A Stronger Theorem Against Macro-realism

John-Mark A. Allen[1], Owen J. E. Maroney[2], and Stefano Gogioso[1]

[1]Department of Computer Science, University of Oxford, Wolfson Building, Parks Road, Oxford, OX1 3QD, United Kingdom

[2]Faculty of Philosophy, University of Oxford, Radcliffe Humanities, Woodstock Road, Oxford, OX2 6GG, United Kingdom

July 10, 2017

arXiv:1610.00022v3 [quant-ph] 7 Jul 2017

Macro-realism is the position that certain "macroscopic" observables must always possess definite values: *e.g.* the table is in some definite position, even if we don't know what that is precisely. The traditional understanding is that by assuming macro-realism one can derive the Leggett-Garg inequalities, which constrain the possible statistics from certain experiments. Since quantum experiments can violate the Leggett-Garg inequalities, this is taken to rule out the possibility of macro-realism in a quantum universe. However, recent analyses have exposed loopholes in the Leggett-Garg argument, which allow many types of macro-realism to be compatible with quantum theory and hence violation of the Leggett-Garg inequalities. This paper takes a different approach to ruling out macro-realism and the result is a no-go theorem for macro-realism in quantum theory that is stronger than the Leggett-Garg argument. This approach uses the framework of ontological models: an elegant way to reason about foundational issues in quantum theory which has successfully produced many other recent results, such as the PBR theorem.

1 Introduction

The concept of macro-realism was introduced to the study of quantum theory by Leggett & Garg alongside their eponymous inequalities [1]. Loosely, macro-realism is the philosophical position that certain "macroscopic" quantities always possess definite values. The Leggett-Garg inequalities (LGIs) are inequalities on observed measurement statistics that are derived by assuming a particular form of macro-realism and can be violated by measurements on quantum systems. The purpose of the LGIs is therefore to prove that quantum theory and macro-realism are incompatible. However, since its introduction the exact meaning of "macro-realism" has been the subject of debate [2–8].

Recently, there has been a surge of interest in violation of the LGIs from both physical and philosophical angles. The review in Ref. [9] comprehensively covers experimental and theoretical work up to 2014. More recent experimental work has focussed on noise tolerance and closing experimental loopholes [10–12]. Also, several theoretical investigations have aimed to interrogate and clarify exactly what is required to derive the LGIs and what is implied by their experimental violation [7, 8, 13–16]. This paper follows the clarifying work of Ref. [8].

Macro-realism is an ontological position; that is, the statement that a certain "macroscopic" quantity is macro-realist is a statement about the real state of affairs, the *ontology*, of the universe. In the field of quantum foundations, the framework of *ontological models* has been developed as a way to analyse such statements in generality, making as few assumptions as possible. To use an ontological model to describe a system requires just two core assumptions. First, that the system being described has some *ontological state*—the real fact about how the system actually is (of course, which ontological state is currently occupied is generally unknown). Second, that standard probability theory can be applied to the ontological states. Their generality has made ontological models very useful and they have been used to derive and clarify many important results in quantum foundations including: Bell's theorem [17], the PBR theorem [18], and excess baggage [19]. Reference [20] comprehensively reviews many of these results.

By using ontological models it is possible to illuminate and classify various definitions of macro-realism precisely [8]. This analysis reveals some fundamental loopholes in the Leggett-Garg argument for the incompatibility of quantum theory and macro-realism. In particular, it shows that violation of the LGIs serves only to rule out

John-Mark A. Allen: john-mark.allen@cs.ox.ac.uk, Corresponding author

one subset of macro-realist models and that there are other macro-realist models of quantum theory which are compatible with the LGIs. For example, Bohmian quantum theory [21–25] can be viewed as a macro-realist model which reproduces all predictions of quantum theory and therefore cannot be ruled out by the Leggett-Garg argument. These loopholes are not experimental but logical; the only way to close them is to fundamentally change the argument.

In this paper, a stronger theorem for the incompatibility between quantum theory and macro-realism is presented. This theorem closes a loophole in the Leggett-Garg argument and establishes that quantum theory is incompatible with a larger subset of macro-realist models. It does not prove incompatibility of quantum theory with all macro-realist models since this is not possible; such a result would be in conflict with the existence of the theory of Bohmian mechanics (section 5). The theorem proceeds in a very different manner than the Leggett-Garg argument and is related to the main theorem from Ref. [26]. It thereby circumvents many of the controversies of the original Leggett-Garg approach.

It should be noted that mathematically there is no meaning to the stipulation that macro-realism is about "macroscopic" quantities, as opposed to other physical quantities that aren't "macroscopic". Philosophically, however, it is easy to understand the desire for macro-realism applying to "macroscopic" quantities. The types of physical quantity that humans experience are all considered macroscopic and they certainly appear to possess definite values. On the other hand, it is much easier to imagine that microscopic quantities that aren't directly observed behave in radically different ways. So while there is nothing in the structure of quantum theory to pick-out "macroscopic" versus "microscopic", the motivation for considering macro-realism does come from considering macroscopic quantities, hence the name.

This paper is organised as follows. The framework of ontological models is introduced in section 2. This is then used in section 3 to give the precise definitions of macro-realism and its three sub-sets. Both of these sections are without reference to quantum theory and are entirely independent of it. This is as it should be, since those concepts apply to descriptions of any physical system based on philosophical assumptions and do not depend on any specific physical theory (of course their most common applications are with quantum systems). In section 4 macro-realism is applied to quantum theory and several useful definitions and lemmas about quantum ontological models are presented. Section 5 discusses the Leggett-Garg argument against macro-realism and explains why the existence of loopholes [8] restrict it to only a simple class of macro-realist models. Section 6 then proves the main theorem which introduces a new approach, ruling out a larger class of macro-realist models than Leggett-Garg while using weaker assumptions. A discussion follows in section 7 where conclusions are drawn, together with discussion of further research directions including the possibility for experiments based on this result.

2 Ontological Models

When we debate types of "realism" in quantum theory, we are normally making an ontological argument. We're trying to say something about the underlying actual state of affairs: whether such a thing exists, how it can or can't behave, and so on. So it is with macro-realism. The macro-realist, loosely, believes that the underlying ontology in some definite sense *possesses* a value for certain macroscopic quantities at all times. The framework of ontological models [27, 28] has been developed to make discussions about ontology in physics precise and is the natural arena for such discussions.

An ontological model is exactly that: a bare-bones model for the underlying ontology of some physical system. The system may also be correctly described by some other, higher, theory—such as Newtonian mechanics or a quantum theory—in which case the ontological model must be constrained to reproduce the predictions of that theory. By combining these constraints with the very general framework of ontological models, interesting and general conclusions can be drawn about the nature of the ontology. It is important to note that, while ontological models are normally used to discuss quantum ontology, the framework itself is entirely independent from quantum theory. The presentation of ontological models here follows Ref. [20], which contains a much more thorough discussion.

