \in \mathcal{L}(U,\mathbb{R}), \omega A\Pi_{H(x)}=0, \omega\tau=1\}, \tag{5}$$

if there exists a negative witness, and $w_-(x)=\infty$ otherwise. If $w_+(x)$ is finite, we say that x is positive *(wrt. P), and if $w_-(x)$ is finite, we say that x is* negative*. We let P_1 denote the set of positive inputs, and P_0 the set of negative inputs for P. In this way, the span program defines a partition (P_0,P_1) of $[N]$.*

$f : X \to \{0,1\}$, $X \subseteq \{0,1\}^N$, *P decides f* $f^{-1}(0) \subseteq P_0$ $f^{-1}(1) \subseteq P_1$. P f, $x \in X$ $\mathcal{O}_x : |i,b\rangle \mapsto |i,b\oplus x_i\rangle$.

Theorem 10 ([21]). *Fix $X \subseteq \{0,1\}^N$ and $f : X \to \{0,1\}$, and let P be a span program on $\{0,1\}^N$ that decides f. Let $W_+(f,P) = \max_{x \in f^{-1}(1)} w_+(x,P)$ and $W_-(f,P) = \max_{x \in f^{-1}(0)} w_-(x,P)$. Then there is a bounded error quantum algorithm that decides f with quantum query complexity $O(\sqrt{W_+(f,P)W_-(f,P)})$.*

2.3 Boolean Formulas

A
$x_1, \ldots, x_N$,
$\{\wedge, \vee, \neg\}$.
$\neg$ (NOT),
$\wedge$ (AND)
$\vee$ (OR),
fan-in
depth
AND-OR *formula*
monotone formula
$\wedge$
$\vee$.
AND-OR
NOT
NOT
formula,
AND-OR *read-once formula*.
x_i
$x \in \{0,1\}^N$
ϕ,
$\phi(x)$
ϕ
$\phi = x_i$
$\phi(x) = x_i$.
ϕ
ϕ
$\phi_1, \ldots, \phi_l$,
$\phi = \phi_1 \wedge \cdots \wedge \phi_l$,
$\wedge$,
$\phi = \phi_1 \vee \cdots \vee \phi_l$,
$\vee$.
$\phi(x) = \phi_1(x) \wedge \cdots \wedge \phi_l(x)$,
$\phi(x) = \phi_1(x) \vee \cdots \vee \phi_l(x)$.
$\phi = \phi_N$
N
$\textsc{Eval}_\phi$,
$x \in \{0,1\}^N$,
$\phi_N(x)$.
$\phi(x) = 0$,
x is a 0-
$\phi(x) = 1$, x is a
$\phi_1 \circ \phi_2$,
ϕ_1
ϕ_2.
$\phi_1 : \{0,1\}^{N_1} \to \{0,1\}$
$\phi_2 : \{0,1\}^{N_2} \to \{0,1\}$,
$\phi_1 \circ \phi_2 : \{0,1\}^{N_1 N_2} \to \{0,1\}$
$\phi_1 \circ \phi_2(x) = \phi_1(\phi_2(x^1), \ldots, \phi_2(x^{N_1}))$,
$x = (x^1, \ldots, x^{N_1})$
$x^i \in \{0,1\}^{N_2}$.
NAND
NAND
d —
d
$\vee$
$\wedge$
NAND_d
NAND
d.
NAND_d
d,
d
AND
OR
NAND_d
$x \in \{0,1\}^N$,
$N = 2^d$.
$\text{NAND}_2(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$
NAND
2. NAND_0
A NAND_d
$x \in \{0,1\}^{2^d}$
NAND_d,
x_i,
A
B)
1,
A
0,
B

Figure 1: Example of how to derive $\overline{G}$, $\overline{G}^\dagger$, and G' from a planar graph G where s and t are on the same face. $\overline{G}$ is obtained from G by adding an edge $(\{s,t\},\emptyset)$. $\overline{G}^\dagger$ is the planar dual of $\overline{G}$. (In the diagram labeled by $\overline{G}^\dagger$, $\overline{G}$ is the gray graph, while $\overline{G}^\dagger$ is black). G' is obtained from $\overline{G}^\dagger$ by removing the edge $(\{s,t\},\emptyset)^\dagger$. Note that dual edges inherit their labels (in this case $1,2,3,4,\emptyset$) from the primal edge.

Lemma 11. *Let G be a planar multigraph with $s,t \in V(G)$ such that $G \cup \{\{s,t\}\}$ is also planar, and let c be a weight function on $E(G)$. Let $x \in \{0,1\}^{E(G)}$. Then $w_+(x, P_{G,c}) = \frac{1}{2}R_{s,t}(G(x),c)$ and $w_-(x, P_{G,c}) = 2R_{s',t'}(G'(x),c')$.*

Theorem 12. *Let G be a planar multigraph with $s,t \in V(G)$ such that $G \cup \{\{s,t\}\}$ is also planar. Then the bounded error quantum query complexity of evaluating st-$\mathrm{CONN}_{G,D}$ is*

$$O\left(\min_c \sqrt{\max_{x\in D: st\text{-}\mathrm{CONN}_G(x)=1} R_{s,t}(G(x),c) \times \max_{x\in D: st\text{-}\mathrm{CONN}_G(x)=0} R_{s',t'}(G'(x),c')}\right) \tag{10}$$

where the minimization is over all positive real-valued functions c on $E(G)$.

$$U_{G,c} : |u\rangle|0\rangle \mapsto \frac{1}{\sqrt{\sum_{v,\lambda:(u,v,\lambda)\in\vec{E}(G)} c(\{u,v\},\lambda)}} \sum_{v,\lambda:(u,v,\lambda)\in\vec{E}(G)} \sqrt{c(\{u,v\},\lambda)}|u\rangle|u,v,\lambda\rangle. \tag{11}$$

Theorem 13. *Let $P_{G,c} = (H, U, A, \tau)$ be defined as in (6). Let $S_{G,c}$ be an upper bound on the time complexity of implementing $U_{G,c}$. If G has the property that $G \cup \{\{s,t\}\}$ is planar, then the time complexity of deciding st-CONN$_{G,D}$ is at most*

$$O\left(\min_c S_{G,c}\sqrt{\max_{x\in D: s,t \text{ are connected}} R_{s,t}(G(x),c) \times \max_{x\in D: s,t \text{ are not connected}} R_{s',t'}(G'(x),c')}\right). \tag{12}$$

$U_{G,c}$ in $S_{G,c}$ is $S'_{G,c}$, $O(\max\{\log|E(G)|, \log|V(G)|\} + S'_{G,c})$.

3.1 Comparison to Previous Quantum Algorithm

$G \cup \{\{s,t\}\}$ (10) (8), c 1

$$C_{s,t}(G(x)) = (\text{shortest path length from } s' \text{ to } t' \text{ in } G'(x)) \geq R_{s't'}(G'(x)). \tag{13}$$

(13), $s't'$ $G'(x)$ s' t' $C_{s,t}(G(x))$ $R_{s',t'}(G(x))$

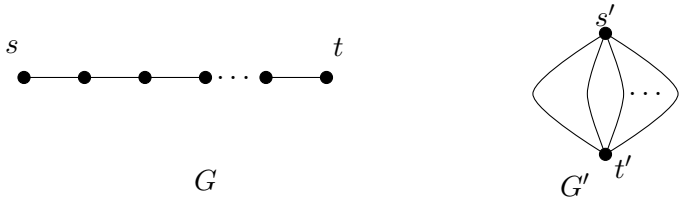


Figure 2: Example of graph for which our analysis does better than the analysis of [4], even with $c = 1$ for all edges, under the promise that $G'(x)$ always contains at least h edges, if s' and t' are connected.

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=1} R_{s,t}(G(x)) = N \tag{14}$$

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=0} R_{s',t'}(G'(x)) \leq 1/h, \tag{15}$$

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=0} C_{s,t}(G(x)) = 1. \tag{16}$$

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=1} R_{s,t}(G(x), c) = 2N, \tag{17}$$

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=0} R_{s',t'}(G'(x), c') \leq 1, \tag{18}$$

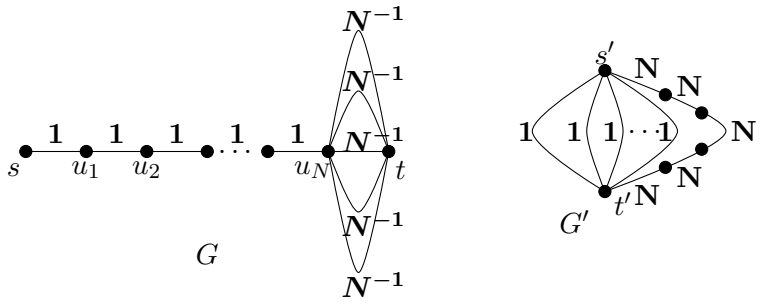


Figure 3: Example of graph for which our analysis does quadratically better than the analysis of [4] by taking advantage of a non-trivial weight function c. The values of c for each edge of G, and of c' for each edge of G', are shown in boldface.

[illegible] s' to t'. [illegible]
N [illegible] N, [illegible] $1/N$ [illegible]

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=0} C_{s,t}(G(x)) = N \tag{19}$$

[illegible] u_N and t, and

$$\max_{x \in D : st\text{-}\mathrm{CONN}_G(x)=1} R_{s,t}(G(x)) = N + 1, \tag{20}$$

[illegible] u_N and t is [illegible] $O(N)$ [illegible]
[illegible] $O(N^{1/2})$.
[illegible]

4 AND-OR Formulas and st-Connectivity

[illegible] AND-OR [illegible] *st*-[illegible] [illegible] $x = (x_1, \ldots, x_N)$ [illegible]
[illegible] O_x, [illegible] $O_x|i\rangle|b\rangle = |i\rangle|b \oplus x_i\rangle$.
[illegible] AND-OR [illegible] ϕ [illegible] N [illegible]
[illegible] G_ϕ, [illegible] G_ϕ [illegible] s and t [illegible]
[illegible] G_ϕ [illegible] $\{x_i\}_{i\in[N]}$. [illegible] $\phi = x_i$ [illegible]
[illegible] G_ϕ [illegible] s and t, [illegible] x_i.
[illegible] $E(G_\phi) = \{(\{s,t\}, x_i)\}$ and $V(G_\phi) = \{s,t\}$.
[illegible] $\phi = \phi_1 \wedge \cdots \wedge \phi_l$. [illegible] G_ϕ [illegible]
$G_{\phi_1}, \ldots, G_{\phi_l}$ [illegible] t in G_{ϕ_i} [illegible] s in $G_{\phi_{i+1}}$,
[illegible] $i = 1, \ldots, l-1$, [illegible] s in G_{ϕ_1} by s, [illegible]
t in G_{ϕ_l} by t. [illegible] $G_{\phi_1}, \ldots, G_{\phi_l}$ *in series*, [illegible]
[illegible] G_ϕ, [illegible]
[illegible] $\phi = \phi_1 \vee \cdots \vee \phi_l$. [illegible] G_ϕ
[illegible] $G_{\phi_1}, \ldots, G_{\phi_l}$ [illegible] s, [illegible]

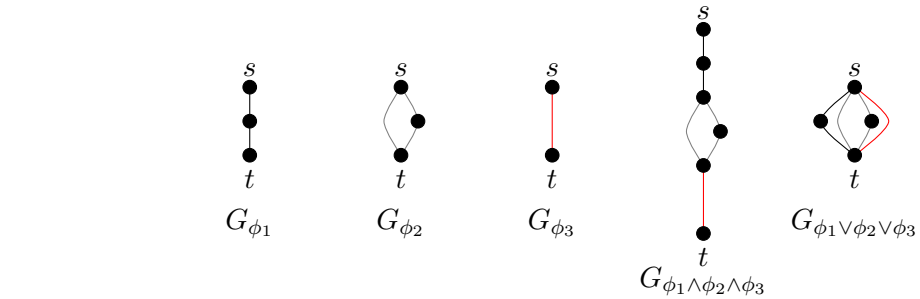


Figure 4: Let $\phi_1 = x_1 \wedge x_2$, $\phi_2 = x_3 \vee (x_4 \wedge x_5)$, and $\phi_3 = x_6$. Then we obtain $G_{\phi_1 \wedge \phi_2 \wedge \phi_3}$ by connecting G_{ϕ_1}, G_{ϕ_2}, and G_{ϕ_3} in series, and $G_{\phi_1 \vee \phi_2 \vee \phi_3}$ by connecting them in parallel.

Lemma 14. *Let ϕ be any* AND-OR *formula on N variables. For every $x \in \{0,1\}^N$, there exists a path from s to t in $G_\phi(x)$ if and only if $\phi(x) = 1$. Furthermore, for every $x \in \{0,1\}^N$, there exists a path from s' to t' in $G'_\phi(x)$ if and only if $\phi(x) = 0$.*

Theorem 15. *For any family ϕ of* AND-OR *formulas, the bounded error quantum query complexity of* $\textsc{Eval}_\phi$ *when the input is promised to come from a set D is*

$$O\left(\min_c \sqrt{\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x), c) \times \max_{x \in D:\phi(x)=0} R_{s',t'}(G'_\phi(x), c')}\right), \tag{21}$$

where the minimization is over all positive real-valued functions c on $E(G_\phi)$.

Proof. By Lemma 14, the query complexity of $\textsc{Eval}_\phi$ on D is at most the query complexity of st-$\textsc{conn}_{G_\phi,D}$. Since G_ϕ is planar, and has s and t on the same face, we can apply Theorem 12, which immediately implies the result. □

4.1 Comparison to Existing Boolean Formula Algorithms

N

$O(\sqrt{N})$

Theorem 16. *Let ϕ be a read-once formula on N variables. Then there exists a choice of c on $E(G_\phi)$ such that the quantum algorithm obtained from the span program $P_{G_\phi,c}$ computes* $\textsc{Eval}_\phi$ *with bounded error in $O(\sqrt{N})$ queries.*

Claim 17. *If $\phi = \phi_1 \vee \phi_2 \vee \cdots \vee \phi_l$, then $G'_\phi(x)$ is formed by composing $\{G'_{\phi_i}(x)\}_i$ in series, and if $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_l$, then $G'_\phi(x)$ is formed by composing $\{G'_{\phi_i}(x)\}_i$ in parallel.*

G'_ϕ

G'_ϕ

ϕ

N

ϕ'

N

$x \in \{0,1\}^N$, $\phi(x) = \neg\phi'(\bar{x})$,

$\bar{x}$

x.

$G_{\phi'} = G'_\phi$,

x, $G_{\phi'}(\bar{x}) = G'_\phi(x)$

Proof of Theorem 16. We will make use of the following fact: for any network (G, c), and any positive real number W:

$$R_{s,t}(G, c/W) = \min_\theta \sum_{e\in E(G)} \frac{\theta(e)^2}{c(e)/W} = W \min_\theta \sum_{e\in E(G)} \frac{\theta(e)^2}{c(e)} = W R_{s,t}(G, c). \tag{22}$$

We now proceed with the proof. For any formula ϕ in $\{\wedge, \vee, \neg\}$, by repeated applications of de Morgan's law, we can push all NOT-gates to distance-1 from a leaf. Since x_i and $\neg x_i$ can both be learned in one query, we can restrict our attention to AND-OR formulas.