As noted above, the framework of ontological models relies on just two core assumptions: 1) that the system of interest has some ontological state λ and 2) that standard probability theory may be applied to these states. Together, these bring us to consider the ontology of some physical system as represented by some measurable set Λ of ontic states $\lambda \in \Lambda$ which the system might occupy. The requirement that Λ be measurable guarantees that probabilities over Λ can be defined.

In the lab, a system can be prepared, transformed, and measured in certain ways. Each of these operational processes needs to be describable in the ontological model.

Preparation must result in the system ending up in some ontic state λ, though the exact state need not be known. Thus, each preparation P gives rise to some *preparation measure* μ over Λ which is a probability measure ($\mu(\emptyset) = 0$, $\mu(\Lambda) = 1$). For every measurable subset $\Omega \subseteq \Lambda$, $\mu(\Omega)$ gives the probability that the resulting λ is in Ω.

Similarly, a transformation T of the system will generally change the ontic state from $\lambda' \in \Lambda$ to a new $\lambda \in \Lambda$. Recalling that the ontic state λ' represents the entirety of the actual state of affairs before the transformation, then the final state can only depend on λ' (and not the preparation method or any previous ontic states, except as mediated through λ'). The transformations must therefore be described as *stochastic maps* γ on Λ. A stochastic map consists of a probability measure $\gamma(\cdot|\lambda')$ for each initial ontic state, such that for any measurable $\Omega \subseteq \Lambda$, $\gamma(\Omega|\lambda')$ is the probability that the final λ lies in Ω given that the initial state was λ' [1].

Finally, a measurement M may give rise to some outcome E. Again, which outcome is obtained can only depend on the current ontic state λ'. Therefore a measurement M gives rise to a conditional probability distribution $\mathbb{P}(E|\lambda')$ [2]. For this paper it is only necessary to consider measurements that have countable sets of possible outcomes E.

Putting these parts together: if we have a system where a preparation P is performed followed by some transformation T and some measurement M then the ontological model for that system must have some preparation measure μ, stochastic map γ, and conditional probability distribution $\mathbb{P}$ such that the probability of obtaining outcome E is

$$\int_\Lambda \mathrm{d}\nu(\lambda)\, \mathbb{P}(E \,|\, \lambda) \tag{1}$$

where

$$\nu(\Omega) \stackrel{\text{def}}{=} \int_\Lambda \mathrm{d}\mu(\lambda)\, \gamma(\Omega \,|\, \lambda) \tag{2}$$

is the effective preparation measure obtained by preparation P followed by transformation T.

Note that ontological models are required to be closed under transformations. That is, for any preparation μ and transformation γ in the model then the preparation ν defined by Eq. (2) must also exist in the model (since a preparation followed by a transformation is itself a type of preparation).

So an ontological model for some physical system does the following:

1. defines a measurable set Λ of ontic states for the system;
2. for each possible transformation method T defines a stochastic map γ from Λ to itself;
3. for each possible preparation method P defines a preparation measure μ over Λ, ensuring closure under the actions of the stochastic maps as in Eq. (2);
4. for each possible measurement method M defines a conditional probability distribution $\mathbb{P}$ over the outcomes given $\lambda \in \Lambda$;

and then produces probabilities for measurement outcomes via Eqs. (1,2). The possible ontological models for a system can be constrained by requiring that these probabilities match those given by other theories known to accurately describe it (or probabilities obtained by experiment).

It should be noted that ontological models are not usually defined in the measure-theoretic way presented here, but are often presented using probability *distributions* rather than measures. However it has been noted in Ref. [20] that this simplification precludes many reasonably ontological models, including the archetypal Beltrametti-Bugajski model [29] which simply takes ontic states to be quantum states. In order to do justice to macro-realism the more accurate approach has therefore been taken here.

3 Macro-realism

Exactly what is meant by "macro-realism" has been a subject of contention ever since its introduction alongside the LGIs. This controversy has fed into more recent work on understanding the violation of the LGIs [13–15]. In Ref. [8], uses of the term "macro-realism" are analysed and the concept is illuminated using ontological models. One

[1] These stochastic maps, viewed as functions $\gamma(\Omega|\cdot) : \Lambda \to [0,1]$ (one for each measurable $\Omega \subseteq \Lambda$), must be *measurable functions*. That is, for any measurable set $\mathcal{S} \subseteq [0,1]$ and any $\gamma(\Omega|\lambda')$, then $\{\lambda' \in \Lambda : \gamma(\Omega|\lambda') \in \mathcal{S}\} \subseteq \Lambda$ is a measurable set.

[2] These probability distributions, viewed as functions $\Lambda \to [0,1]$, must also be measurable functions.

result of that paper is that the "macro-realism" intended by Leggett and Garg, as well as many subsequent authors, can be made precise in a reasonable way with the definition:

"A macroscopically observable property with two or more distinguishable values available to it will at all times determinately possess one or other of those values." [8]

Throughout this paper, "distinguishable" will be taken to mean "in principle perfectly distinguishable by a single measurement in the noiseless case". Note that macro-realism is defined with respect to some specific property Q. A macro-realist model might (and generally will) be macro-realist for some properties and not others. This property will have some values $\{q\}$ and to be "observable" must correspond to at least one measurement M_Q with corresponding outcomes E_q.

Reference [8] fleshes out this definition using ontological models and as a result describes three sub-categories of macro-realism. In order to discuss these it will be necessary to first define an *operational eigenstate* in ontological models.

An operational eigenstate Q_q of any value q of an observable property Q is a set of preparation procedures $\{P_q\}$. This set is defined so that immediately following any P_q with any measurement of the quantity Q will result in the outcome E_q with certainty. In other words, an operational eigenstate is simply an extension of the concept of a quantum eigenstate to ontological models: the preparations which, when appropriately measured, always return a particular value of a particular property. Note that if two values q, q' have operational eigenstates then they can sensibly be called "distinguishable", since any system prepared in a corresponding operational eigenstate can be identified to have one value and not the other with certainty.

The three sub-categories of macro-realism for some quantity Q are then:

1. *Operational eigenstate mixture macro-realism (EMMR)* – The only preparations in the model are operational eigenstates of Q or statistical mixtures of operational eigenstates. That is, for each preparation $P_{q,i}$ of each operational eigenstate Q_q let $\mu_{q,i}$ be the preparation measure and let $\{c_{q,i}\}_{q,i}$ be a set of positive numbers summing to unity. In EMMR every preparation measure can be written in the form $\nu = \sum_q \sum_i c_{q,i}\mu_{q,i}$. Note that this means that the space of ontic states Λ need only include those λ accessible by preparing some operational eigenstate of Q, as no other ontic states can ever be prepared.

2. *Operational eigenstate support macro-realism (ESMR)* – Like EMMR, every ontic state $\lambda \in \Lambda$ accessible by preparing some operational eigenstate but, unlike EMMR, there are preparation measures in the model that are not statistical mixtures of operational eigenstate preparations for Q. That is, let Ω be a measurable subset for which $\mu_q(\Omega) = 1$ for every operational eigenstate preparation μ_q. Then for every preparation measure ν in an ESMR model, $\nu(\Omega) = 1$ for all such Ω. Moreover, there exists some preparation procedure not in the mixture form required by EMMR. In other words, if you're certain to prepare an ontic state from some subset Ω when preparing an operational eigenstate of Q, then you're also certain to prepare an ontic state from Ω from any other preparation measure in the model.

3. *Supra eigenstate support macro-realism (SSMR)* – Every ontic state λ in the model will produce some specific value q_λ of Q when a measurement of Q is made, but some of those ontic states are not accessible by preparing any operational eigenstate of Q. That is, for every $\lambda \in \Lambda$ then there is some value q_λ of Q such that $\mathbb{P}_M(E_{q_\lambda} \,|\, \lambda) = 1$ for every $\mathbb{P}_M$ corresponding to a measurement of Q. Moreover, there exists some measurable subset $\Omega \subset \Lambda$ and preparation measure ν such that $\nu(\Omega) > 0$ while $\mu_q(\Omega) = 0$ for every preparation measure μ_q of an operational eigenstate Q_q.

To help unpack these definitions, they are illustrated in Fig. 1.

In each of these cases, every ontic state λ (up to possible measure-zero sets of exceptions[3]) is associated with a specific value q_λ of Q, such that it can be sensibly said that λ "possesses" q_λ. This is why they are all considered types of macro-realism. Let's consider this for each case in turn.

In an EMMR model, any preparation can be viewed as a choice between preparations of operational eigenstates for values of Q, so any resulting ontic state λ "possesses" the corresponding value q since it could have been obtained by preparing an operational eigenstate of q.

In an ESMR model, for every ontic state λ accessible by preparing some non-operational-eigenstate measure ν, λ can also be prepared by an operational eigenstate of exactly one value of Q (up to measure-zero sets of exceptions) and so similarly each ontic state "possesses" the corresponding value of Q.

[3]One persuasive advantage to defining ontological models with probability distributions, rather than measures, is that it eliminates the need to append "up to possible measure-zero sets of exceptions" to many otherwise-clear statements. We apologise for the necessary repetition of this phrase in this paper.

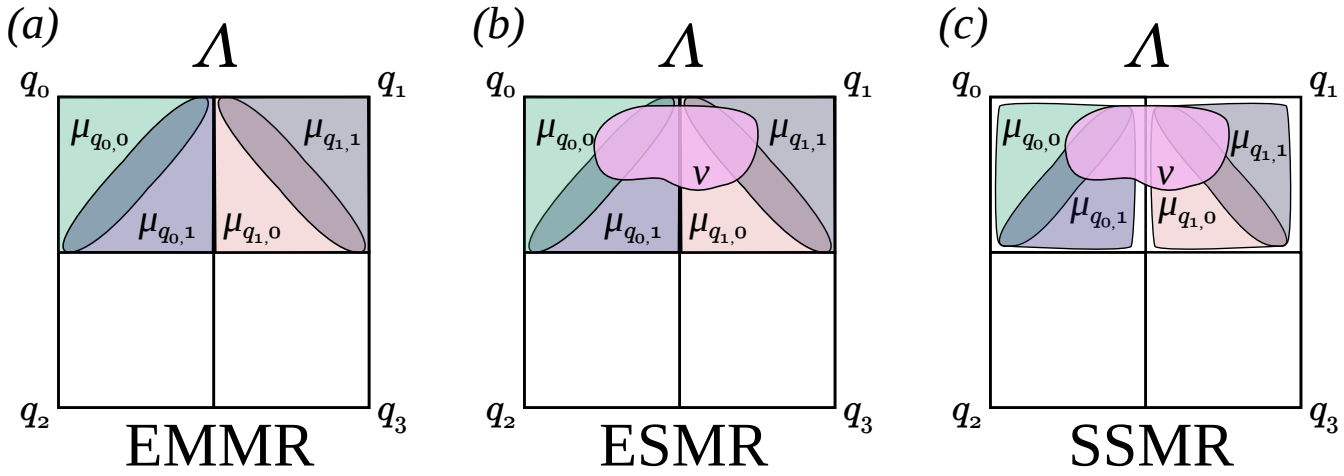

Figure 1: Illustration of the three sub-categories of macro-realism as defined in the text. In each case the large square represents the whole ontic state space Λ, the four smaller squares indicate those subspaces of ontic states associated with each value $q_{0..3}$ of some quantity Q, and the shaded regions represent those ontic states accessible by preparing some select preparation measures.
(a) illustrates EMMR, where the squares for each q_i are all ontic states preparable via some operational eigenstate preparation $\mu_{q_i,j}$ and all other allowed preparation measures are simply statistical mixtures of these, *e.g.* $\nu = \frac{1}{3}\left(\mu_{q_0,0} + \mu_{q_0,1} + \mu_{q_1,0}\right)$ is permissible.
(b) illustrates ESMR, where the state space is exactly as in EMMR, but now more general preparation measures, such as the ν illustrated, are permitted.
(c) illustrates SSMR, where now every λ in the box for q_i must produce outcome q_i in any appropriate measurement of Q, but the operational eigenstates no longer fill these boxes. That is, there are ontic states that lie outside the preparations for operational eigenstates. General preparation measures over the boxes are still permitted.

In SSMR models the link between each $\lambda \in \Lambda$ and the corresponding q_λ is explicit in the definition. For any λ the q_λ is that value for which $\mathbb{P}_M(E_{q_\lambda} \,|\, \lambda) = 1$ as specified in the above definition. Thus, λ "possesses" the value which it must return with certainty in any relevant measurement.

Note that these three sub-categories of macro-realism are defined such that they are mutually exclusive, but they still have a natural hierarchy to them. EMMR can be seen as a more restrictive variation on ESMR, since you can make an EMMR model into an ESMR model simply by including a single preparation measure that is not a statistical mixture of operational eigenstate preparations (the ontic state space and everything else can remain unchanged). Similarly, SSMR can be seen as a less restrictive variation on ESMR. In ESMR, every ontic state λ can be obtained by preparing an operational eigenstate preparation for a value of Q, by definition of operational eigenstate it follows that a measurement of Q will therefore return some specific value for each ontic state (up to measure-zero sets of exceptions), which is the requirement on the ontic states for SSMR.