If ϕ has only $N = 1$ variable, it's easy to see that $W_+(P_{G_\phi,c})W_-(P_{G_\phi,c}) \leq N$ for c taking value 1 on the single edge in G_ϕ. We will prove by induction that this is true for any ϕ, for some choice of c, completing the proof, since the complexity of our algorithm obtained from $P_{G_\phi,c}$ is $O\left(\sqrt{W_+(P_{G_\phi,c})W_-(P_{G_\phi,c})}\right)$.

Suppose $\phi = \phi_1 \wedge \cdots \wedge \phi_l$ for formulas ϕ_i on N_i variables, so ϕ has $N = \sum_i N_i$ variables. For $x \in \{0,1\}^N$, we will let $x^i \in \{0,1\}^{N_i}$ denote the $(N_1 + \cdots + N_{i-1} + 1)$-th to $(N_1 + \cdots + N_i)$-th bits of x. For each G_{ϕ_i}, by the induction hypothesis, there is some weight function c_i on $E(G_{\phi_i})$ such that $W_+(P_{G_{\phi_i},c_i})W_-(P_{G_{\phi_i},c_i}) \leq N_i$.

Using our construction, G_ϕ is formed by composing $\{G_{\phi_i}\}_i$ in series. Thus every edge $(\{u, v\}, \lambda) \in E(G_\phi)$ corresponds to an edge $(\{u, v\}, \lambda) \in E(G_{\phi_i})$ for some i. We create a weight function $c : E(G_\phi) \to \mathbb{R}^+$ such that $c(\{u, v\}, \lambda) = \frac{c_i(\{u,v\},\lambda)}{W_-(P_{G_{\phi_i},c_i})}$ if $(\{u, v\}, \lambda)$ is an edge originating from the graph G_{ϕ_i}. That is, our new weight function is the same as combining all of the old weight functions, up to scaling factors $\{W_-(P_{G_{\phi_i},c_i})\}_i$.

Using Lemma 11, Claim 5, and Eq. (22), for any 1-instance x,

$$\begin{aligned} w_+(x, P_{G_phi,c}) &= \frac{1}{2} R_{s,t}(G_\phi(x), c) = \frac{1}{2} \sum_{i=1}^{l} R_{s,t}\left(G_{\phi_i}(x), \frac{c_i}{W_-(P_{G_{\phi_i},c_i})}\right) \\ &= \frac{1}{2} \sum_{i=1}^{l} W_-(P_{G_{\phi_i},c_i}) R_{s,t}\left(G_{\phi_i}(x), c_i\right) \leq \sum_{i=1}^{l} W_-(\phi_i, P_{G_{\phi_i},c_i}) W_+(P_{G_{\phi_i},c_i}). \end{aligned} \tag{23}$$

Thus

$$W_+(P_{G_\phi,c}) \leq \sum_{i=1}^{l} W_-(P_{G_{\phi_i},c_i}) W_+(P_{G_{\phi_i},c_i}) \leq \sum_{i=1}^{l} N_i = N. \tag{24}$$

Recall that for a weight function c on G_ϕ, we define a weight function c' on G'_ϕ by $c'(e^\dagger) = 1/c(e)$. Then for an edge $e \in E(G_{\phi_i}(x^i))$, we have $c'(e^\dagger) = W_-(P_{G_{\phi_i},c_i})/c_i(e) = W_-(P_{G_{\phi_i},c_i}) c'_i(e^\dagger)$. By Claim 17, G'_ϕ is formed by composing $\{G'_{\phi_i}\}_i$ in parallel, so by Lemma 11, Claim 5, and Eq. (22):

$$\begin{aligned} w_-(x, P_{G_\phi,c}) &= 2R_{s',t'}(G'_\phi(x), c') = 2\left(\sum_{i=1}^{l} \frac{1}{R_{s',t'}(G'_{\phi_i}(x^i), c')}\right)^{-1} \\ &= 2\left(\sum_{i=1}^{l} \frac{1}{R_{s',t'}(G'_{\phi_i}(x^i), W_-(P_{G_{\phi_i},c_i}) c'_i)}\right)^{-1} = 2\left(\sum_{i=1}^{l} \frac{W_-(P_{G_{\phi_i},c_i})}{R_{s',t'}(G'_{\phi_i}(x^i), c'_i)}\right)^{-1} \\ &= 2\left(\sum_{i=1}^{l} \frac{W_-(P_{G_{\phi_i},c_i})}{w_-(x^i, P_{G_{\phi_i},c_i})}\right)^{-1}. \end{aligned} \tag{25}$$

Whenever x is a 0-instance of ϕ, the set $S \subseteq [l]$ of i such that x^i is a 0-instance of ϕ_i is non-empty. This is exactly the set of i such that $w_-(x^i, P_{G_{\phi_i},c_i}) < \infty$. Continuing from above, we have:

$$w_-(x, P_{G_\phi,c}) = \left(\sum_{i\in S} \frac{W_-(P_{G_{\phi_i},c_i})}{w_-(x^i, P_{G_{\phi_i},c_i})}\right)^{-1} \leq \left(\sum_{i\in S} \frac{W_-(P_{G_{\phi_i},c_i})}{W_-(P_{G_{\phi_i},c_i})}\right)^{-1} = \frac{1}{|S|} \leq 1. \tag{26}$$

Thus $W_-(P_{G_{\phi_i},c_i}) \leq 1$. Combining this with Eq. (24) we have $W_+(P_{G_\phi,c}) W_-(P_{G_\phi,c}) \leq N$, as desired.

The proof for the case $\phi = \phi_1 \vee \cdots \vee \phi_l$ is similar. □

Corollary 18. *Deciding st-connectivity on subgraphs of two-terminal series-parallel graphs of N edges can be accomplished using $O(\sqrt{N})$ queries, if s and t are chosen to be the two terminal nodes.*

$N, h \in \mathbb{Z}^+$, $D_{N,h} = \{x \in \{0,1\}^N : |x| = N \text{ or } |x| \leq N - h\}$ $D'_{N,h} = \{x \in \{0,1\}^N : |x| = 0 \text{ or } |x| \geq h\}$.

AND-OR AND) OR) AND $|_{D_{N,h}}$ $D'_{N,h}$ OR $|_{D'_{N,h}}$ OR 1, h.

Theorem 19. *Let $\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_l$, where for each $i \in [l]$, $\phi_i = \text{OR}|_{D'_{N_i,h_i}}$ or $\phi_i = \text{AND}|_{D_{N_i,h_i}}$. Then the randomized bounded-error query complexity of evaluating ϕ is $\Omega\left(\prod_{i=1}^{l} N_i/h_i\right)$, and the bounded-error quantum query complexity of evaluating ϕ is $O\left(\prod_{i=1}^{l} \sqrt{N_i/h_i}\right)$.*

[illegible] ϕ_1 [illegible] $D_1 \subseteq \{0,1\}^{N_1}$ [illegible] ϕ_2 [illegible] $D_2 \subseteq \{0,1\}^{N_2}$, $\phi = \phi_1 \circ \phi_2$ [illegible] $x = (x^1, \ldots, x^{N_1}) \in \{0,1\}^{N_1 N_2}$ [illegible] $x^i \in D_2$ [illegible] $i \in [N_1]$, [illegible] $(\phi_2(x^1), \ldots, \phi_2(x^{N_1})) \in D_1$.

[illegible]

$$\frac{\prod_{i=1}^{l} N_i}{\prod_{i=1}^{l} h_i} = \left(\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x))\right)\left(\max_{x \in D:\phi(x)=0} R_{s,t}(G'_\phi(x))\right), \tag{27}$$

[illegible] ϕ. [illegible]

[illegible] *st*- [illegible] $\text{AND}|_{D_{N,h}}$, [illegible] *st*-

5 NAND-tree Results

5.1 Query Separations

[illegible] NAND [illegible] k [illegible] F_k^d [illegible] NAND_d [illegible] k [illegible]

Theorem 20. *Using the st-connectivity approach to formula evaluation (Theorem 15), one can solve $\text{EVAL}_{\text{NAND}_d}$ when the input is promised to be from $F^d_{\log d}$ with $O(d)$ queries, while any classical algorithm requires $\Omega(d^{\log\log(d)})$ queries.*

[illegible] k, [illegible] *st*- [illegible]

Theorem 21. *Consider the problem st-$\text{CONN}_{G_{\text{NAND}_d},F_1^d}$. The analysis of [4] gives a bound of $O(N^{1/4})$ quantum queries to decide this problem (where $N = 2^d$ is the number of edges in G_{NAND_d}), while our analysis shows this problem can be decided with $O(1)$ quantum queries.*

[illegible] k [illegible] NAND [illegible] NAND [illegible]

NAND_d. $G_{\mathrm{NAND}_d}(x)$ $G'_{\mathrm{NAND}_d}(x)$.

NAND_d $x \in \{0,1\}^{2^d}$, NAND-

NAND A B p

NAND x is Z

$\mathcal{P}_Z(x)$ Z Z

$\mathcal{P}_A(x)$ $\mathcal{P}_B(x)$

A B

B A

A B

$\mathcal{P}_A$. $\nu_Z(p)$

p Z $\nu_A(p)$ $\nu_B(p))$

p > 0

v a *fault* A

B

$f_Z(p)$ $\nu_Z(p)$,

$\mathcal{F}(x)$ x [1] $\mathcal{F}(x) = \min\{\mathcal{F}_A(x), \mathcal{F}_B(x)\}$,

$$\mathcal{F}_Z(x) = \begin{cases} 2^{\max_{p \in \mathcal{P}_Z(x)} f_Z(p)} & \text{if } x \text{ is } Z\text{-winnable} \\ \infty & \text{otherwise.} \end{cases} \tag{28}$$

$k = 0, \ldots, d/2$, k F_k^d, $x \in \{0,1\}^{2^d}$

$\log \mathcal{F}(x) \leq k$. k

NAND x $\mathcal{F}(x)$.

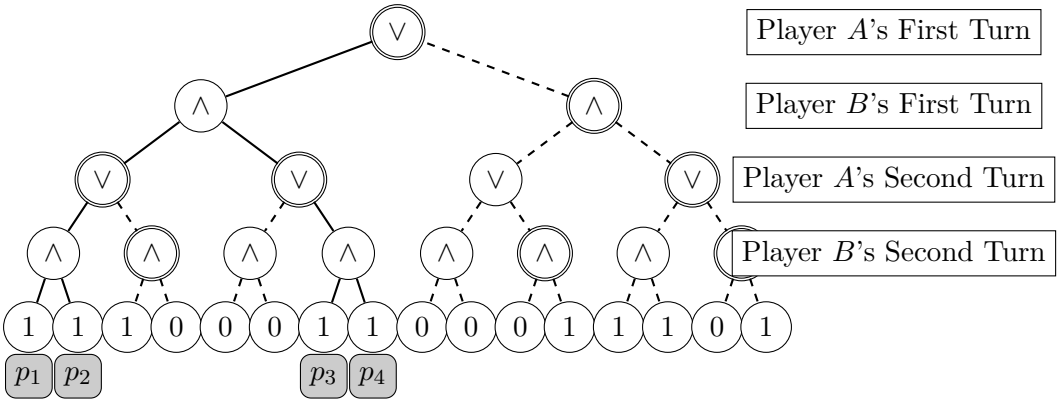


Figure 5: Depiction of a depth-4 NAND-tree as a two-player game. Let x be the input to NAND_4 shown in the figure. This instance is A-winnable, and $\mathcal{P}_A(x)$ consists of the paths $\{p_1, p_2, p_3, p_4\}$, shown using solid lines. Fault nodes are those with double circles. Each path in $\mathcal{P}_A(x)$ encounters two faults at nodes where Player A makes decisions. Therefore, $\mathcal{F}_A(x) = 4$.

$G_{\mathrm{NAND}_d}(x)$ $G'_{\mathrm{NAND}_d}(x)$ $\mathcal{F}(x)$:

[1] We have actually used the more refined definition of k-fault from [19].

Lemma 22. *For any $x \in \{0,1\}^{2^d}$, if d is even, then we have $R_{s,t}(G_{\mathrm{NAND}_d}(x)) \leq \mathcal{F}_A(x)$ and $R_{s',t'}(G'_{\mathrm{NAND}_d}(x)) \leq \mathcal{F}_B(x)$, while if d is odd, we have $R_{s,t}(G_{\mathrm{NAND}_d}(x)) \leq 2\mathcal{F}_A(x)$ and $R_{s',t'}(G'_{\mathrm{NAND}_d}(x)) \leq 2\mathcal{F}_B(x)$.*

[illegible]

Corollary 23. *The span program P_{G_ϕ} for $\phi = \mathrm{NAND}_d$ decides $\mathrm{EVAL}_{\mathrm{NAND}_d}$ restricted to the domain X in $O(\max_{x\in X} \mathcal{F}(x))$ queries. In particular, it decides k-fault trees (on domain F_k^d) in $O(2^k)$ queries.*

Proof of Theorem 20. Theorem 20 is now an immediate consequence of Corollary 23, with k set to $\log(d)$, along with the fact from [27] that the classical query complexity of evaluating such formulas is $\Omega(d^{\log\log(d)})$. □

[illegible]

Claim 24. *Let two graphs G_1 and G_2 each have nodes s and t and let $x^1 \in \{0,1\}^{E(G_1)}$ and $x^2 \in \{0,1\}^{E(G_2)}$. Suppose we create a new graph G by identifying the s nodes and the t nodes (i.e. connecting the graphs in parallel), then*

$$C_{s,t}(G(x^1,x^2)) = C_{s,t}(G_1(x^1)) + C_{s,t}(G_2(x^2)) \tag{29}$$

If we create a new graph G by identifying the t node of G_1 with the s node of G_2 and relabeling this node $v \notin \{s,t\}$ (i.e. connecting the graphs in series), then

$$C_{s,t}(G(x^1,x^2)) = \min\{C_{s,t}(G_1(x^1)), C_{s,t}(G_2(x^2))\}. \tag{30}$$

Proof of Theorem 21. Using Corollary 23, our analysis shows that $st\text{-CONN}_{G_{\mathrm{NAND}_d},F_1^d}$ can be decided in $O(1)$ queries.

We apply Eq. (8) to compare to the analysis of [4]. We must characterize the quantity $\max\{C_{s,t}(G_{\mathrm{NAND}_d}(x)) : x \in F_1^d, \text{and } s,t \text{ are not connected}\}$, since we already have

$$\max_{x\in F_1^d : s,t \text{ are connected}} R_{s,t}(G(x)) = O(1). \tag{31}$$

We now prove that for *every* $x \in \{0,1\}^{2^d}$ such that $\mathrm{NAND}_d(x) = 0$, $C_{s,t}(G_{\mathrm{NAND}_d}(x)) = 2^{\lfloor d/2 \rfloor}$. Thus, for any promise D on the input, as long as there exists some $x \in D$ such that $\mathrm{NAND}_d(x) = 0$, we have $\max_{x\in D:\mathrm{NAND}_d(x)=0} C_{s,t}(G_{\mathrm{NAND}_d}(x)) = 2^{\lfloor d/2 \rfloor}$. Intuitively, this is because every st-cut on any subgraph of G_{NAND_d} cuts across $2^{\lfloor d/2 \rfloor}$ edges of G_{NAND_d}.

The proof is by induction on d. For the base even case, $d = 0$ and G_{NAND_0} is a single edge connecting s and t. The only input $x \in \{0,1\}^{2^0}$ such that $\mathrm{NAND}_d(x) = 0$ is $x = 0$, in which case, the st-cut is $\kappa(s) = 1$ and $\kappa(t) = 0$, so the cut is across the unique edge in G_{NAND_0}, so $C_{s,t}(G_{\mathrm{NAND}_0}(x)) = 1$.