4 Macro-realism in Quantum Theory

Now that macro-realism and ontological models have been defined independently from quantum theory, it is time to apply these definitions to quantum systems. Doing this will enable precise discussion of the loopholes in the Leggett-Garg argument and proof of a stronger theorem. It will also be necessary to state some useful definitions and lemmas from ontological models.

4.1 Ontological Models for Quantum Systems

The postulates of quantum theory assign some Hilbert space $\mathcal{H}$ to a quantum system with dimension d so that the set of physical pure quantum states is $\mathcal{P}(\mathcal{H}) = \{|\psi\rangle \in \mathcal{H} \,:\, \||\psi\rangle\| = 1, |\psi\rangle \sim \mathrm{e}^{i\theta}|\psi\rangle\}$. They also assign unitary operators on $\mathcal{H}$ to transformations and orthonormal bases over $\mathcal{H}$ to measurements. For simplicity, consider only systems with $d < \infty$. With this in mind, ontological models for quantum systems can be described in generality.

An ontological model for a quantum system is defined by some ontic state space Λ as well as the relevant preparation measures, stochastic maps, and conditional probability distributions. For each state $|\psi\rangle \in \mathcal{P}(\mathcal{H})$ there

must be a set $\Delta_{|\psi\rangle}$ of preparation measures $\mu_{|\psi\rangle}$—potentially one for each distinct method for preparing $|\psi\rangle$. Similarly, for each unitary operator U on $\mathcal{H}$ there is a set Γ_U of stochastic maps γ_U and for each basis measurement $M = \{|i\rangle\}_{i=0}^{d}$ there is a set Ξ_M of conditional probability distributions $\mathbb{P}_M$—again, potentially one stochastic map/probability distribution for each experimental method for transforming/measuring.

In order to investigate the properties of possible ontological models it is required that the ontological model is capable of reproducing the predictions of quantum theory. That is, for any $|\psi\rangle \in \mathcal{P}(\mathcal{H})$, $\mu \in \Delta_{|\psi\rangle}$, U, $\gamma \in \Gamma_U$, basis M, and $\mathbb{P}_M \in \Xi_M$ it is required that

$$|\langle i|U|\psi\rangle|^2 = \int_\Lambda \mathrm{d}\nu(\lambda)\, \mathbb{P}_M(|i\rangle\,|\,\lambda) \quad \forall |i\rangle \in M \tag{3}$$

where ν is defined as in Eq. (2). Note also that $\nu \in \Delta_{U|\psi\rangle}$ since preparing the quantum state $|\psi\rangle$ (via any ontological preparation $\mu \in \Delta_{|\psi\rangle}$) followed by performing the quantum transformation U (via any $\gamma \in \Gamma_U$) is simply a way to prepare the quantum state $U|\psi\rangle$.

4.2 State Overlaps

In order to properly discuss macro-realism in quantum theory it is necessary to discuss how to quantify state overlaps in ontological models. In quantum theory any pair of non-orthogonal states $|\psi\rangle, |\phi\rangle \in \mathcal{P}(\mathcal{H})$ overlap by an amount quantified by the Born rule probability $|\langle\psi|\phi\rangle|^2$. That is, for a system prepared in state $|\psi\rangle$ the probability for it to behave (for all intents and purposes) like it was prepared in state $|\phi\rangle$ is $|\langle\psi|\phi\rangle|^2$.

Adapting this logic to an ontological model for the quantum system, consider the probability that a system prepared according to measure μ behaves like it was prepared according to ν. That is, the probability that the ontic state obtained from μ could also have been obtained from ν. This quantity is called the *asymmetric overlap* and is mathematically defined as [26, 30–32]

$$\varpi(\nu\,|\,\mu) \stackrel{\text{def}}{=} \inf\{\mu(\Omega) \,:\, \Omega \subseteq \Lambda,\, \nu(\Omega) = 1\}, \tag{4}$$

recalling that the infimum of a subset of real numbers is the greatest lower bound of that set. This is because a preparation of ν has unit probability of producing a λ from each measurable subset $\Omega \subseteq \Lambda$ that satisfies $\nu(\Omega) = 1$ [4]. Therefore by taking the minimum such Ω, $\mu(\Omega)$ gives the desired probability.

It is not difficult to see that—for an ontological model of a quantum system—the Born rule upper bounds the asymmetric overlap. Almost all[5] ontic states that can be obtained by preparing $|\phi\rangle$ will also return outcome $|\phi\rangle$ in any relevant measurement and so if a preparation of $|\psi\rangle$ results in a λ that could have been obtained by preparing $|\phi\rangle$ then we know that this ontic state will almost surely return $|\phi\rangle$ in a relevant measurement. It follows that for any $\mu \in \Delta_{|\psi\rangle}$ and $\nu \in \Delta_{|\phi\rangle}$

$$\varpi(\nu\,|\,\mu) \leq |\langle\phi|\psi\rangle|^2. \tag{5}$$

A full proof of this is provided in Appendix B.

It is useful to overload the definition of asymmetric overlap to include the probability that preparing an ontic state via μ will produce a λ accessible by preparing some *quantum state* $|\phi\rangle$. This corresponds to

$$\varpi(|\phi\rangle\,|\,\mu) \stackrel{\text{def}}{=} \inf\left\{\mu(\Omega) \,:\, \nu(\Omega) = 1,\ \forall \nu \in \Delta_{|\phi\rangle}\right\}, \tag{6}$$

which is clearly also upper bounded by the Born rule $\varpi(|\phi\rangle\,|\,\mu) \leq |\langle\phi|\psi\rangle|^2$.

The next useful generalisation is the overlap of some preparation measure μ with *two* quantum states $|0\rangle, |\phi\rangle$. This can be thought of as the *union* of the overlaps expressed by $\varpi(|\phi\rangle\,|\,\mu)$ and $\varpi(|0\rangle\,|\,\mu)$ and is mathematically defined as

$$\varpi(|0\rangle, |\phi\rangle\,|\,\mu) \stackrel{\text{def}}{=} \inf\left\{\mu(\Omega) \,:\, \nu(\Omega) = \chi(\Omega) = 1,\ \forall \nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}\right\} \tag{7}$$

or, equivalently as

$$\varpi(|0\rangle, |\phi\rangle\,|\,\mu) = \inf\left\{\mu(\Omega_\phi \cup \Omega_0) \,:\, \nu(\Omega_\phi) = \chi(\Omega_0) = 1,\ \forall \nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}\right\}. \tag{8}$$

[4] When defining ϖ in this paper, the subsets $\Omega \subseteq \Lambda$ that are extremised over are all taken to be *measurable* subsets of Λ. This detail is omitted in the text for clarity.

[5] Here, as elsewhere in measure theory, "almost" is used to mean "up to measure-zero sets of exceptions".

So in this way $\varpi(|0\rangle, |\phi\rangle \,|\, \mu)$ expresses the probability that sampling from μ produces a λ accessible by preparing either $|0\rangle$ or $|\phi\rangle$.

Since $\varpi(|0\rangle, |\phi\rangle \,|\, \mu)$ expresses the probability of a disjunction of two events that have probabilities $\varpi(|0\rangle \,|\, \mu)$ and $\varpi(|\phi\rangle \,|\, \mu)$, it follows (by Boole's inequality) that it is bounded as follows

$$\varpi(|0\rangle, |\phi\rangle \,|\, \mu) \leq \varpi(|0\rangle \,|\, \mu) + \varpi(|\phi\rangle \,|\, \mu). \tag{9}$$

There are special triples of quantum states $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ for which the bound Eq. (9) is necessarily saturated for all $\mu \in \Delta_{|\psi\rangle}$. *Anti-distinguishable triples*[6] have this property. They are triples $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ which have a quantum measurement with outcomes $E_{\neg\psi}, E_{\neg\phi}, E_{\neg 0}$ where the probability of obtaining outcome $E_{\neg\psi}$ for a system in state $|\psi\rangle$ is zero (and similarly for $E_{\neg\phi}, E_{\neg 0}$). In other words, there is a measurement which can, with certainty, identify one state that was definitely not prepared. For this to be possible, almost no ontic states can be accessible by preparing all three states in the triple, because any such ontic state wouldn't be able to return any of the outcomes in the measurement. A full proof of this is provided in Appendix D.

It is known that[7] triples of states $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ with inner products $a = |\langle\psi|\phi\rangle|^2$, $b = |\langle\psi|0\rangle|^2$, and $c = |\langle\phi|0\rangle|^2$ satisfying

$$a + b + c < 1 \quad \text{and} \quad (1 - a - b - c)^2 \geq 4abc \tag{10}$$

are necessarily anti-distinguishable by a projective measurement.

Finally, it is useful to consider the effect that unitary transformations have on asymmetric overlaps. If some transformation U is applied, via some stochastic map $\gamma \in \Gamma_U$, to a system prepared according to measure $\mu \in \Delta_{|\psi\rangle}$ then the overlap with some quantum state $|\phi\rangle$ cannot decrease. That is

$$\varpi(U|\phi\rangle \,|\, \mu') \geq \varpi(|\phi\rangle \,|\, \mu) \tag{11}$$

where $\mu' \in \Delta_{U|\psi\rangle}$ is the preparation measure obtained by applying γ to μ as in Eq. (2). This is because any ontic state accessible by preparing μ will be mapped by γ onto ontic states accessible by preparing μ' (since preparing μ then applying γ is a preparation of μ'). Similarly, ontic states accessible by preparing $|\phi\rangle$ map onto states preparable by $U|\phi\rangle$. Thus ontic states in the overlap of μ and $|\phi\rangle$ are mapped by γ onto states in the overlap of μ' and $U|\phi\rangle$. Once again, a full proof is provided in Appendix C.

4.3 Macro-realism for Quantum Systems

Having laid the groundwork the above definition of macro-realism can now be applied to quantum systems.

First consider what can count as a "macroscopically observable" quantity Q. To be observable Q must correspond to some quantum measurement M_Q. Therefore, there is some orthonormal basis $\mathcal{B}_Q$ so that for each value q of Q the corresponding outcome of M_Q is a state in $\mathcal{B}_Q$. In order to make sense of the above definitions Q must also have operational eigenstates for each value q of Q. Fortunately this is straightforward in quantum theory: every state in $\mathcal{B}_Q$ is an operational eigenstate. Moreover, because the elements of $\mathcal{B}_Q$ are orthogonal it follows that preparations corresponding to different values q, q' of Q are therefore distinguishable.

Now consider an ESMR or EMMR model for a quantum system. For any state $|\psi\rangle \in \mathcal{P}(\mathcal{H})$ and any eigenstate $|0\rangle \in \mathcal{B}_Q$ the asymmetric overlap $\varpi(|0\rangle \,|\, \mu)$ for any $\mu \in \Delta_{|\psi\rangle}$ must be maximal. That is,

$$\varpi(|0\rangle \,|\, \mu) = |\langle 0|\psi\rangle|^2, \quad \forall \mu \in \Delta_{|\psi\rangle}. \tag{12}$$

A full proof of this is provided in Appendix E and only an outline provided here. Since each state in $\mathcal{B}_Q$ is orthogonal to every other, no ontic state (up to measure-zero exceptions) can be accessible by preparing more than one state in $\mathcal{B}_Q$. Moreover—by definition of ESMR and EMMR—every ontic state must be accessible by preparing *some* operational eigenstate of Q. Therefore, any ontic state accessible by preparing μ must also be accessible by preparing *exactly one* state in $\mathcal{B}_Q$ (up to measure-zero sets of exceptions). So the sum $\sum_i \varpi(|i\rangle|\mu) = 1$ because each overlap is disjoint (up to measure-zero sets of exceptions) and by Eq. (5) each must be maximal, giving Eq. (12).

Equation (12) is the key consequence of ESMR and EMMR that leads to the no-go theorem with quantum systems presented in section 6.

[6] Anti-distinguishability was introduced in Ref. [33] under the name "PP-incompatibility" and was given the more informative name of anti-distinguishability in Ref. [20].

[7] This result was proved in Ref. [33] but Ref. [34] points out and corrects a typographical error in their result (the original had the second inequality as a strict inequality, which is incorrect).

5 Loopholes in the Leggett-Garg Proof

The aim of the LGIs has always been to rule out macro-realist ontologies for quantum theory when the inequalities are violated. However, in light of the above precise definition of macro-realism, some loopholes in the argument can be identified.

The first loophole is that violation of the LGIs cannot rule out SSMR models of quantum systems. Indeed, no argument that rests on compatibility with quantum predictions can completely rule out SSMR models since there exists a well-known SSMR model for quantum systems that reproduces all quantum predictions: Bohmian mechanics [23–25].

To see that Bohmian mechanics provides an SSMR ontology consider, for example, the Bohmian description of a single spinless point particle in three-dimensional space (the argument for more general systems is analogous). Bohmian mechanics has the ontic state as a pair $\lambda = (\vec{r}, |\psi\rangle) \in \mathbb{R}^3 \times \mathcal{P}(\mathcal{H})$ where $\vec{r}$ is the actual position of the particle and $|\psi\rangle$ is the quantum state (or "pilot wave"). Note that the quantum state is part of the ontology here. The "macroscopically observable property" is the position of the particle, $\vec{r}$, and any sharp measurement of position will reveal the true value of $\vec{r}$ with certainty. Thus, for any ontic state λ there is some value of the macroscopically observable property (that is, $\vec{r}$) which is obtained with certainty from any appropriate measurement. Thus, Bohmian mechanics provides an SSMR ontological model.