For the induction step, we treat even and odd separately. Suppose $d > 0$ is odd. Then $\mathrm{NAND}_d(x) = \mathrm{NAND}_{d-1}(x^0) \wedge \mathrm{NAND}_{d-1}(x^1)$, where $x^0 = (x_1, \ldots, x_{2^{d-1}})$ and $x^1 = (x_{2^{d-1}+1}, \ldots, x_{2^d})$. Thus $G_{\mathrm{NAND}_d}(x)$ involves composing two graphs $G_{\mathrm{NAND}_{d-1}}(x^0)$ and $G_{\mathrm{NAND}_{d-1}}(x^1)$ in series. Since we are assuming s and t are not connected in $G_{\mathrm{NAND}_d}(x)$, at least one of $G_{\mathrm{NAND}_{d-1}}(x^0)$ and $G_{\mathrm{NAND}_{d-1}}(x^1)$ must not be connected. Without loss of generality, suppose $G_{\mathrm{NAND}_{d-1}}(x^0)$ is not connected and $C_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0)) \leq C_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))$. By induction, $C_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0)) = 2^{(d-1)/2}$. Thus using Eq. (30) in Claim 24,

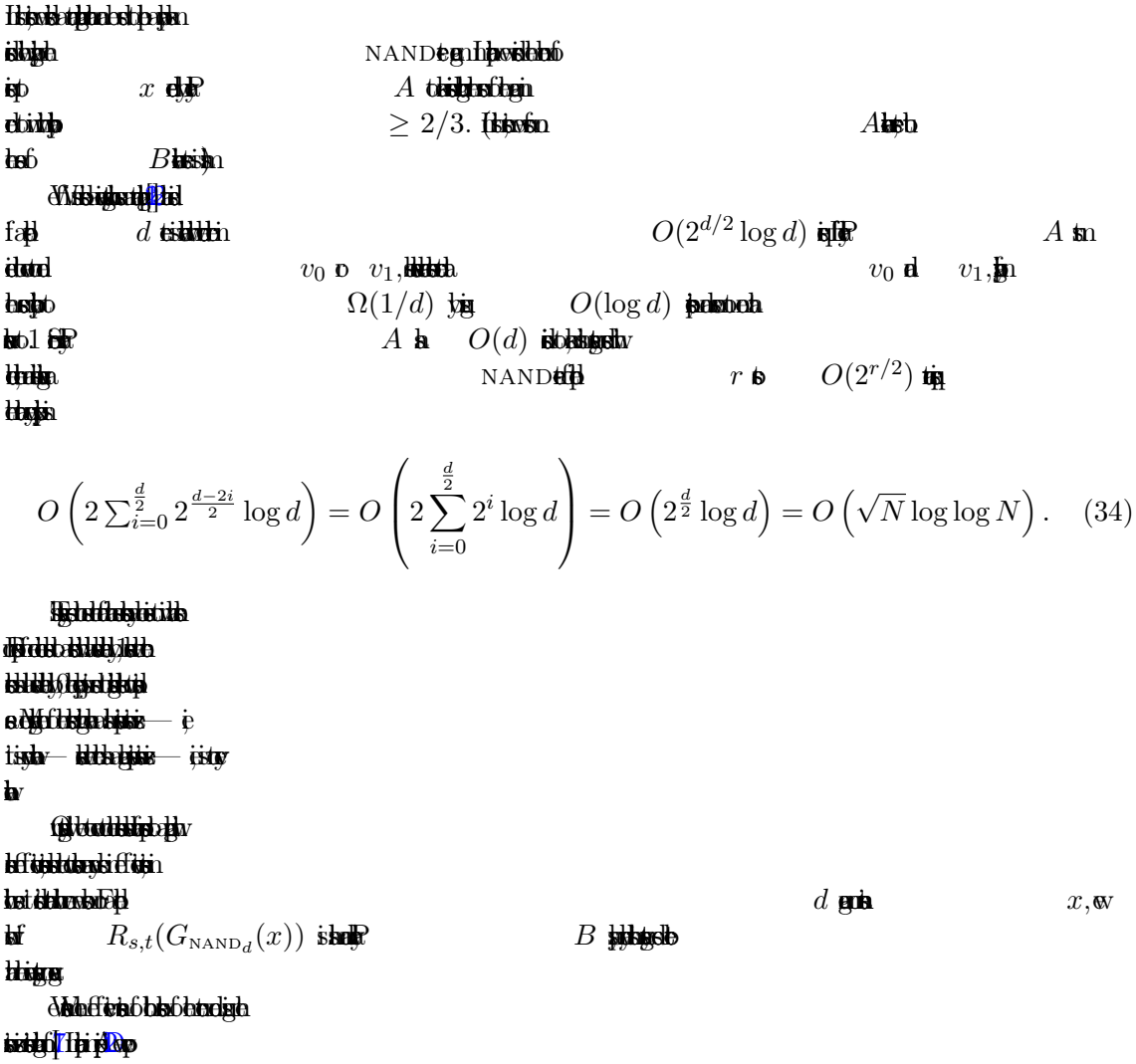
$$C_{s,t}(G_{\mathrm{NAND}_d}(x)) = 2^{(d-1)/2} = 2^{\lfloor d/2 \rfloor}. \tag{32}$$

Now suppose $d > 0$ is even. Then $\text{NAND}_d(x) = \text{NAND}_{d-1}(x^0) \vee \text{NAND}_{d-1}(x^1)$. Thus $G_{\text{NAND}_d}(x)$ involves composing two graphs $G_{\text{NAND}_{d-1}}(x^0)$ and $G_{\text{NAND}_{d-1}}(x^1)$ in parallel. Since we are assuming s and t are not connected in $G_{\text{NAND}_d}(x)$, both of $G_{\text{NAND}_{d-1}}(x^0)$ and $G_{\text{NAND}_{d-1}}(x^1)$ must not be connected, and so by induction, we have $C_{s,t}(G_{\text{NAND}_{d-1}}(x^0)) = C_{s,t}(G_{\text{NAND}_{d-1}}(x^1)) = 2^{\lfloor (d-1)/2 \rfloor} = 2^{d/2-1}$, since d is even. Thus using Eq. (29) in Claim 24,

$$C_{s,t}(G_{\text{NAND}_d}(x)) = 2^{d/2-1} + 2^{d/2-1} = 2^{d/2} = 2^{\lfloor d/2 \rfloor}. \tag{33}$$

Therefore, using Eq. (8), we have that the analysis of [4] for d-depth NAND-trees with inputs in F_1^d gives a query complexity of $O(\sqrt{2^{\lfloor d/2 \rfloor}}) = O(N^{1/4})$, where $N = 2^d$ is the number of input variables. Comparing with our analysis, which gives a query complexity of $O(1)$, we see there is a polynomial to constant improvement. □

5.2 Winning the NAND-tree Game



$$O\left(2\sum_{i=0}^{\frac{d}{2}} 2^{\frac{d-2i}{2}} \log d\right) = O\left(2\sum_{i=0}^{\frac{d}{2}} 2^{i} \log d\right) = O\left(2^{\frac{d}{2}} \log d\right) = O\left(\sqrt{N} \log\log N\right). \tag{34}$$

Lemma 25 (Est Algorithm)**.** *Let ϕ be an* AND-OR *formula with constant fan-in l, $\vee$-depth $d_\vee$ and $\wedge$-depth $d_\wedge$. Then the quantum query complexity of estimating $R_{s,t}(G_\phi(x))$ (resp. $R_{s,t}(G'_\phi(x))$) to relative accuracy ϵ is $\widetilde{\text{O}}\left(\frac{1}{\epsilon^{3/2}}\sqrt{R_{s,t}(G_\phi(x))l^{d_\vee}}\right)$ (resp. $\widetilde{\text{O}}\left(\frac{1}{\epsilon^{3/2}}\sqrt{R_{s,t}(G'_\phi(x))l^{d_\wedge}}\right)$).*

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A Analysis of the Span Program for *st*-Connectivity

$P_{G,c}$

G.

Definition 28 (Circulation)**.** *A* circulation *on a graph G is a function $\theta : \overrightarrow{E}(G) \to \mathbb{R}$ such that:*

1. *For all $(u, v, \lambda) \in \overrightarrow{E}(G)$, $\theta(u, v, \lambda) = -\theta(v, u, \lambda)$;*
2. *for all $u \in V(G)$, $\sum_{v,\lambda:(u,v\lambda)\in\overrightarrow{E}(G)} \theta(u, v, \lambda) = 0$.*

Claim 29. *Let θ be a unit st-flow in some multigraph G. We can consider the corresponding vector $|\theta\rangle = \sum_{(u,v,\lambda)\in\overrightarrow{E}(G)} \theta(u, v, \lambda)|u, v, \lambda\rangle$. Then $|\theta\rangle$ can be written as a linear combination*

of vectors corresponding to self-avoiding st-paths and cycles that are edge-disjoint from these paths.

Let σ be a circulation on G. Then $|\sigma\rangle$ can be written as a linear combination of cycles in G. Furthermore, $|\sigma\rangle$ can be written as a linear combination of cycles such that each cycle goes around a face of G.

[illegible] *st*-flow

Claim 30. *Fix a span program $P_{G,c}$ as in (6). Call $|w\rangle \in H$ a positive witness in $P_{G,c}$ if $A|w\rangle = \tau$ (note that such a $|w\rangle$ is not necessarily a positive witness for any particular input x). Then if θ is a unit st-flow in G, $\frac{1}{2}\sum_{(u,v,\lambda)\in\vec{E}(G)} \frac{\theta(u,v,\lambda)}{\sqrt{c(\{u,v\},\lambda)}}|u,v,\lambda\rangle$ is a positive witness in $P_{G,c}$, and furthermore, if $|w\rangle$ is a positive witness in $P_{G,c}$, then $\theta(u,v,\lambda) = \sqrt{c(\{u,v\},\lambda)}(\langle w|u,v,\lambda\rangle - \langle w|v,u,\lambda\rangle)$ is a unit st-flow in G.*

Proof. The proof is a straightforward calculation. Let θ be a unit st-flow on G. Then

$$\begin{aligned}
&A\left(\frac{1}{2}\sum_{(u,v,\lambda)\in\vec{E}(G)} \frac{\theta(u,v,\lambda)}{\sqrt{c(\{u,v\},\lambda)}}|u,v,\lambda\rangle\right) \\
&= \frac{1}{2}\sum_{(u,v,\lambda)\in\vec{E}(G)} \theta(u,v,\lambda)(|u\rangle - |v\rangle) \\
&= \frac{1}{2}\sum_{u\in V(G)}\left(\sum_{v,\lambda:(u,v,\lambda)\in\vec{E}(G)} \theta(u,v,\lambda)\right)|u\rangle + \frac{1}{2}\sum_{v\in V(G)}\left(\sum_{u,\lambda:(v,u,\lambda)\in\vec{E}(G)} \theta(v,u,\lambda)\right)|v\rangle \\
&= \frac{1}{2}(|s\rangle - |t\rangle) + \frac{1}{2}(|s\rangle - |t\rangle) = \tau. \qquad (35)
\end{aligned}$$

Above we have used that $\theta(u,v,\lambda) = -\theta(v,u,\lambda)$, and $\sum_{v,\lambda:(u,v,\lambda)\in\vec{E}(G)} \theta(u,v,\lambda) = 0$ when $u \notin \{s,t\}$, 1 when $u = s$, and -1 when $u = t$.

To prove the second half of the claim, let $|w\rangle$ be such that $A|w\rangle = \tau$, and define $\theta(u,v,\lambda) = \sqrt{c(\{u,v\},\lambda)}(\langle w|u,v,\lambda\rangle - \langle w|v,u,\lambda\rangle)$. We immediately see that $\theta(u,v,\lambda) = -\theta(v,u,\lambda)$ for all (u,v,λ). Furthermore, we have:

$$\begin{aligned}
|s\rangle - |t\rangle = A|w\rangle &= \sum_{u\in V(G)}\left(\sum_{(u,v,\lambda)\in\vec{E}(G)} \sqrt{c(\{u,v\},\lambda)}\langle u,v,\lambda|w\rangle\right)|u\rangle \\
&\quad - \sum_{v\in V(G)}\left(\sum_{(u,v,\lambda)\in\vec{E}(G)} \sqrt{c(\{u,v\},\lambda)}\langle u,v,\lambda|w\rangle\right)|v\rangle \\
&= \sum_{u\in V(G)}\left(\sum_{(u,v,\lambda)\in\vec{E}(G)} \sqrt{c(\{u,v\},\lambda)}(\langle u,v,\lambda|w\rangle - \langle v,u,\lambda|w\rangle)\right)|u\rangle \\
&= \sum_{u\in V(G)}\left(\sum_{(u,v,\lambda)\in\vec{E}(G)} \theta(u,v,\lambda)\right)|u\rangle. \qquad (36)
\end{aligned}$$

Thus, for all $u \in V(G(x))\backslash\{s,t\}$, $\sum_{(u,v,\lambda)\in\vec{E}(G)} \theta(u,v,\lambda) = 0$, and $\sum_{(s,v,\lambda)\in\vec{E}(G)} \theta(s,v,\lambda) = \sum_{(v,t,\lambda)\in\vec{E}(G)} \theta(v,t,\lambda) = 1$. Thus, θ is a unit st-flow on G. □

[illegible] $s't'$-flow

Claim 31. *For a planar graph G, fix a span program $P_{G,c}$ as in (6). Call a linear function $\omega : V(G) \to \mathbb{R}$ a negative witness if $\omega\tau = 1$. Then $\theta((u,v,\lambda)^\dagger) = \omega(u) - \omega(v)$ is a unit $s't'$-flow on G', and furthermore, for every $s't'$-flow θ on G' there is a negative witness ω such that $\theta((u,v,\lambda)^\dagger) = \omega(u) - \omega(v)$ for all $(u,v,\lambda) \in \overrightarrow{E}(G)$.*

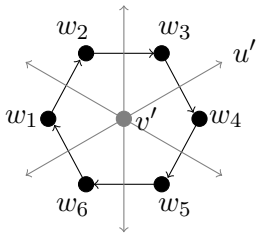

Figure 6: The duality between a cycle and a star.

Proof. When we consider the edges of G as directed edges, we assign edge directions to the dual by orienting each dual edge $\pi/2$ radians counter-clockwise from the primal edge.

Note that without loss of generality, if ω is a negative witness, we can assume $\omega(s) = 1$ and $\omega(t) = 0$. This is because $\|\omega A\|$ and $\left\|\omega A \Pi_{H(x)}\right\|$ are invariant under affine transformations of ω.

We first show that if ω is a negative witness in $P_{G,c}$, then $\theta : \overrightarrow{E}(G') \to \mathbb{R}$ defined $\theta((u,v,\lambda)^\dagger) = \omega(u) - \omega(v)$ is a unit $s't'$-flow on G'. To begin with, we will define $\theta'((u,v,\lambda)^\dagger) = \omega(u) - \omega(v)$ on $\overrightarrow{E}(\overline{G}^\dagger)$, so θ' agrees with θ everywhere θ is defined, and in addition, $\theta'(s',t',\emptyset) = \theta'((s,t,\emptyset)^\dagger) = \omega(s) - \omega(t) = 1$, and $\theta'(t',s',\emptyset) = -1$. Then clearly we have $\theta'(u',v',\lambda) = -\theta'(v',u',\lambda)$ for all $(\{u',v'\},\lambda) \in E(\overline{G}^\dagger)$.