The second loophole is that LGI violation is also unable to rule out ESMR ontological models. This is also demonstrated through a counter-example in the form of the Kochen-Specker model for the qubit [35], which is an ontological model satisfying ESMR[8]. The Kochen-Specker model exactly reproduces quantum predictions for $d = 2$ dimensional Hilbert Spaces. As the LGIs are defined in $d = 2$ the Kochen-Specker model will violate them.

A key question is *why* these counter-examples evade the Leggett-Garg argument. To derive the LGIs, one needs an additional assumption: *non-invasive measurability*. The Leggett-Garg approach compares the non-invasiveness of the measurement process on operational eigenstates with the invasiveness on preparations that are not operational eigenstates. Their violation shows that these other preparations cannot be expressed as mixtures of operational eigenstates. In neither ESMR nor SSMR can generic preparations be related to mixtures of operational eigenstates, so the Leggett Garg approach generically has loopholes for these types of macro-realism [8]. Bohmian mechanics and the Kochen-Specker model are both examples: they contain measurement disturbances that violate the non-invasive measurability assumption, while still satisfying SSMR and ESMR respectively. The crux is that both SSMR and ESMR models can include measurements that don't disturb the distribution over Λ if the system is prepared in an operational eigenstate, but still disturb the distribution over Λ for systems prepared in other ways.

EMMR, by contrast, requires that all preparations are represented by statistical mixtures of operational eigenstates. If it can be demonstrated that operational eigenstates are not disturbed by a given measurement, then according to EMMR no preparations can be disturbed by that measurement. It is this feature that prevents EMMR models from violating the LGI (see Ref. [8] for a more extensive discussion of this point).

Recent experiments [10, 11] following Ref. [16] have sought to address a "clumsiness loophole" in Leggett-Garg. They drop non-invasive measurability as an assumption by incorporating control experiments to check the disturbance of the measurement on the operational eigenstates. These approaches follow the Leggett-Garg argument quite closely and show that the disturbance on some general preparation cannot be explained in terms of disturbances on a statistical mixture of operational eigenstates. As a result, they are still only able to rule out EMMR models.

So the Leggett-Garg proof, even taking into account the clumsiness loophole, only rules out EMMR macro-realism and leaves loopholes for SSMR and ESMR. Moreover, the loophole for SSMR models cannot be fully plugged by any proof because Bohmian mechanics exists as a counter-example. Similarly, the loophole for ESMR cannot be fully pluged in $d = 2$ dimensions, since the Kochen-Specker model exists as a counter-example. This leaves a clear question: can the ESMR loophole be closed by another theorem for any $d > 2$? Answering this question needs a different approach, one which does not make use of the measurement disturbance assumptions at all.

[8] Strictly speaking, the Kochen-Specker model was not defined with a post-measurement update rule and so cannot deal with sequences of measurements (and therefore Leggett-Garg experiments). However, it is simple to append the obvious update rule "prepare a new state corresponding to the measurement outcome" and this fixes this issue.

6 A Stronger No-Go Theorem

Using the machinery developed above, it is possible to prove a theorem that rules out both ESMR and EMMR models for quantum systems with "macroscopically observable" properties with $n > 3$ values. This theorem is therefore stronger than the Leggett-Garg proof as it rules out ESMR models as well as EMMR models.

First, assume there is an ontological model for a quantum system with $d > 3$ dimensions which is ESMR or EMMR for quantity Q with $n = d > 3$ values. By applying Eqs. (9,11,12) to specially chosen quantum states it is possible to prove a contradiction.

Let $|0\rangle \in \mathcal{B}_Q$ be the eigenstate of some value q of Q and let $|\psi\rangle \in \mathcal{P}(\mathcal{H})$ be any other state of the system such that $|\langle 0|\psi\rangle|^2 \in (0, \frac{1}{2})$. Since quantum states in $\mathcal{P}(\mathcal{H})$ are equivalent up to global phase, $\langle 0|\psi\rangle$ can be taken to be a positive real without loss of generality. Now select another orthonormal basis $\mathcal{B}' = \{|0\rangle\} \cup \{|i'\rangle\}_{i=1}^{d-1}$ for $\mathcal{H}$ such that

$$|\psi\rangle = \alpha|0\rangle + \beta|1'\rangle + \tau|2'\rangle, \tag{13}$$

$$\alpha \in \left(0, \frac{1}{\sqrt{2}}\right), \tag{14}$$

$$\beta \stackrel{\text{def}}{=} \sqrt{2}\,\alpha^2, \tag{15}$$

such a basis always exists since, for any α, a $\tau \in (0, 1)$ exists such that $|\psi\rangle$ is normalised. Define another state with respect to the same basis

$$|\phi\rangle \stackrel{\text{def}}{=} \delta|0\rangle + \eta|1'\rangle + \kappa|3'\rangle, \tag{16}$$

$$\delta \stackrel{\text{def}}{=} 1 - 2\alpha^2, \tag{17}$$

$$\eta \stackrel{\text{def}}{=} \sqrt{2}\,\alpha. \tag{18}$$

These states have been chosen such that $\langle 0|\psi\rangle = \alpha = \langle\phi|\psi\rangle$ and therefore there is some unitary U satisfying $U|0\rangle = |\phi\rangle$ and $U|\psi\rangle = |\psi\rangle$. Moreover, the inner products of $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ satisfy Eq. (10) meaning that $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ is an anti-distinguishable triple and therefore satisfies Eq. (9) with equality.

Choose any preparation measure $\mu' \in \Delta_{|\psi\rangle}$ for $|\psi\rangle$ and any stochastic map $\gamma \in \Gamma_U$ for U. If $|\psi\rangle$ is prepared according to μ' and then transformed according to γ such that $\mu \in \Delta_{|\psi\rangle}$ is the resulting preparation measure (as in Eq. (2)) then

$$\varpi(|\phi\rangle, |0\rangle \,|\, \mu) = \varpi(|\phi\rangle \,|\, \mu) + \varpi(|0\rangle \,|\, \mu) \tag{19}$$

$$\geq \varpi(|0\rangle \,|\, \mu') + \varpi(|0\rangle \,|\, \mu) \tag{20}$$

$$= 2|\langle 0|\psi\rangle|^2 = 2\alpha^2 \tag{21}$$

where the first line follows from anti-distinguishability, the second from Eq. (11), and the third from ESMR/EMMR via Eq. (12).

Now consider that $\varpi(|\phi\rangle, |0\rangle \,|\, \mu) \leq \mathbb{P}_{\mathcal{B}'}(|0\rangle, |1'\rangle \,|\, |\psi\rangle)$. That is, $\varpi(|\phi\rangle, |0\rangle \,|\, \mu)$ is a lower bound on the probability that a quantum basis measurement in $\mathcal{B}'$ has outcome $|0\rangle$ or $|1'\rangle$ for a preparation of $|\psi\rangle$. To see this, consider that any ontic state preparable by both $|\psi\rangle$ and $|\phi\rangle$ must return a measurement outcome compatible with both preparations: thus the outcome must be either $|0\rangle$ or $|1'\rangle$. Similarly any ontic state preparable by both $|\psi\rangle$ and $|0\rangle$ must return the outcome $|0\rangle$ in such a measurement. A full proof of this is provided in Appendix F.