Next, every $v' \in V(\overline{G}^\dagger)$ corresponds to a face $f_{v'}$ of $\overline{G}$, and the edges coming out of v' are dual to edges going clockwise around the face $f_{v'}$ (see Figure 6). If $(w_1,w_2,\lambda_1),\ldots,(w_k,w_{k+1},\lambda_k)$, for $w_{k+1} = w_1$, are the directed edges going clockwise around $f_{v'}$, then we have:

$$0 = \sum_{i=1}^{k} (\omega(w_i) - \omega(w_{i+1})) = \sum_{i=1}^{k} \theta'((w_i,w_{i+1},\lambda_i)^\dagger) = \sum_{\substack{u',\lambda: \\ (\{v',u'\},\lambda)\in E(\overline{G}')}} \theta'(v',u',\lambda). \tag{37}$$

Thus, θ' is a circulation. Then, since $\theta'(s',t',\emptyset) = 1$, if we remove the flow on this edge, which recovers θ, we get a unit $s't'$-flow on G'.

Next we will show that if θ is a unit $s't'$-flow on G', then there exists a negative witness ω in $P_{G,c}$ such that for all $(u,v,\lambda) \in \overrightarrow{E}(G)$, $\theta((u,v,\lambda)^\dagger) = \omega(u) - \omega(v)$.

Define θ' to be the circulation on $\overline{G}^\dagger$ obtained from defining $\theta'(u',v',\lambda) = \theta(u',v',\lambda)$ for all $(u',v',\lambda) \in \overrightarrow{E}(G')$, and $\theta'(s',t',\emptyset) = -\theta'(t',s',\emptyset) = 1$. Then if we define $|\theta'\rangle = \sum_{(u,v,\lambda)\in\overrightarrow{E}(\overline{G}^\dagger)} \theta'(u,v,\lambda)|u,v,\lambda\rangle$, we can express $|\theta'\rangle$ as a linear combination of cycles around the faces of $\overline{G}^\dagger$, $|\theta'\rangle = \sum_{f\in F(\overline{G}^\dagger)} \alpha_f |\overrightarrow{C}_f\rangle + \sum_{f\in F(\overline{G}^\dagger)} \alpha'_f |\overleftarrow{C}_f\rangle$, where if $w_{k+1} = w_1$ and $(w_1,w_2,\lambda_1),\ldots,(w_k,w_{k+1},\lambda_k)$ is a clockwise cycle around f, $|\overrightarrow{C}_f\rangle = \sum_{i=1}^{k} |w_i,w_{i+1},\lambda_i\rangle$ is the clockwise cycle around the face f, and $|\overleftarrow{C}_f\rangle = \sum_{i=1}^{k} |w_{i+1},w_i,\lambda_i\rangle$ is the counter-clockwise cycle around f. There is a one-to-one correspondance between vertices in $V(\overline{G}) = V(G)$ and faces in $F(\overline{G}^\dagger)$, so we can define $\omega : V(G) \to \mathbb{R}$ by $\omega(v_f) = \frac{1}{2}(\alpha_f - \alpha'_f)$.

We claim that for all $(u,v,\lambda) \in \overrightarrow{E}(\overline{G})$, $\omega(u) - \omega(v) = \theta'((u,v,\lambda)^\dagger)$. Let (u',v',λ) be any edge in $\overrightarrow{E}(\overline{G}^\dagger)$. This edge is part of a clockwise cycle around one face in $\overline{G}^\dagger$, call it f,

and a counter clockwise cycle around one face in $\overline{G}^\dagger$, call it g. Since these are the only two faces containing the edge (u', v', λ), we must have $\theta'(u', v', \lambda) = \langle u', v', \lambda | \theta' \rangle = \alpha_f + \alpha'_g$. Since $\theta'(u', v', \lambda) = -\theta'(v', u', \lambda)$, we have $\alpha_f + \alpha'_g = -\alpha'_f - \alpha_g$. Thus:

$$\omega(v_f) - \omega(v_g) = \frac{1}{2}\left(\alpha_f - \alpha'_f - \alpha_g + \alpha'_g\right) = \frac{1}{2}\left(\theta'(u', v', \lambda) - \theta'(v', u', \lambda)\right) = \theta'((v_f, v_g, d)^\dagger). \tag{38}$$

In particular, this means that $\omega(s) - \omega(t) = \theta'((s, t, \emptyset)^\dagger) = \theta'(s', t', \emptyset) = 1$, so ω is a negative witness, and for all $(u, v, \lambda) \in \overrightarrow{E}(G)$, $\omega(u) - \omega(v) = \theta((u, v, \lambda)^\dagger)$. □

Lemma 11. *Let G be a planar multigraph with $s, t \in V(G)$ such that $G \cup \{\{s, t\}\}$ is also planar, and let c be a weight function on $E(G)$. Let $x \in \{0, 1\}^{E(G)}$. Then $w_+(x, P_{G,c}) = \frac{1}{2} R_{s,t}(G(x), c)$ and $w_-(x, P_{G,c}) = 2R_{s',t'}(G'(x), c')$.*

Proof. If x is a 1-instance, s and t are connected in $G(x)$, so there exists a unit st-flow on $G(x)$, which is a unit st-flow on G that is supported only on $\overrightarrow{E}(G(x))$. Let θ be the flow on $G(x)$ such that $R_{s,t}(G(x), c) = \sum_{(\{u,v\},\lambda) \in E(G(x))} \frac{\theta(u,v,\lambda)^2}{c(\{u,v\},\lambda)}$. By Claim 30, $|w\rangle = \frac{1}{2} \sum_{(u,v,\lambda)} \frac{\theta(u,v,\lambda)}{\sqrt{c(\{u,v\},\lambda)}} |u, v, \lambda\rangle$ is a positive witness in $P_{G,c}$, and since θ is supported on $\overrightarrow{E}(G(x))$, $|w\rangle \in H(x)$, and so $|w\rangle$ is a positive witness for x in $P_{G,c}$. Thus

$$w_+(x, P_{G,c}) \leq \||w\rangle\|^2 = \frac{1}{4} \sum_{(u,v,\lambda) \in \overrightarrow{E}(G(x))} c(\{u, v\}, \lambda)\theta(u, v, \lambda)^2 = \frac{1}{2} R_{s,t}(G(x), c). \tag{39}$$

On the other hand, let $|w\rangle$ be an optimal positive witness for x. By Claim 30, $\theta(u, v, \lambda) = \sqrt{c(\{u, v\}, \lambda)}(\langle u, v, \lambda | w\rangle - \langle v, u, \lambda | w\rangle)$ is a unit st-flow on G, and since $|w\rangle \in H(x)$, $\theta(u, v, \lambda)$ is only non-zero on $\overrightarrow{E}(G(x))$, so θ is a unit st-flow on $G(x)$. Thus,

$$\begin{aligned} R_{s,t}(G(x), c) &\leq \sum_{(\{u,v\},\lambda) \in E(G(x))} \frac{\theta(u, v, \lambda)^2}{c(\{u, v\}, \lambda)} = \frac{1}{2} \sum_{(u,v,\lambda) \in \overrightarrow{E}(G(x))} \left(\langle u, v, \lambda | w\rangle - \langle v, u, \lambda | w\rangle\right)^2 \\ &= \sum_{(u,v,\lambda) \in \overrightarrow{E}(G(x))} \langle u, v, \lambda | w\rangle^2 - \sum_{(u,v,\lambda) \in \overrightarrow{E}(G(x))} \langle u, v, \lambda | w\rangle \langle v, u, \lambda | w\rangle \leq 2 \||w\rangle\|^2 \end{aligned} \tag{40}$$

where the last inequality is by Cauchy-Schwarz. Thus, $w_+(x, P_{G,c}) = \frac{1}{2} R_{s,t}(G(x))$.

Now we prove that $w_-(x, P_{G,c}) = 2R_{s',t'}(G'(x), c')$. Let $x \in \{0, 1\}^{E(G)}$ be such that s and t are not connected in $G(x)$. Fix an optimal negative witness ω for x. By Claim 31 the linear function $\theta : \overrightarrow{E}(G') \to \mathbb{R}$ defined by $\theta((u, v, \lambda)^\dagger) = \omega(u) - \omega(v)$ is a unit $s't'$ -flow on G'. Since ω is a negative witness for x, we also have:

$$\begin{aligned} 0 = \left\|\omega A \Pi_{H(x)}\right\|^2 &= \sum_{(u,v,\lambda) \in \overrightarrow{E}(G(x))} c(\{u, v\}, \lambda)(\omega(u) - \omega(v))^2 \\ &= \sum_{(u,v,\lambda) \in \overrightarrow{E}(G(x))} c(\{u, v\}, \lambda)\theta((u, v, \lambda)^\dagger)^2 \\ &= \sum_{(u',v',\lambda) \in \overrightarrow{E}(G') \setminus \overrightarrow{E}(G'(x))} \frac{\theta(u', v', \lambda)^2}{c'(\{u', v'\}, \lambda)}, \end{aligned} \tag{41}$$

since $(u,v,\lambda) \in \overrightarrow{E}(G(x))$ exactly when $(u,v,\lambda)^\dagger \notin \overrightarrow{E}(G'(x))$. So θ is only supported on $\overrightarrow{E}(G'(x))$, and so it is a unit $s't'$-flow on $G'(x)$. Thus

$$w_-(x, P_{G,c}) = \|\omega A\|^2 = \sum_{(u,v,\lambda)\in\overrightarrow{E}(G)} c(\{u,v\},\lambda)(\omega(u)-\omega(v))^2$$
$$= \sum_{(u',v',\lambda)\in\overrightarrow{E}(G'(x))} \frac{\theta(u',v',\lambda)^2}{c'(\{u',v'\},\lambda)} \geq 2R_{s',t'}(G'(x),c'). \qquad (42)$$

For the other direction, let θ be an $s't'$-flow in $G'(x)$ with minimal energy. By Claim 31, there is a negative witness ω such that $\theta((u,v,\lambda)^\dagger) = \omega(u) - \omega(v)$. Since θ is supported on edges $(u',v',\lambda) \in \overrightarrow{E}(G'(x))$, which are exactly those edges such that $(u',v',\lambda)^\dagger \notin \overrightarrow{E}(G(x))$, we have

$$0 = \sum_{\substack{(u,v,\lambda)\\ \in\overrightarrow{E}(G(x))}} c(\{u,v\},\lambda)\theta((u,v,\lambda)^\dagger)^2 = \sum_{\substack{(u,v,\lambda)\\ \in\overrightarrow{E}(G(x))}} c(\{u,v\},\lambda)(\omega(u)-\omega(v))^2 = \left\|\omega A \Pi_{H(x)}\right\|^2, \qquad (43)$$

so ω is a negative witness for x in $P_{G,c}$. Thus:

$$w_-(x, P_{G,c}) \leq \|\omega A\|^2 = \sum_{(u,v,\lambda)\in\overrightarrow{E}(G)} c(\{u,v\},\lambda)(\omega(u)-\omega(v))^2$$
$$= \sum_{(u',v',\lambda)\in\overrightarrow{E}(G'(x))} \frac{\theta(u',v',\lambda)^2}{c'(\{u',v'\},\lambda)} = 2R_{s',t'}(G'(x),c'), \qquad (44)$$

completing the proof. □

A.1 Time and Space Analysis of the Span Program Algorithm for *st*-Connectivity

[illegible] st-CONN$_G$ [illegible] G. [illegible] is $O(\max\{\log|E(G)|, \log|V(G)|\})$.

[illegible] Π_S [illegible] S. [illegible] $P = (H, U, A, \tau)$, [illegible] $(2\Pi_{H(x)} - I)(2\Pi_{\ker A} - I)$ [illegible] $|w_0\rangle = A^+\tau$, [illegible] $\Pi_{H(x)}$ [illegible] $H(x)$, [illegible] $\Pi_{\ker A}$ [illegible] A, [illegible] A^+ [illegible] A. [illegible] f [illegible] D, [illegible]

$$O\left(\sqrt{\max_{x\in D:f(x)=1} w_+(x) \times \max_{x\in D:f(x)=0} w_-(x)}\right).$$

[illegible] st [illegible] $P_{G,c}$ in (6), [illegible] $2\Pi_{H(x)} - I$ [illegible] $O(1)$ [illegible] x. [illegible] $2\Pi_{\ker A} - I$ [illegible] x, [illegible] G. [illegible] $2\Pi_{\ker A} - I$ [illegible] G, [illegible] G [illegible] G is [illegible] n [illegible] $O(\log n)$ [illegible]

[illegible] G [illegible] c, [illegible] *quantum walk step on G* [illegible] [illegible] $U_{G,c}$ [illegible] $u \in V(G)$:

$$U_{G,c} : |u\rangle|0\rangle \mapsto \frac{1}{\sqrt{\sum_{v,\lambda:(u,v,\lambda)\in\overrightarrow{E}(G)} c(\{u,v\},\lambda)}} \sum_{v,\lambda:(u,v,\lambda)\in\overrightarrow{E}(G)} \sqrt{c(\{u,v\},\lambda)}|u\rangle|u,v,\lambda\rangle. \tag{45}$$

Theorem 13. *Let $P_{G,c} = (H, U, A, \tau)$ be defined as in (6). Let $S_{G,c}$ be an upper bound on the time complexity of implementing $U_{G,c}$. If G has the property that $G \cup \{\{s,t\}\}$ is planar, then the time complexity of deciding st-CONN$_{G,D}$ is at most*

$$O\left(\min_c S_{G,c}\sqrt{\max_{x\in D: s,t \text{ are connected}} R_{s,t}(G(x),c) \times \max_{x\in D: s,t \text{ are not connected}} R_{s',t'}(G'(x),c')}\right). \tag{12}$$

[illegible]

Lemma 32. *Let A be defined as in (6). Let $S_{G,c}$ be an upper bound on the time complexity of implementing $U_{G,c}$. Then $2\Pi_{\ker A} - I$ can be implemented in time complexity $O(S_{G,c})$.*

Proof. This analysis follows [4] (see also [17]). Let

$$d(u) = \sum_{v,\lambda:(u,v,\lambda)\in\overrightarrow{E}(G)} c(\{u,v\},\lambda). \tag{46}$$

Define spaces Z and Y as follows.

$$Z = \text{span}\left\{|z_u\rangle := \sum_{v,\lambda:(u,v,\lambda)\in\overrightarrow{E}(G)} \frac{\sqrt{c(\{u,v\},\lambda)}}{\sqrt{2d(u)}} (|0,u,u,v,\lambda\rangle + |1,u,v,u,\lambda\rangle) : u \in V(G)\right\} \tag{47}$$

$$Y = \text{span}\left\{|y_{u,v,\lambda}\rangle := \frac{|0,u,u,v,\lambda\rangle - |1,v,u,v,\lambda\rangle}{\sqrt{2}} : (u,v,\lambda) \in \overrightarrow{E}(G)\right\} \tag{48}$$

Define isometries whose column-spaces are Z and Y respectively:

$$M_Z = \sum_{u\in V(G)} |z_u\rangle\langle u| \quad \text{and} \quad M_Y = \sum_{(u,v,\lambda)\in\overrightarrow{E}(G)} |y_{u,v,\lambda}\rangle\langle u,v,\lambda|. \tag{49}$$

Now we note that for any $(\{u,v\},\lambda) \in E(G)$, we have the following:

$$\langle z_u|y_{u,v,\lambda}\rangle = \frac{\sqrt{c(\{u,v\},\lambda)}}{2\sqrt{d(u)}}, \quad \text{and} \quad \langle z_v|y_{u,v,\lambda}\rangle = -\frac{\sqrt{c(\{u,v\},\lambda)}}{2\sqrt{d(v)}}. \tag{50}$$

Thus, we can calculate:

$$\begin{aligned} M_Z^\dagger M_Y &= \sum_{(u,v,\lambda)\in\overrightarrow{E}(G)} \left(\frac{|u\rangle}{2\sqrt{d(u)}} - \frac{|v\rangle}{2\sqrt{d(v)}}\right)\sqrt{c(\{u,v\},\lambda)}\langle u,v,\lambda| \\ &= \sum_{u'\in V(G)} \frac{|u'\rangle\langle u'|}{2\sqrt{d(u')}} \sum_{(u,v,\lambda)\in\overrightarrow{E}(G)} \sqrt{c(\{u,v\},\lambda)}(|u\rangle - |v\rangle)\langle u,v,\lambda| \\ &= \sum_{u'\in V(G)} \frac{|u'\rangle\langle u'|}{2\sqrt{d(u')}} A. \end{aligned} \tag{51}$$

Note that the rows of $M_Z^\dagger M_Y$ are non-zero multiples of the rows of A, so $\mathrm{row}(M_Z^\dagger M_Y) = \mathrm{row}(A)$, and thus $\ker(M_Z^\dagger M_Y) = \ker A$.

Define $W = (2\Pi_Z - I)(2\Pi_Y - I)$. We now claim that M_Y maps $\ker A$ to the (-1)-eigenspace of W, and $(\ker A)^\perp$ to the 1-eigenspace of W, so that $2\Pi_{\ker A} - I = M_Y^\dagger W M_Y$. To see this, note that if $|\psi\rangle \in \ker A$, then $|\psi\rangle \in \ker(M_Z^\dagger M_Y)$ so $M_Y|\psi\rangle \in \ker M_Z^\dagger = Z^\perp$. Thus $M_Y|\psi\rangle \in Y \cap Z^\perp$, which is in the (-1)-eigenspace of W.

Next, suppose $|\psi\rangle \in (\ker A)^\perp = (\ker(M_Z^\dagger M_Y))^\perp$, so since M_Y is an isometry, $M_Y|\psi\rangle \in (\ker M_Z)^\perp = \mathrm{row} M_Z = Z$. Thus $M_Y|\psi\rangle \in Y \cap Z$, which is in the 1-eigenspace of W.

Thus, we can implement $2\Pi_{\ker A} - I$ by $M_Y^\dagger W M_Y$. It only remains to argue that each of M_Y, $2\Pi_Z - I$ and $2\Pi_Y - I$ can be implemented in time complexity at most $O(S_{G,c})$.

We first show that we can implement the isometry M_Y, or rather a unitary U_Y that acts as $|0\rangle|0\rangle|u,v,\lambda\rangle \mapsto M_Y|u,v,\lambda\rangle = |y_{u,v,\lambda}\rangle$. First, use HX on the first qubit to perform the map:

$$|0\rangle|0\rangle|u,v,\lambda\rangle \mapsto |-\rangle|0\rangle|u,v,\lambda\rangle. \tag{52}$$

Conditioned on the value of the first register, copy either u or v into the second register to get:

$$\frac{1}{\sqrt{2}}(|0,u,u,v,\lambda\rangle - |1,v,u,v,\lambda\rangle) = |y_{u,v,\lambda}\rangle. \tag{53}$$

Thus, we can implement U_Y in the time it takes to write down a vertex of G, $O(\log|V(G)|)$, which is at most $O(S_{G,c})$. Using the ability to implement U_Y, we can implement $2\Pi_Y - I$ as $U_Y R_Y U_Y^\dagger$, where R_Y is the reflection that acts as the identity on computational basis states of the form $|0\rangle|0\rangle|u,v,\lambda\rangle$, and reflects computational basis states without this form.

Next, we implement a unitary U_Z that acts as $|0\rangle|u\rangle|0\rangle \mapsto M_Z|u\rangle = |z_u\rangle$. First, use the quantum walk step $U_{G,c}$, which can be implemented in time $S_{G,c}$, to perform:

$$|+\rangle|u\rangle|0\rangle \mapsto \frac{1}{2\sqrt{d(u)}} \sum_{v,\lambda:(u,v,\lambda)\in\vec{E}(G)} \sqrt{c(\{u,v\},\lambda)}(|0\rangle + |1\rangle)|u\rangle|u,v,\lambda\rangle. \tag{54}$$

Conditioned on the bit in the first register, swap the third and fourth registers, to get:

$$\frac{1}{2\sqrt{d(u)}} \sum_{v,\lambda:(u,v,\lambda)\in\vec{E}(G)} \sqrt{c(\{u,v\},\lambda)}(|0\rangle|u\rangle|u,v,\lambda\rangle + |1\rangle|u\rangle|v,u,\lambda\rangle) = |z_u\rangle. \tag{55}$$

The total cost of implementing U_Z is $O(S_{G,c} + \log|V(G)|) = O(S_{G,c})$. Thus, we can implement $2\Pi_Z - I$ in $O(S_{G,c})$ quantum gates. □

Lemma 33. *Let A and τ be defined as in (6). Let $S_{G,c}$ be an upper bound on the complexity of implementing $U_{G,c}$. Then the initial state of the algorithm, $\frac{|w_0\rangle}{\||w_0\rangle\|}$ where $|w_0\rangle = A^+\tau$, can be approximated in time $O(S_{G,c})$.*

Proof. Without loss of generality, we can assume that G includes the edge $(\{s,t\},\emptyset)$ (we can simply not include it in any subgraph). Furthermore, we set $c(\{s,t\},\emptyset) = 1/r$, for some positive r to be specified later, so that $A|s,t,\emptyset\rangle = r^{-1/2}\tau$. This has no effect on other edges in G. Note that

$$\Pi_{(\ker A)^\perp}|s,t,\emptyset\rangle = A^+A|s,t,\emptyset\rangle = r^{-1/2}A^+\tau = r^{-1/2}|w_0\rangle, \tag{56}$$

so

$$|s,t,\emptyset\rangle = r^{-1/2}|w_0\rangle + |w_0^{\perp}\rangle \tag{57}$$

for some $|w_0^{\perp}\rangle \in \ker A$. Thus, constant precision phase estimation on $2\Pi_{\ker A} - I$ maps $|s,t,\emptyset\rangle$ to

$$r^{-1/2}|0\rangle|w_0\rangle + |1\rangle|w_0^{\perp}\rangle. \tag{58}$$

Using quantum amplitude amplification [7], we can amplify the amplitude on the $|0\rangle|w_0\rangle$ part of this arbitrarily close to 1 using a number of calls to $2\Pi_{\ker A} - I$ proportional to $\left\| r^{-1/2}|w_0\rangle \right\|^{-1}$.

In fact, it is straightforward to show that for any $|\mu\rangle \in \mathrm{row}A$, the vector $|\nu\rangle$ with smallest norm that satisfies $A|\nu\rangle = |\mu\rangle$, is $A^{+}|\mu\rangle$ [17]. Using this fact along with Claim 30 and Definition 9, we have $\||w_0\rangle\|^2 = R_{s,t}(G,c)$.

Let $R = R_{s,t}(G \setminus \{(\{s,t\},\emptyset)\},c)$ be the effective resistance of G without the edge $(s,t,\emptyset)$. Now we can think of $(\{s,t\},\emptyset)$ and $G \setminus \{(\{s,t\},\emptyset)\}$ as two graphs in parallel, so using Claim 5, we have

$$\||w_0\rangle\|^2 = \frac{1}{1/R + 1/r}. \tag{59}$$

Setting $r = R$, we have $\||w_0\rangle\|^2 = R/2$ and $\|r^{-1/2}|w_0\rangle\|^{-1} = O(1)$. Thus, using $O(1)$ calls to $2\Pi_{\ker A} - I$, we can approximate the initial state $|w_0\rangle$. □

$U_{G,c}$, is $O(\max\{\log|E(G)|, \log|V(G)|\})$. U_Y $4|V(G)|^2|E(G)|$, $O(\max\{\log|E(G)|, \log|V(G)|\})$ $U_{G,c}$ $S'_{G,c}$ $(2\Pi_{H(x)} - I)(2\Pi_{\ker A} - I)$

$$O\left(\min_c \sqrt{\max_{x\in D:\phi(x)=1} R_{s,t}(G_\phi(x),c) \times \max_{x\in D:\phi(x)=0} R_{s',t'}(G'_\phi(x),c')}\right) = O(|E(G)|). \tag{60}$$

$O(\log(|E(G)|)$

B Formula Evaluation and *st*-Connectivity

ϕ, G_ϕ. G_ϕ.

Definition 34 (G_ϕ)**.** *If $\phi = x_i$ is a single-variable formula, then $V(G_\phi) = \{s,t\}$ and $E(G_\phi) = \{(\{s,t\},x_i)\}$.*

If $\phi = \phi_1 \wedge \cdots \wedge \phi_l$, then define $V(G_\phi) = \{(i,v) : i \in [l], v \in V(G_{\phi_i}) \setminus \{s,t\}\} \cup \{s, s_2, \ldots, s_l, t\}$ and, letting $s_1 = s$ and $s_{l+1} = t$, define:

$$\begin{aligned} E(G_\phi) = &\{(\{(i,u),(i,v)\},x_j) : i \in [l], u,v \in V(G_{\phi_i}) \setminus \{s,t\}, (\{u,v\},x_j) \in E(G_{\phi_i})\} \\ &\cup \{(\{(i,u),s_i\},x_j) : i \in [l], u \in V(G_{\phi_i}), (\{s,u\},x_j) \in E(G_{\phi_i})\} \\ &\cup \{(\{(i,u),s_{i+1}\},x_j) : i \in [l], u \in V(G_{\phi_i}), (\{t,u\},x_j) \in E(G_{\phi_i})\}. \end{aligned} \tag{61}$$

If $\phi = \phi_1 \vee \cdots \vee \phi_l$ *define* $V(G_\phi) = \{(i,v) : i \in [l], v \in V(G_{\phi_i}) \setminus \{s,t\}\} \cup \{s,t\}$ *and*

$$\begin{aligned} E(G_\phi) =& \{(\{(i,u),(i,v)\}, x_j) : i \in [l], u,v \in V(G_{\phi_i}) \setminus \{s,t\}, (\{u,v\}, x_j) \in E(G_{\phi_i})\} \\ &\cup \{(\{(i,u),s\}, x_j) : i \in [l], u \in V(G_{\phi_i}), (\{u,s\}, x_j) \in E(G_{\phi_i})\} \\ &\cup \{(\{(i,u),t\}, x_j) : i \in [l], u \in V(G_{\phi_i}), (\{u,t\}, x_j) \in E(G_{\phi_i})\}. \end{aligned} \tag{62}$$

Lemma 35. *For an* AND-OR *formula* ϕ *on* $\{0,1\}^N$*, define* ϕ' *to be the formula obtained by replacing* $\vee$*-nodes with* $\wedge$*-nodes and* $\wedge$*-nodes with* $\vee$*-nodes in* ϕ*. Then for all* $x \in \{0,1\}^N$*, if* $\bar{x}$ *denotes the bitwise complement of* x*, then* $\phi'(x) = \neg\phi(\bar{x})$*. Furthermore up to an isomorphism that maps* s *to* s'*,* t *to* t'*, and an edge labeled by any label* λ *to an edge labeled by* λ*, we have* $G'_\phi = G_{\phi'}$ *and* $G'_\phi(x) = G_{\phi'}(\bar{x})$*.*

Proof. The first part of the proof is by induction. Suppose ϕ has depth 0, so $\phi = x_i$ for some variable x_i. Then $\phi'(x) = \phi(x) = \neg(\phi(\bar{x}))$. So suppose $\phi = \phi_1 \wedge \cdots \wedge \phi_l$. Then $\phi' = \phi'_1 \vee \cdots \vee \phi'_l$. Then by the induction hypothesis,

$$\phi'(x) = \phi'_1(x) \vee \cdots \vee \phi'_l(x) = (\neg\phi_1(\bar{x})) \vee \cdots \vee (\neg\phi_l(\bar{x})) = \neg(\phi_1(\bar{x}) \wedge \cdots \wedge \phi_l(\bar{x})) = \neg\phi(\bar{x}) \tag{63}$$

where the second to last equality is de Morgan's law. The case $\phi = \phi_1 \vee \cdots \vee \phi_l$ is similar.

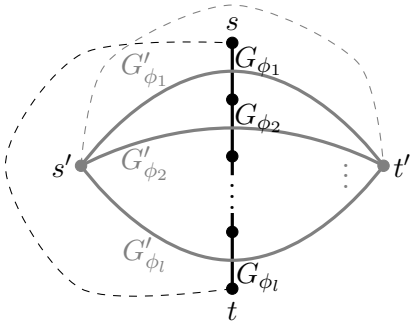

Figure 7: $\overline{G}_\phi$ shown in black, and its dual, $\overline{G}_{\phi'}$, shown in grey. The thick lines represent graphs. Edges in G_{ϕ_i} are dual to edges in G'_{ϕ_i}, and the dotted edge $(\{s,t\},\emptyset)$ is dual to $(\{s',t'\},\emptyset)$.

We will now prove that $\overline{G}^\dagger_\phi = \overline{G}_{\phi'}$, and furthermore, dual edges have the same label, by induction on the depth of ϕ, from which the result follows immediately.

If $\phi = x_i$ is a depth-0 formula, then $\phi' = x_i$. In that case, G_ϕ is just an edge from s to t, labeled by x_i, and G'_ϕ is just an edge from s' to t' labeled x_i, so $G'_\phi = G_{\phi'}$.

For the inductive step, to show that $\overline{G}_\phi$ and $\overline{G}_{\phi'}$ are dual, and therefore $G'_\phi = G_{\phi'}$. It suffices to exhibit a bijection $\zeta : V(\overline{G}_{\phi'}) \to F(\overline{G}_\phi)$ such that $(\{u,v\}, x_j) \in E(\overline{G}_{\phi'})$ if and only if the faces $\zeta(u)$ and $\zeta(v)$ are separated by an edge in $E(\overline{G}_\phi)$ with the label x_j. We first consider the case that $\phi = \phi_1 \wedge \cdots \wedge \phi_l$, so $\phi' = \phi'_1 \vee \cdots \vee \phi'_l$. Then, $\overline{G}_\phi$ consists of the graphs $G_{\phi_1}, \ldots, G_{\phi_l}$, chained together in series as in Figure 7, with an additional edge from s to t, so the faces of $\overline{G}_\phi$ are exactly all the interior faces of each G_{ϕ_i}, as well as the two faces on either side of the st-edge $(\{s,t\},\emptyset)$, which we will denote by $f^{s'}$ and $f^{t'}$. That is, adding an i to the label of each internal face of G_{ϕ_i}:

$$F(\overline{G}_\phi) = \{(i,f) : i \in [l], f \in F(\overline{G}_{\phi_i}) \setminus \{f^{s'}, f^{t'}\}\} \cup \{f^{s'}, f^{t'}\}, \tag{64}$$

since $F(\overline{G}_{\phi_i}) \setminus \{f^{s'}, f^{t'}\} = F(G_{\phi_i}) \setminus \{f^E\}$, where f^E is the external face. Since $\phi' = \phi'_1 \vee \cdots \vee \phi'_l$ we also have

$$V(\overline{G}_{\phi'}) = V(G_{\phi'}) = \{(i, v) : i \in [l], v \in V(G_{\phi'_i}) \setminus \{s, t\}\} \cup \{s', t'\}, \tag{65}$$

where we will use the labels s' and t' in anticipation of the isometry between G'_ϕ and $G_{\phi'}$.