Putting these inequalities together, one finds

$$2\alpha^2 \leq \mathbb{P}_{\mathcal{B}'}(|0\rangle, |1'\rangle \,|\, |\psi\rangle) = \alpha^2 + \beta^2 = \alpha^2\left(1 + 2\alpha^2\right) \tag{22}$$

implying

$$\alpha \geq \frac{1}{\sqrt{2}}. \tag{23}$$

But α was defined to be in the range $(0, \frac{1}{\sqrt{2}})$ and so this is a contradiction.

The following summarises the assumptions which together imply this contradiction:

1. The ontology satisfies ESMR or EMMR.
2. The "macroscopically observable property" Q has $n > 3$ distinguishable values (requiring that the quantum system has $d \geq n > 3$ dimensions).

3. An eigenstate $|0\rangle$ of Q can be chosen such that some quantum state $|\psi\rangle$ satisfying $|\langle 0|\psi\rangle| \in (0, \frac{1}{\sqrt{2}})$ can be prepared.

4. The quantum transformation U and quantum measurement $\mathcal{B}'$ described above can be performed.

Assumptions (i-ii) are about possible underlying ontological models, while (iii-iv) are implications of standard quantum theory. The conclusion must therefore be that either ESMR/EMMR ontologies are impossible for $n > 3$ distinguishable values, or that quantum theory is not correct. Quantum theory is therefore incompatible with ESMR or EMMR macro-realism.

7 Discussion

The theorem in this paper proves that quantum theory is incompatible with ESMR and EMMR macro-realist ontologies where the macroscopically observable property has $n > 3$ distinguishable values. This is stronger than the argument from the Leggett-Garg inequalities, which is only able to rule out EMMR ontologies. Therefore, only SSMR models are left as possibilities for macro-realist quantum ontologies.

As noted above, no argument is able to rule out all SSMR ontologies because Bohmian mechanics is an SSMR theory which reproduces all predictions of quantum theory. It may be possible, however, to produce a theorem that rules out some subset of SSMR theories.

For example, Bohmian mechanics is a *ψ-ontic* theory [20]. That is, each ontic state $\lambda = (\vec{r}, |\psi\rangle)$ can only be accessed by preparing one quantum state, namely $|\psi\rangle$—there is no ontic overlap between different quantum states. It may therefore be possible to prove the incompatibility of quantum theory and all SSMR ontologies that aren't ψ-ontic. This, together with the result presented here, would essentially say that to be macro-realist you must have an ontology consisting of the full quantum state plus extra information. Many would consider this a very strong argument against macro-realism. For example, such models might reasonably be accused of simply artificially adding macro-realism on top of quantum theory, rather than providing an understanding of quantum theory that respects macro-realism. Of course, those sympathetic to Bohmian mechanics would not be swayed by any such arguments, as Bohmian mechanics is already macro-realist.

Papers on the Leggett-Garg argument, including those addressing the clumsiness loophole [10, 11, 16], have concentrated on $d = 2$ dimensional systems. As a result of closely following the Leggett-Garg assumptions, they are still unable to rule out any models outside of EMMR. They certainly could not rule out all ESMR models, since the Kochen-Specker model satisfies ESMR and exists in $d = 2$. By contrast, the theorem in this paper works when $d \geq n > 3$.

The next stage is the development of an experimental test of the improved theorem. This requires a detailed, noise-tolerant analysis, as any experiment is unavoidably subject to non-zero noise. The asymmetric overlap measure, which is used to characterise the different categories of macro-realism, is an inherently noise-intolerant quantity. To bring the theorem into an experimentally testable form therefore requires a more noise-tolerant alternative and suitable adjusted characterisations of the different categories of macro-realism. This is possible, but it is not a simple process. Two different approaches for such noise-tolerant replacements are currently in development and will be the subject of future papers [36, 37].

It is interesting to note that experiments based on this result will be an entirely new avenue for tests of macro-realism. Experimental tests based on the Leggett-Garg argument will always have certain features and difficulties in common (such as the clumsiness loophole noted in section 5). However, since the approach of this work is so different in character one can expect the resulting experiments to be similarly different, hopefully avoiding many of the difficulties common to Leggett-Garg while requiring challenging new high-precision tests of quantum theory in $d > 2$ Hilbert spaces.

As is common in such foundational works, this paper has considered only the case of finite-dimensional quantum systems. It is hoped that an extension to infinite-dimensional cases should be possible. Due to the fact that quantum states become integrals over bases in the infinite case, a further layer of measure-theoretic complexity would likely be required. Actually developing such an extension therefore remains an interesting open problem.

Finally, one should note that in this paper the "macro" quantity Q was taken to correspond to a measurement of basis $\mathcal{B}_Q$ in the quantum case. A more general approach might allow Q to correspond to a POVM measurement instead. That is, for each value q of Q there would be some POVM element E_q and the operational eigenstates $|\psi\rangle$ of q would be those satisfying $\langle\psi|E_q|\psi\rangle = 1$. We are confident that the results presented here can be fairly directly

extended to such a case and this would be another interesting avenue for further work. Such an extension would likely add significant complexity to the proofs without changing the fundamental ideas, however.

Acknowledgements

We would like to thank Jonathan Barrett, Dominic C. Horsman, Matty Hoban, Ciarán Lee, Chris Timpson, Ronnie Hermens, and Andrew Briggs for insightful discussions as well as anonymous referees for constructive feedback.
JMAA is supported by the Engineering and Physical Sciences Research Council, the NQIT Quantum Hub, the FQXi Large Grant "Thermodynamic vs information theoretic entropies in probabilistic theories", and the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science.
OJEM is supported by the Templeton World Charity Foundation.
SG is supported by the Engineering and Physical Sciences Research Council and Trinity College, Oxford (Williams Scholarship).