By the induction hypothesis, for each $i \in [l]$, there exists a bijection $\zeta_i : V(\overline{G}_{\phi'_i}) \to F(\overline{G}_{\phi_i})$ such that for all $u, v \in V(\overline{G}_{\phi'_i}) = V(G_{\phi'_i})$, $(\{u, v\}, x_j) \in E(\overline{G}_{\phi'_i})$ if and only if $\zeta_i(u)$ and $\zeta_i(v)$ are faces separated by an edge with the label x_j. We define ζ by $\zeta(i, v) = (i, \zeta_i(v))$ for all $i \in [l]$ and $v \in V(G_{\phi'_i}) \setminus \{s, t\}$, $\zeta(s') = f^{s'}$, and $\zeta(t') = f^{t'}$. By the induction hypothesis, we can see that for any edge $(\{u, v\}, x_j) \in E(\overline{G}_{\phi'}) \setminus (\{s', t'\}, \emptyset)$, $\zeta(u)$ and $\zeta(v)$ are separated by an edge labeled x_j. This is because this edge is in one of the $G_{\phi'_i}$, and so it has a dual edge in G_{ϕ_i}, by the induction hypothesis (see Figure 7). The only other edge in $\overline{G}_{\phi'}$ is the edge $(\{s', t'\}, \emptyset)$, and $\zeta(s')$ and $\zeta(t')$ are exactly those faces on either side of $(\{s, t\}, \emptyset)$ in $\overline{G}_\phi$, completing the proof that $\overline{G}^\dagger_\phi = \overline{G}_{\phi'}$.

If $\phi = \phi_1 \vee \cdots \vee \phi_l$, then $\phi' = \phi'_1 \wedge \cdots \wedge \phi'_l$, and a nearly identical proof shows that $\overline{G}^\dagger_\phi = \overline{G}_{\phi'}$.

Now that we have shown an isomorphism between G'_ϕ and $G_{\phi'}$, note that $G'_\phi(x)$ is the subgraph of G'_ϕ that includes all those edges where $x_e = 0$. On the other hand $G_{\phi'}(x)$ is the graph that includes all those edges where $x_e = 1$. Taking the bitwise negation of x, we find that $G'_\phi(x) = G_{\phi'}(\bar{x})$. □

Claim 17. *If $\phi = \phi_1 \vee \phi_2 \vee \cdots \vee \phi_l$, then $G'_\phi(x)$ is formed by composing $\{G'_{\phi_i}(x)\}_i$ in series, and if $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_l$, then $G'_\phi(x)$ is formed by composing $\{G'_{\phi_i}(x)\}_i$ in parallel.*

Proof. If $\phi = \phi_1 \vee \cdots \vee \phi_l$, then $\phi' = \phi'_1 \wedge \cdots \wedge \phi'_l$. From Lemma 35, $G'_\phi(x) = G_{\phi'}(\bar{x})$, which using Definition 34 is composed of $\{G_{\phi'_i}(\bar{x})\}_{i=1}^l$ in series. But using the isomorphism of Lemma 35 again, this is just $\{G'_{\phi_i}(x)\}_{i=1}^l$ composed in series. The proof for $\phi = \phi_1 \wedge \cdots \wedge \phi_l$ is similar. □

$G_\phi(x)$ ɒ $G'_\phi(x)$

$\phi(x)$:

Lemma 14. *Let ϕ be any* AND-OR *formula on N variables. For every $x \in \{0, 1\}^N$, there exists a path from s to t in $G_\phi(x)$ if and only if $\phi(x) = 1$. Furthermore, for every $x \in \{0, 1\}^N$, there exists a path from s' to t' in $G'_\phi(x)$ if and only if $\phi(x) = 0$.*

Proof. We will prove the statement by induction on the depth of ϕ. If $\phi = x_j$ has depth 0, then G_ϕ is just an edge $(\{s, t\}, x_j)$, and G'_ϕ is just an edge $(\{s', t'\}, x_j)$. Thus s and t are connected in $G_\phi(x)$ if and only if $x_j = 1$, in which case ϕ evaluates to 1, and s' and t' are connected in $G_{\phi'}$ if and only if $x_j = 0$, in which case ϕ evaluates to 0.

If $\phi = \phi_1 \wedge \cdots \wedge \phi_l$, then G_ϕ consists of $G_{\phi_1}, \ldots, G_{\phi_l}$ connected in series from s to t, and moreover, $G_\phi(x)$ consists of $G_{\phi_1}(x), \ldots, G_{\phi_l}(x)$ connected in series from s to t. Thus an st-path in $G_\phi(x)$ consists of an st-path in $G_{\phi_1}(x)$, followed by an st-path in $G_{\phi_2}(x)$, etc., up to an st-path in $G_{\phi_l}(x)$. Thus, s and t are connected in $G_\phi(x)$ if and only if s and t are connected in each $G_{\phi_1}(x), \ldots, G_{\phi_l}(x)$, which happens if and only if $\phi_1(x) \wedge \cdots \wedge \phi_l(x) = 1$.

On the other hand, by Claim 17, G'_ϕ consists of $G'_{\phi_1}, \ldots, G'_{\phi_l}$ connected in parallel between s' and t'. So any $s't'$-path in $G'_\phi(x)$ is an $s't'$-path in one of the $G'_{\phi_i}(x)$, which is equivalent to an st-path in one of $G_{\phi'_i}(\bar{x})$. Thus, by Lemma 35 s' and t' are connected in

$G'_\phi(x)$ if and only if $\phi'_1(\bar{x}) \vee \cdots \vee \phi'_l(\bar{x}) = \neg\phi_1(x) \vee \cdots \vee \neg\phi_l(x) = 1$. By de Morgan's law is true if and only if $\phi(x) = \phi_1(x) \wedge \cdots \wedge \phi_l(x) = 0$.
The case when $\phi = \phi_1 \vee \cdots \vee \phi_l$ is similar. □

C Classical Lower Bound on Class of Promise Boolean Formulas

[illegible] AND-OR [illegible]

[illegible] f, [illegible] $R(f)$, [illegible] $R(f) = \Omega(RS(f))$, [illegible] $RS(f)$ [illegible]

[illegible] $f \circ g$, $RS(f \circ g) \geq (RS(f)RS(g))$.

[illegible] $f : D \to \{0,1\}$, [illegible] $D \subseteq \{0,1\}^N$, [illegible] $f_{\text{sab}} : D_{\text{sab}} \to \{0,1\}$, [illegible]

$$D_{\text{sab}} = \{x \in \{0,1,*\}^N \cup \{0,1,\dagger\}^N : x \text{ is consistent with } y, y' \in D, \text{ s.t. } f(y) \neq f(y')\}. \tag{66}$$

[illegible] $x \in \{0,1,*,\dagger\}^N$ [illegible] $y \in \{0,1\}^N$ [illegible] $x_i = y_i$ [illegible] $i \in [N]$ [illegible] $x_i \in \{0,1\}$. [illegible] $f_{\text{sab}}(x) = 1$ [illegible] $x \in \{0,1,*\}^N$, [illegible] $f_{\text{sab}}(x) = 0$ [illegible] $x \in \{0,1,\dagger\}^N$. [illegible] $RS(f) = R_0(f_{\text{sab}})$, [illegible] $R_0(f)$ [illegible] f. [illegible]

[illegible] $\text{AND}|_{D_{N,h}}$ [illegible] $\text{OR}|_{D'_{N,h}}$:

Lemma 36. *$RS\left(\text{OR}|_{D'_{N,h}}\right) = RS\left(\text{AND}|_{D_{N,h}}\right) = \Omega(N/h)$.*

Proof. For $x \in [D'_{N,h}]_{\text{sab}}$ to be consistent with $y, y' \in D'_{N,h}$ such that $\text{OR}(y) \neq \text{OR}(y')$, we must have that $x \in \{0,*\}^N \cup \{0,\dagger\}^N$. Furthermore, the number of $*$'s or $\dagger$'s in x must be at least h. Thus the sabotaged problem reduces to finding at least one marked item out of n, promised there are at least h marked items. The randomized bounded-error query complexity of this task is $\Omega(N/h)$, and so by Theorem 3 in [5],

$$RS\left(\text{OR}|_{D'_{N,h}}\right) = R_0\left((\text{OR}|_{D'_{N,h}})_{\text{sab}}\right) = \Omega\left(R\left((\text{OR}|_{D'_{N,h}})_{\text{sab}}\right)\right) = \Omega(N/h). \tag{67}$$

The proof for AND is similar. □

[illegible]

Corollary 37. *Let $\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_l$, where for each $i \in [l]$, $\phi_i = \text{OR}|_{D'_{N_i,h_i}}$ or $\phi_i = \text{AND}|_{D_{N_i,h_i}}$. Then $R(\phi) = \Omega\left(\prod_{i=1}^l N_i/h_i\right)$.*

[illegible] AND-OR [illegible]

Lemma 38. *Let $\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_l$, where for each $i \in [l]$, $\phi_i = \text{OR}|_{D'_{N_i,h_i}}$ or $\phi_i = \text{AND}|_{D_{N_i,h_i}}$. Let D be the domain of ϕ. Then*

$$\frac{\prod_{i=1}^l N_i}{\prod_{i=1}^l h_i} = \left(\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x))\right)\left(\max_{x \in D:\phi(x)=0} R_{s,t}(G'_\phi(x))\right). \tag{68}$$

Proof. The proof follows by induction on the number of compositions. First suppose that $\phi = \text{OR}|_{D'_{N,h}}$. Then G_ϕ consists of N edges connected in parallel between s and t, and G'_ϕ consists of N edges connected in series. The only input x such that $\phi(x) = 0$ is the all zeros input. Therefore $\max_{x \in D:\phi(x)=0} R_{s,t}(G'_\phi(x)) = N$. Now notice (using Claim 5) that $R_{s,t}(G_\phi(x)) = 1/|x|$. However because of the domain of OR_{N_i,h_i}, inputs x have $|x| \geq h$, so $\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x)) = 1/h$. Thus

$$N/h = \left(\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x)) \right) \left(\max_{x \in D:\phi(x)=0} R_{s,t}(G'_\phi(x)) \right). \tag{69}$$

A similar analysis holds for the base case $\phi = \text{AND}|_{D_{N,h}}$.

Now for the inductive step, let $\phi = \phi_1 \circ \xi$ for $\xi = \phi_2 \circ \cdots \circ \phi_l$, where for each i, ϕ_i is either $\text{OR}|_{D'_{N_i,h_i}}$ or $\text{AND}|_{D_{N_i,h_i}}$. Let D_ξ be the domain of ξ and let $x^j \in D_\xi$ denote the bits of x that are input to the j^{th} copy of ξ. Suppose first that $\phi_1 = \text{OR}|_{D'_{N_1,h_1}}$. G'_ϕ is formed by taking the N_1 graphs G'_ξ and connecting them in series. The only way $\phi(x) = 0$ is if the input $x^j \in D_\xi$ to each of the ξ functions satisfies $\xi(x^j) = 0$, so by Claim 5

$$\max_{x \in D:\phi(x)=0} R_{s,t}(G'_\phi(x)) = N_1 \max_{y \in D_\xi:\xi(y)=0} R_{s,t}(G'_\xi(y)). \tag{70}$$

On the other hand, G_ϕ is formed by taking N_1 graphs G_ξ and connecting them in parallel. Using Claim 5, if $x^j \in D_\xi$ is the input to j^{th} function ξ, we have

$$R_{s,t}(G_\phi(x)) = \left(\sum_{j=1}^{N} \frac{1}{R_{s,t}(G_\xi(x^j))} \right)^{-1}. \tag{71}$$

Thus larger values for $R_{s,t}(G_\phi(x))$ come from cases where $R_{s,t}(G_\xi(x^j))$ are large. Now

$$\max_{x \in D_\xi} R_{s,t}(G_\xi(x)) = \infty, \tag{72}$$

which occurs when $\xi(y) = 0$. Because of the promise on the domain of ϕ_1, there must be at least h_1 of the N_1 subformulas ξ that evaluate to 1. On each of those subformulas, we want to have an input $x^j \in D_\xi$ that maximizes the effective resistance of that subformula. Therefore, we have

$$\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x)) = \left(\frac{h_1}{\max_{y \in D_\xi:\xi(y)=1} R_{s,t}(G_{\xi_j}(y)} \right)^{-1} = \frac{\max_{y \in D_\xi:\xi(y)=1} R_{s,t}(G_{\xi_j}(y))}{h_1}. \tag{73}$$

Therefore, using the inductive assumption,

$$\begin{aligned} &\left(\max_{x \in D:\phi(x)=1} R_{s,t}(G_\phi(x)) \right) \left(\max_{x \in D:\phi(x)=0} R_{s,t}(G'_\phi(x)) \right) \\ =&\frac{N_1}{h_1} \max_{y \in D_\xi:\xi(y)=1} R_{s,t}(G_{\xi_j}(y^j)) \max_{y \in D_\xi:\xi(y)=0} R_{s,t}(G'_\xi(y)) = \frac{\prod_{i=1}^{l} N_i}{\prod_{i=1}^{l} h_i}. \end{aligned} \tag{74}$$

The inductive step for $\phi_1 = \text{AND}|_{D_{N,h}}$ is similar. □

D NAND-tree Proofs

D.1 Relationship Between Faults and Effective Resistance

Lemma 22. *For any $x \in \{0,1\}^{2^d}$, if d is even, then we have $R_{s,t}(G_{\mathrm{NAND}_d}(x)) \leq \mathcal{F}_A(x)$ and $R_{s',t'}(G'_{\mathrm{NAND}_d}(x)) \leq \mathcal{F}_B(x)$, while if d is odd, we have $R_{s,t}(G_{\mathrm{NAND}_d}(x)) \leq 2\mathcal{F}_A(x)$ and $R_{s',t'}(G'_{\mathrm{NAND}_d}(x)) \leq 2\mathcal{F}_B(x)$.*

Proof. We will give a proof for $\mathcal{F}_A(x)$; the case of $\mathcal{F}_B(x)$ is similar.

First, $R_{s,t}(G_{\mathrm{NAND}_d}(x)) = \infty$ if and only if s and t are not connected in $G_{\mathrm{NAND}_d}(x)$, which, by Lemma 14, occurs if and only if x is a 0-instance. This means exactly that x is not A-winnable, which, by Eq. (28), holds if and only if $\mathcal{F}_A(x) = \infty$. Thus, suppose this is not the case, so $\mathcal{F}_A(x) < \infty$.

The rest of the proof is by induction. We need to look at both odd and even cases. For the case of $d = 0$, the only A-winnable input in $\{0,1\}^{2^0}$ is $x = 1$. In that case, using Eq. (28), $\mathcal{F}_A(x) = 1$, since there are no decision nodes for Player A, and since $G_{\mathrm{NAND}_0}(x)$ is just a single edge from s to t, $R_{s,t}(G_{\mathrm{NAND}_0}(x)) = 1$.

Let $x \in \{0,1\}^{2^d}$ be any A-winnable input with $d > 1$. We let x^0 be the first 2^{d-1} bits of x and x^1 be the last 2^{d-1} bits of x, so $x = (x^0, x^1)$.

We first consider odd $d > 1$. Using the definition of G_ϕ from Section 4, and the fact that for d odd, the root node is an $\wedge$-node, we see that $G_{\mathrm{NAND}_d}(x)$ consists of $G_{\mathrm{NAND}_{d-1}}(x^0)$ and $G_{\mathrm{NAND}_{d-1}}(x^1)$ connected in series, so by Claim 5

$$R_{s,t}(G_{\mathrm{NAND}_d}(x)) = R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0)) + R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1)). \tag{75}$$

Now the root can not be a fault, because it is a decision node for Player B, but we know the tree is A-winnable, so no choice Player B makes would allow her to win the game. Therefore, both subtrees connected to the root node must be A-winnable. Using Eq. (28) we have

$$\mathcal{F}_A(x^0) + \mathcal{F}_A(x^1) \leq \max_{b\in\{0,1\}} 2\mathcal{F}_A(x^b) = 2\mathcal{F}_A(x). \tag{76}$$

Combining Eqs. (75) and (76) and the inductive assumption for even depth trees, we have for odd d,

$$R_{s,t}(G_{\mathrm{NAND}_d}(x)) \leq 2\mathcal{F}_A(x). \tag{77}$$

Now we consider the case that d is even, so the root is a decision node for Player A. Consequently, the root node is a $\vee$-node, so by Claim 5

$$R_{s,t}(G_{\mathrm{NAND}_d}(x)) = \left(\frac{1}{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0))} + \frac{1}{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))}\right)^{-1}. \tag{78}$$

Suppose the root is a fault. Without loss of generality, let's assume the subtree with input x^0 is not A-winnable. Then $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0)) = \infty$ so Eq. (78) becomes

$$R_{s,t}(G_{\mathrm{NAND}_d}(x)) = R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1)). \tag{79}$$

Using the inductive assumption for odd depth trees, Eq. (28), and the fact that the root is a fault, we have

$$R_{s,t}(G_{\mathrm{NAND}_d}(x)) \leq 2\mathcal{F}(x^1) = \mathcal{F}(x). \tag{80}$$

If the root is not a fault, then both $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0))$ and $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))$ are finite, so from (78), and using the inductive assumption, we have

$$\begin{aligned} R_{s,t}(G_{\mathrm{NAND}_d}(x)) &\leq \frac{1}{2}\max\{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0)), R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))\} \\ &\leq \max\{\mathcal{F}(x^0), \mathcal{F}(x^1)\} = \mathcal{F}(x). \end{aligned} \tag{81}$$

A similar analysis for $\mathcal{F}_B(x)$ completes the proof. □

D.2 Estimating Effective Resistances

[illegible] $\mathcal{O}_x$. [illegible] $G_\phi(x)$ [illegible] $G'_\phi(x)$.

[illegible] *approximate positive and negative witness sizes,* $\tilde{w}_+(x,P)$ [illegible] $\tilde{w}_-(x,P)$. [illegible] $|w\rangle \in H(x)$ [illegible] $\omega A \Pi_{H(x)} = 0$ [illegible]

Definition 39 (Approximate Positive Witness)**.** *For any span program* P *on* $\{0,1\}^N$ *and* $x \in \{0,1\}^N$*, we define the* positive error of x in P *as:*

$$e_+(x) = e_+(x,P) := \min\left\{\left\|\Pi_{H(x)^\perp}|w\rangle\right\|^2 : A|w\rangle = \tau\right\}. \tag{82}$$

We say $|w\rangle$ *is an* approximate positive witness *for* x *in* P *if* $\left\|\Pi_{H(x)^\perp}|w\rangle\right\|^2 = e_+(x)$ *and* $A|w\rangle = \tau$*. We define the* approximate positive witness size *as*

$$\tilde{w}_+(x) = \tilde{w}_+(x,P) := \min\left\{\||w\rangle\|^2 : A|w\rangle = \tau, \left\|\Pi_{H(x)^\perp}|w\rangle\right\|^2 = e_+(x)\right\}. \tag{83}$$

[illegible] $x \in P_1$, [illegible] $e_+(x) = 0$. [illegible] x [illegible] $\tilde{w}_+(x) = w_+(x)$. [illegible] 0

Definition 40 (Approximate Negative Witness)**.** *For any span program* P *on* $\{0,1\}^N$ *and* $x \in \{0,1\}^N$*, we define the* negative error of x in P *as:*

$$e_-(x) = e_-(x,P) := \min\left\{\left\|\omega A \Pi_{H(x)}\right\|^2 : \omega \in \mathcal{L}(U,\mathbb{R}), \omega\tau = 1\right\}. \tag{84}$$

Any ω *such that* $\left\|\omega A \Pi_{H(x)}\right\|^2 = e_-(x,P)$ *is called an* approximate negative witness *for* x *in* P*. We define the* approximate negative witness size *as*

$$\tilde{w}_-(x) = \tilde{w}_-(x,P) := \min\left\{\|\omega A\|^2 : \omega \in \mathcal{L}(U,\mathbb{R}), \omega\tau = 1, \left\|\omega A \Pi_{H(x)}\right\|^2 = e_-(x,P)\right\}. \tag{85}$$

[illegible] $x \in P_0$, [illegible] $e_-(x) = 0$. [illegible] x [illegible] $\tilde{w}_-(x) = w_-(x)$. [illegible] 0

Theorem 41 ([17]). *Fix $X \subseteq \{0,1\}^N$ and $f : X \to \mathbb{R}_{\geq 0}$. Let P be a span program such that for all $x \in X$, $f(x) = w_+(x, P)$ and define $\widetilde{W}_- = \widetilde{W}_-(P, f) = \max_{x \in X} \tilde{w}_-(x, P)$. There exists a quantum algorithm that estimates f to relative error ε and that uses $\widetilde{O}\left(\frac{1}{\varepsilon^{3/2}}\sqrt{w_+(x)\widetilde{W}_-}\right)$ queries. Similarly, let P be a span program such that for all $x \in X$, $f(x) = w_-(x, P)$ and define $\widetilde{W}_+ = \widetilde{W}_+(P, f) = \max_{x \in X} \tilde{w}_+(x, P)$. Then there exists a quantum algorithm that estimates f to accuracy ε and that uses $\widetilde{O}\left(\frac{1}{\varepsilon^{3/2}}\sqrt{w_-(x)\widetilde{W}_+}\right)$ queries.*

[illegible] $P_{G,c}$ [illegible] (6), [illegible] $G = G_\phi$. [illegible] c [illegible] G. [illegible] $P_{G,c}$ [illegible] P_G. [illegible] $\widetilde{W}_+(P_{G_\phi})$ [illegible] $\widetilde{W}_-(P_{G_\phi})$. [illegible]

Lemma 42. *For any formula ϕ, its $\wedge$-depth is the largest number of $\wedge$-labeled nodes on any path from the root to a leaf. Let ϕ be any* AND-OR *formula with maximum fan-in l, $\wedge$-depth $d_\wedge$, and $\vee$-depth $d_\vee$. Then $\widetilde{W}_+(P_{G_\phi}) \leq \frac{1}{2}l^{d_\wedge}$ and $\widetilde{W}_-(P_{G_\phi}) \leq 2l^{d_\vee}$.*

[illegible]

Lemma 25 (`Est` Algorithm). *Let ϕ be an* AND-OR *formula with constant fan-in l, $\vee$-depth $d_\vee$ and $\wedge$-depth $d_\wedge$. Then the quantum query complexity of estimating $R_{s,t}(G_\phi(x))$ (resp. $R_{s,t}(G'_\phi(x))$) to relative accuracy ϵ is $\widetilde{O}\left(\frac{1}{\varepsilon^{3/2}}\sqrt{R_{s,t}(G_\phi(x))l^{d_\vee}}\right)$ (resp. $\widetilde{O}\left(\frac{1}{\varepsilon^{3/2}}\sqrt{R_{s,t}(G'_\phi(x))l^{d_\wedge}}\right)$).*

Proof of Lemma 25. By Theorem 41, since $R_{s,t}(G_\phi(x)) = \frac{1}{2}w_+(x, P_{G_\phi})$ (Lemma 11), we can estimate this quantity using a number of queries that depends on $\widetilde{W}_-(P_{G_\phi})$. By Lemma 42, we have that $\widetilde{W}_-(P_{G_\phi}) \leq 2l^{d_\vee}$, so we can estimate $w_+(x) = R_{s,t}(G_\phi(x))$ in $\widetilde{O}\left(\frac{1}{\varepsilon^{2/3}}\sqrt{w_+(x)\widetilde{W}_-^{1/2}}\right) = \widetilde{O}\left(\frac{1}{\varepsilon^{2/3}}\sqrt{R_{s,t}(G_\phi(x))l^{d_\vee}}\right)$ queries. Similarly, $R_{s,t}(G'_\phi(x)) = 2w_-(x, P_{G_\phi})$ for all 0-instances, and $\widetilde{W}_+ \leq \frac{1}{2}l^{d_\wedge}$, so we can estimate $R_{s,t}(G'_\phi(x))$ in $\widetilde{O}\left(\frac{1}{\varepsilon^{2/3}}\sqrt{R_{s,t}(G'_\phi(x))l^{d_\wedge}}\right)$ queries. □

[illegible] *st* [illegible] G_ϕ, [illegible] $\wedge$ [illegible] ϕ. [illegible]

Claim 43. *Let ϕ be an* AND-OR *formula with constant fan-in l. If ϕ has $\wedge$-depth $d_\wedge$, then the longest self-avoiding path connecting s and t in G_ϕ has length at most $l^{d_\wedge}$.*

Proof. We will prove the statement by induction. If ϕ has $\wedge$-depth $d_\wedge = 0$, then it has no $\wedge$-nodes. Thus, it is easy to see that G_ϕ has only two vertices, s and t, with some number of edges connecting them, so every st-path has length 1.

Suppose ϕ has $\wedge$-depth $d_\wedge > 0$. First, suppose $\phi = \phi_1 \wedge \cdots \wedge \phi_l$. Then since G_ϕ consists of $G_{\phi_1}, \ldots, G_{\phi_l}$ connected in series, any st-path in G_ϕ consists of an st-path in G_{ϕ_1}, followed by an st-path in G_{ϕ_2}, etc. up to an st-path in G_{ϕ_l}, so if $d_\wedge(\phi_i)$ is the $\wedge$-depth of ϕ_i, then the longest st-path in G_ϕ has length at most:

$$l^{d_\wedge(\phi_1)} + \cdots + l^{d_\wedge(\phi_l)} \leq ll^{d_\wedge - 1} = l^{d_\wedge}. \tag{86}$$

If $\phi = \phi_1 \vee \cdots \vee \phi_l$, then $\max_i d_\wedge(\phi_i) = d_\wedge(\phi) = d_\wedge$, and G_ϕ consists of $G_{\phi_1}, \ldots, G_{\phi_l}$, connected in parallel. Any self-avoiding st-path must include exactly one edge adjacent to s and one edge adjacent to t. However, any path that includes an edge from G_{ϕ_i} and G_{ϕ_j} for $i \neq j$ must go through s or t, so it must have more than one edge adjacent to s, or more than one edge adjacent to t, so such a path can never be a self-avoiding st-path. Thus, any self-avoiding st-path must be contained completely in one of the G_{ϕ_i}. The longest such path is thus the longest self-avoiding st-path in any of the G_{ϕ_i}, which, by induction, is $\max_i l^{d_\wedge(\phi_i)} = l^{d_\wedge}$. □

Proof of Lemma 42. To begin, we will prove the upper bound on $\widetilde{W}_+$. Suppose $|\tilde{w}\rangle$ is an optimal approximate positive witness for x. By Claim 30, if $|\tilde{w}\rangle$ is an approximate positive witness, then since $A|\tilde{w}\rangle = \tau$, and c has unit value on all edges of G, $\theta(u,v,\lambda) = \langle u,v,\lambda|\tilde{w}\rangle - \langle v,u,\lambda|\tilde{w}\rangle$ is a unit flow on G. Since $|\tilde{w}\rangle$ is an approximate positive witness for x, it has minimal error for x, so it minimizes $\left\|\Pi_{H(x)^\perp}|\tilde{w}\rangle\right\|^2$, and since it is optimal, it minimizes $\||\tilde{w}\rangle\|^2$ over all approximate positive witnesses. Define $|\theta\rangle = \sum_{(u,v,\lambda)\in\vec{E}(G)} \theta(u,v,\lambda)|u,v,\lambda\rangle$, so we know that $\frac{1}{2}|\theta\rangle$ also maps to τ under A, so is also a positive witness in P_{G_ϕ}.Then we have

$$\left\|\Pi_{H(x)^\perp}|\theta\rangle\right\|^2 = 2 \sum_{\substack{(u,v,\lambda)\in \\ \vec{E}(G)\setminus\vec{E}(G(x))}} \langle u,v,\lambda|\tilde{w}\rangle^2 - 2 \sum_{\substack{(u,v,\lambda)\in \\ \vec{E}(G)\setminus\vec{E}(G(x))}} \langle u,v,\lambda|\tilde{w}\rangle\langle v,u,\lambda|\tilde{w}\rangle \leq \left\|2\Pi_{H(x)^\perp}|\tilde{w}\rangle\right\|^2, \tag{87}$$

where the last inequality uses Cauchy-Schwarz, so $\frac{1}{2}|\theta\rangle$ is also an approximate positive witness for x. Similarly,

$$\||\theta\rangle\|^2 \leq \|2|\tilde{w}\rangle\|^2, \tag{88}$$

so $\frac{1}{2}|\theta\rangle$ is optimal.

By Claim 29, we can consider a decomposition of $|\theta\rangle$ into self-avoiding paths p_i and cycles c_i such that all cycles are disjoint from all paths, $|\theta\rangle = \sum_{i=1}^r \alpha_i|p_i\rangle + \sum_{i=1}^{r'} \beta_i|c_i\rangle$, where for each i,

$$|p_i\rangle = \sum_{j=1}^{L_i} |u_j^{(i)}, u_{j+1}^{(i)}, \lambda_{i,j}\rangle - \sum_{j=1}^{L_i} |u_{j+1}^{(i)}, u_j^{(i)}, \lambda_{i,j}\rangle, \tag{89}$$

$$|c_i\rangle = \sum_{j=1}^{L_i'} |v_j^{(i)}, v_{j+1}^{(i)}, \lambda'_{i,j}\rangle - \sum_{j=1}^{L_i'} |v_{j+1}^{(i)}, v_j^{(i)}, \lambda'_{i,j}\rangle \tag{90}$$

where $v_{L_j'+1}^{(i)} = v_1^{(i)}$ and $\{\lambda_{i,j}\}_{i,j} \cap \{\lambda'_{i,j}\}_{i,j} = \emptyset$. It's easy to see (in the case of unit edge weights) that $A|c_i\rangle = 0$ for all i, so

$$A\frac{1}{2}\sum_{i=1}^r \alpha_i|p_i\rangle = A\frac{1}{2}|\theta\rangle = \tau. \tag{91}$$

Let $|\theta'\rangle = \sum_{i=1}^r \alpha_i|p_i\rangle$. Then since c_i and p_j have no common edges, we have $\langle c_i|p_j\rangle = 0$, and also $\langle c_i|(I - \Pi_{H(x)})|p_j\rangle = 0$, so the error of $\frac{1}{2}|\theta'\rangle$ is $\frac{1}{4}\left\|\Pi_{H(x)^\perp}|\theta'\rangle\right\|^2 \leq \frac{1}{4}\left\|\Pi_{H(x)^\perp}|\theta\rangle\right\|^2$, so $\frac{1}{2}|\theta'\rangle$ also has minimal error. Furthermore, $\||\theta'\rangle\|^2 \leq \||\theta\rangle\|^2$, with equality if and only if there are no cycles in the decomposition. By the optimality of $\frac{1}{2}|\theta\rangle$ as an approximate

positive witness for x, we can conclude that $|\theta\rangle = \sum_{i=1}^{r} \alpha_i |p_i\rangle$, and since $A|p_i\rangle = 2\tau$ for all i, and $A|\theta\rangle = 2\tau$, we have $\sum_{i=1}^{r} \alpha_i = 1$. Then

$$\||\theta\rangle\|^2 \leq \max_i \||p_i\rangle\|^2 = \max_i 2L_i. \tag{92}$$

Since the longest self-avoiding st-path in G_ϕ has length at most $l^{d_\wedge}$, and each L_i is the length of a self-avoiding path in G_ϕ, we have $\tilde{w}_+(x, P_{G_\phi}) \leq \frac{1}{4} 2 l^{d_\wedge} = \frac{1}{2} l^{d_\wedge}$. Thus $\widetilde{W}_+(P_{G_\phi}) \leq \frac{1}{2} l^{d_\wedge}$.

Next we prove the bound on $\widetilde{W}_-$. A min-error approximate negative witness for x in P_{G_ϕ} is a function $\omega : V(G_\phi) \to \mathbb{R}$ such that $\omega\tau = \omega(s) - \omega(t) = 1$, and $\left\|\omega A \Pi_{H(x)}\right\|^2 = \sum_{(u,v,\lambda)\in\overrightarrow{E}(G_\phi(x))} (\omega(u) - \omega(v))^2$ is minimized. By Claim 31, since $\omega\tau = 1$, the function $\theta : \overrightarrow{E}(G'_\phi) \to \mathbb{R}$ defined by $\theta((u, v, \lambda)^\dagger) = \omega(u) - \omega(v)$ is a unit $s't'$-flow on $G'_\phi = G_{\phi'}$, and the witness complexity is

$$\|\omega A\|^2 = \sum_{(u,v,\lambda)\in\overrightarrow{E}(G_\phi)} (\omega(u) - \omega(v))^2 = \sum_{(u',v',\lambda)\in\overrightarrow{E}(G'_\phi)} \theta(u', v', \lambda)^2 = \||\theta\rangle\|^2 \tag{93}$$

where we create $|\theta\rangle$ from θ in the usual way. By an argument similar to the previous argument, if ω is an optimal approximate negative witness for x, then $\||\theta\rangle\|^2$ is upper bounded by twice the length of the longest self-avoiding $s't'$-path in $G'_\phi = G_{\phi'}$. By Lemma 35 and Claim 43, this is upper bounded by $2l^{d_\wedge(\phi')} = 2l^{d_\vee(\phi)}$, where $d_\wedge(\phi')$ is the $\wedge$-depth of ϕ', and $d_\vee = d_\vee(\phi)$ is the $\vee$-depth of ϕ. Thus $\tilde{w}_-(x, P_{G_\phi}) \leq 2l^{d_\vee}$, and so $\widetilde{W}_- \leq 2l^{d_\vee}$. □

D.3 Winning the NAND-tree

[illegible] NAND[illegible]

Lemma 26. *Let $x^0, x^1 \in \{0,1\}^{2^d}$ be instances of NAND_d with at least one of them a 1-instance. Let $N = 2^d$, and $w_{\min} = \min\{R_{s,t}(G_{\mathrm{NAND}_d}(x^0)), R_{s,t}(G_{\mathrm{NAND}_d}(x^1))\}$. Then $\mathtt{Select}(x^0, x^1)$ terminates after $\widetilde{O}\left(N^{1/4}\sqrt{w_{\min}}\right)$ queries to (x^0, x^1) and outputs b such that $R_{s,t}(G_{\mathrm{NAND}_d}(x^b)) \leq 2R_{s,t}(G_{\mathrm{NAND}_d}(x^{\bar{b}}))$ with bounded error.*

Proof. Since at least one of x^0 and x^1 is a 1-instance, using the description of `Select` in Section 5.2, at least one of the programs will terminate. Suppose without loss of generality that $\mathtt{Est}(x^0)$ is the first to terminate, outputting w_0. Then there are two possibilities: $\mathtt{Est}(x^1)$ does not terminate after $p(d)\sqrt{w_0}N^{1/4}$ steps, in which case, $R_{s,t}(G_{\mathrm{NAND}_d}(x^0)) \leq 2R_{s,t}(G_{\mathrm{NAND}_d}(x^1))$, and `Select` outputs 0; or $\mathtt{Est}(x^1)$ outputs w_1 before $p(d)\sqrt{w_0}N^{1/4}$ steps have passed and `Select` outputs b such that $w_b \leq w_{\bar{b}}$.

We will prove the first case by contradiction. Suppose

$$2R_{s,t}(G_{\mathrm{NAND}_d}(x^1)) < R_{s,t}(G_{\mathrm{NAND}_d}(x^0)). \tag{94}$$

Then $\mathtt{Est}(x^1)$ must terminate after

$$p(d)\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x^1))}N^{1/4} \leq \frac{1}{\sqrt{2}} p(d)\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x^0))}N^{1/4} \tag{95}$$

steps. In `Select`, we run `Est` to relative accuracy $\varepsilon = 1/3$, so we have

$$|w_0 - R_{s,t}(G_{\mathrm{NAND}_d}(x^0))| \leq \frac{1}{3} R_{s,t}(G_{\mathrm{NAND}_d}(x^0)), \tag{96}$$

and so

$$w_0 \geq \frac{2}{3} R_{s,t}(G_{\mathrm{NAND}_d}(x^0)). \tag{97}$$

Plugging Eq. (97) into Eq. (95), we have $\texttt{Est}(x^1)$ must terminate after $\frac{1}{\sqrt{2}} p(d) \sqrt{\frac{3}{2} w_0} N^{1/4} < p(d)\sqrt{w_0} N^{1/4}$ steps, which is a contradiction.

Thus, $R_{s,t}(G_{\mathrm{NAND}_d}(x^0)) \leq 2R_{s,t}(G_{\mathrm{NAND}_d}(x^1))$, so outputting 0 is correct. Furthermore, since we terminate after $p(d)\sqrt{w_0}N^{1/4} = \widetilde{O}(\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x^0))}N^{1/4})$ steps, and since $R_{s,t}(G_{\mathrm{NAND}_d}(x^0)) = O(R_{s,t}(G_{\mathrm{NAND}_d}(x^1)))$, the running time is at most $\widetilde{O}\left(N^{1/4}\sqrt{w_{\min}}\right)$.

We now consider the second case, in which both programs output estimates w_0 and w_1, such that $|w_b - R_{s,t}(G_{\mathrm{NAND}_d}(x^b))| \leq \varepsilon R_{s,t}(G_{\mathrm{NAND}_d}(x^b))$ for $b = 0, 1$. Suppose $w_b \leq w_{\bar{b}}$. We then have

$$\frac{R_{s,t}(G_{\mathrm{NAND}_d}(x^b))}{R_{s,t}(G_{\mathrm{NAND}_d}(x^{\bar{b}}))} \leq \frac{R_{s,t}(G_{\mathrm{NAND}_d}(x^b))}{w_b} \frac{w_{\bar{b}}}{R_{s,t}(G_{\mathrm{NAND}_d}(x^{\bar{b}}))} \leq \frac{1+\varepsilon}{1-\varepsilon} = \frac{4/3}{2/3} = 2. \tag{98}$$

Thus $R_{s,t}(G_{\mathrm{NAND}_d}(x^b)) \leq 2R_{s,t}(G_{\mathrm{NAND}_d}(x^{\bar{b}}))$, as required. Furthermore, the running time of the algorithm is bounded by the running time of $\texttt{Est}(x^1)$, the second to terminate. We know that $\texttt{Est}(x^1)$ has running time at most $\widetilde{O}\left(\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x^1))}N^{1/4}\right)$ steps, and by assumption, $\texttt{Est}(x^1)$ terminated after less than $p(d)\sqrt{w_0}N^{1/4} = \widetilde{O}\left(\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x^0))}N^{1/4}\right)$ steps, so the total running time is at most $\widetilde{O}\left(N^{1/4}\sqrt{w_{\min}}\right)$. □

Theorem 27. *Let $x \in \{0,1\}^N$ for $N = 2^d$ be an A-winnable input to NAND_d. At every node v where Player A makes a decision, let Player A use the* `Select` *algorithm in the following way. Let v_0 and v_1 be the two children of v, with inputs to the respective subtrees of v_0 and v_1 given by x^0 and x^1 respectively. Then Player A moves to v_b where b is the outcome that occurs a majority of times when $\texttt{Select}(x^0, x^1)$ is run $O(\log d)$ times. Then if Player B, at his decision nodes, chooses left and right with equal probability, Player A will win the game with probability at least $2/3$, and will use $\widetilde{O}\left(N^{1/4}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}\right)$ queries on average, where the average is taken over the randomness of Player B's choices.*

Proof. First note that Player A must make $O(d)$ choices over the course of the game. We amplify Player A's probability of success by repeating `Select` at each decision node $O(\log d)$ times and taking the majority. Then the probability that Player A chooses the wrong direction at any node is $O(1/d)$, and we ensure that her probability of choosing the wrong direction over the course of the algorithm is $< 1/3$. From here on, we analyze the error free case.

Let $p(d)$ be a non-decreasing polynomial function in d such that `Select`, on inputs $x^0, x^1 \in \{0,1\}^{2^d}$, terminates in at most $p(d)2^{d/4}\sqrt{\min\{R_{s,t}(G_{\mathrm{NAND}_d}(x^0)), R_{s,t}(G_{\mathrm{NAND}_d}(x^1))\}}$ queries. Then we will prove that for trees of odd depth d, the expected number of queries by Player A over the course of the game is at most $p(d)2^{d/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}$, while for even depth trees, it is at most $p(d)2^{d/4+11/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}$, thus proving the main result.

We prove the result by induction on the depth of the tree. For depth zero trees, there are no decisions, $N = R_{s,t}G_{\mathrm{NAND}_0}(x) = 1$, so the result holds.

For the inductive case, we treat odd and even depth cases separately. First consider an instance of NAND_d with $d > 0$, d odd. Thus $\mathrm{NAND}_d(x) = \mathrm{NAND}_{d-1}(x^0) \wedge \mathrm{NAND}_{d-1}(x^1)$,

where $x = (x^0, x^1)$. Because the root is at odd distance from the leaves, the root is a decision node for Player B. Because we are in an A-winnable tree, no matter which choice Player B makes, we will end up at an A-winnable subtree of depth $d-1$, so the inductive assumption holds for those trees. That is, the expected number of queries for Player A must make to win the subtree with input x^b (for $b \in \{0,1\}$) averaged over Player B's choices is at most

$$p(d-1)2^{(d-1)/4+11/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}. \tag{99}$$

We are assuming that Player B chooses left and right with equal probability. Thus, the expected number of queries that Player A must make over Player B's choices throughout the game is at most

$$\begin{aligned}
&\frac{1}{2}\Big(p(d-1)2^{(d-1)/4+11/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0))} \\
&\qquad +p(d-1)2^{(d-1)/4+11/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))}\Big) \\
&\le p(d-1)2^{(d-1)/4+11/2}\sqrt{\frac{1}{2}\left(R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0)) + R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))\right)} \quad \text{by Jensen's ineq.,} \\
&= p(d-1)2^{(d-1)/4+11/2}\sqrt{\frac{1}{2}R_{s,t}(G_{\mathrm{NAND}_d}(x))} \quad \text{by Claim 5,} \\
&\le p\left(d\right)2^{d/4-1/4+11/2-1/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))} \\
&\le p\left(d\right)2^{d/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))},
\end{aligned} \tag{100}$$

proving the case for odd d.

Now consider an instance of NAND_d with $d > 0$, d even. Thus $\mathrm{NAND}_d(x) = \mathrm{NAND}_{d-1}(x^0) \vee \mathrm{NAND}_{d-1}(x^1)$, where $x = (x^0, x^1)$. Because the root is at even distance from the leaves, the root is a decision node for Player A. Player A runs $\texttt{Select}(x^0, x^1)$, which returns $b \in \{0,1\}$ such that (by Lemma 26)

$$R_{s,t}(G_{\mathrm{NAND}^{d-1}}(x^b)) \le 2R_{s,t}(G_{\mathrm{NAND}^{d-1}}(x^{\bar{b}})), \tag{101}$$

which requires at most

$$\min_{b^* \in \{0,1\}} p\left(d-1\right)2^{(d-1)/4}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{b^*}))} \tag{102}$$

queries.

After making the choice to move to the subtree with input x^b, by the inductive assumption, the expected number of queries that Player A need to make throughout the rest of the game (averaged over Player B's choices) is

$$p\left(d-1\right)2^{d/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}. \tag{103}$$

There are two cases to consider. If $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b)) \le R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))$, then combining Eq. (102) and Eq. (103), we have that the total number of queries averaged

over Player B's choices is

$$\begin{aligned}
&p\,(d-1)\,2^{(d-1)/4}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}+p\,(d-1)\,2^{(d-1)/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}\\
\leq&p\,(d-1)\,2^{(d-1)/4}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}(1+2^5)\\
\leq&p\,(d-1)\,2^{(d-1)/4+5+1/16}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}\\
\leq&p\,(d-1)\,2^{(d-1)/4+5+1/16+1/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}\\
\leq&p\,(d)\,2^{d/4+11/2}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}
\end{aligned}\tag{104}$$

where we've used $R_{s,t}(G_{\mathrm{NAND}_d}(x)) = \left(R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0))^{-1}+R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))^{-1}\right)^{-1}$ from Claim 5 and the fact that $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b)) \leq R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))$ to bound the value $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))$ by $2R_{s,t}(G_{\mathrm{NAND}_d}(x))$. This proves the even induction step for this case.

The other case is if $R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b)) > R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))$. In that case, again using the fact that $R_{s,t}(G_{\mathrm{NAND}_d}(x)) = \left(R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^0))^{-1}+R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^1))^{-1}\right)^{-1}$, we have

$$R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}})) = R_{s,t}(G_{\mathrm{NAND}_d}(x))\left(1+\frac{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))}{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}\right)^{-1} \leq \frac{2}{3}R_{s,t}(G_{\mathrm{NAND}_d}(x)),\tag{105}$$

where the inequality follows from Eq. (101). Thus, the average total number of queries is

$$\begin{aligned}
&p\,(d-1)\,2^{(d-1)/4}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))}+p\,(d-1)\,2^{(d-1)/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^b))}\\
\leq&p(d-1)2^{(d-1)/4}\left(\sqrt{R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))}+2^5\sqrt{2R_{s,t}(G_{\mathrm{NAND}_{d-1}}(x^{\bar{b}}))}\right)\\
\leq&p(d-1)2^{(d-1)/4}(1+2^{5+1/2})\sqrt{\frac{2}{3}R_{s,t}(G_{\mathrm{NAND}_d}(x))}\\
\leq&p(d)2^{d/4-1/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}\\
\leq&p(d)2^{d/4+5}\sqrt{R_{s,t}(G_{\mathrm{NAND}_d}(x))}.
\end{aligned}\tag{106}$$

This proves the induction step for the other case. □