References

[1] A. J. Leggett and A. Garg, *Quantum Mechanics versus Macroscopic Realism: Is the Flux There when Nobody Looks?* Physical Review Letters **54**(9), 857–860 (1985). http://dx.doi.org/10.1103/PhysRevLett.54.857

[2] L. E. Ballentine, *Realism and Quantum Flux Tunneling*. Physical Review Letters **59**(14), 1493 (1987). http://dx.doi.org/10.1103/PhysRevLett.59.1493

[3] A. J. Leggett and A. Garg, *Comment on "Realism and Quantum Flux Tunneling"*. Physical Review Letters **59**(14), 1621 (1987). http://dx.doi.org/10.1103/PhysRevLett.59.1621

[4] A. J. Leggett, *Experimental Approaches to the Quantum Measurement Paradox*. Foundations of Physics **18**(9), 939–952 (1988). http://dx.doi.org/10.1007/BF01855943

[5] A. J. Leggett, *Probing Quantum Mechanics Towards the Everyday World: Where do we Stand?* Physica Scripta **2002**(T102), 69 (2002). http://dx.doi.org/10.1238/Physica.Topical.102a00069

[6] A. J. Leggett, *Testing the Limits of Quantum Mechanics: Motivation, State of Play, Prospects*. Journal of Physics: Condensed Matter **14**(15), R415 (2002). http://dx.doi.org/10.1088/0953-8984/14/15/201

[7] J. Kofler and Časlav Brukner, *Condition for Macroscopic Realism Beyond the Leggett-Garg Inequalities*. Physical Review A **87**(5), 052115 (2013). arXiv:1207.3666 [quant-ph] , http://dx.doi.org/10.1103/PhysRevA.87.052115

[8] O. J. E. Maroney and C. G. Timpson, *Quantum- vs. Macro- Realism: What does the Leggett-Garg Inequality Actually Test?* British Journal for the Philosophy of Science **In Production** (2017). arXiv:1412.6139 [quant-ph]

[9] C. Emary, N. Lambert, and F. Nori, *Leggett-Garg Inequalities*. Reports on Progress in Physics **77**(1), 016001 (2014). arXiv:1304.5133 [quant-ph] , http://dx.doi.org/10.1088/0034-4885/77/1/016001

[10] G. C. Knee, K. Kakuyanagi, M.-C. Yeh, Y. Matsuzaki, H. Toida, H. Yamaguchi, S. Saito, A. J. Leggett, and W. J. Munro, *A Strict Experimental Test of Macroscopic Realism in a Superconducting Flux Qubit*. Nature Communications **7**, 13253 (2016). arXiv:1601.03728 [quant-ph] , http://dx.doi.org/10.1038/ncomms13253

[11] E. Huffman and A. Mizel, *Violation of Noninvasive Macrorealism by a Superconducting Qubit: Implementation of a Leggett-Garg Test that Addresses the Clumsiness Loophole*. Physical Review A **95**(3), 032131 (2017). arXiv:1609.05957 [quant-ph] , https://doi.org/10.1103/PhysRevA.95.032131

[12] Z.-Q. Zhou, S. F. Huelga, C.-F. Li, and G.-C. Guo, *Experimental Detection of Quantum Coherent Evolution through the Violation of Leggett-Garg-Type Inequalities*. Physical Review Letters **115**(11), 113002 (2015). arXiv:1209.2176 [quant-ph] , http://dx.doi.org/10.1103/PhysRevLett.115.113002

[13] L. Clemente and J. Kofler, *Necessary and Sufficient Conditions for Macroscopic Realism from Quantum Mechanics*. Physical Review A **91**(6), 062103 (2015). arXiv:1501.07517 [quant-ph] , http://dx.doi.org/10.1103/PhysRevA.91.062103

[14] L. Clemente and J. Kofler, *No Fine Theorem for Macrorealism: Limitations of the Leggett-Garg Inequality*. Physical Review Letters **116**(15), 150401 (2016). arXiv:1509.00348 [quant-ph] , http://dx.doi.org/10.1103/PhysRevLett.116.150401

[15] G. Bacciagaluppi, *Leggett-Garg Inequalities, Pilot Waves and Contextuality*. International Journal of Quantum Foundations **1**, 1–17 (2015). arXiv:1409.4104 [quant-ph]
[16] M. M. Wilde and A. Mizel, *Addressing the Clumsiness Loophole in a Leggett-Garg Test of Macrorealism*. Foundations of Physics **42**(2), 256–265 (2012). arXiv:1001.1777 [quant-ph] , http://dx.doi.org/10.1007/s10701-011-9598-4
[17] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*. Cambridge University Press, Cambridge (1987). http://dx.doi.org/10.1017/CBO9780511815676
[18] M. F. Pusey, J. Barrett, and T. Rudolph, *On the Reality of the Quantum State*. Nature Physics **8**(6), 475–478 (2012). arXiv:1111.3328 [quant-ph] , http://dx.doi.org/10.1038/nphys2309
[19] L. Hardy, *Quantum Ontological Excess Baggage*. Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **35**(2), 267–276 (2004). http://dx.doi.org/10.1016/j.shpsb.2003.12.001
[20] M. S. Leifer, *Is the Quantum State Real? An Extended Review of ψ-ontology Theorems*. Quanta **3**(1), 67–155 (2014). arXiv:1409.1570 [quant-ph] , http://dx.doi.org/10.12743/quanta.v3i1.22
[21] D. Bohm and B. J. Hiley, *The Undivided Universe: An Ontological Interpretation of Quantum Theory*. Routledge (1995)
[22] D. Dürr and S. Teufel, *Bohmian Mechanics: The Physics and Mathematics of Quantum Theory*. Springer, Berlin (2009). http://dx.doi.org/10.1007/b99978
[23] D. Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I*. Physical Review **85**(2), 166–179 (1952). http://dx.doi.org/10.1103/PhysRev.85.166
[24] D. Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II*. Physical Review **85**(2), 180–193 (1952). http://dx.doi.org/10.1103/PhysRev.85.166
[25] L. de Broglie, *The New Dynamics of Quanta*. In *Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference*, eds. G. Bacciagaluppi and A. Valentini, pp. 373–406, Cambridge University Press, Cambridge (2009), ISBN 9780521814218. arXiv:quant-ph/0609184 [quant-ph]
[26] J.-M. A. Allen, *Quantum Superpositions cannot be Epistemic*. Quantum Studies: Mathematics and Foundations **3**(2), 161–177 (2016). arXiv:1501.05969 [quant-ph] , http://dx.doi.org/10.1007/s40509-015-0066-2
[27] N. Harrigan and T. Rudolph, *Ontological Models and the Interpretation of Contextuality* (2007). arXiv:0709.4266 [quant-ph]
[28] N. Harrigan and R. W. Spekkens, *Einstein, Incompleteness, and the Epistemic View of Quantum States*. Foundatations of Physics **40**(2), 125 (2010). arXiv:0706.2661 [quant-ph] , http://dx.doi.org/10.1007/s10701-009-9347-0
[29] E. G. Beltrametti and S. Bugajski, *A Classical Extension of Quantum Mechanics*. Journal of Physics A **28**(12), 3329 (1995). http://dx.doi.org/10.1088/0305-4470/28/12/007
[30] L. Ballentine, *Ontological Models in Quantum Mechanics: What do they Tell Us?* (2014). arXiv:1402.5689 [quant-ph]
[31] M. S. Leifer and O. J. E. Maroney, *Maximally Epistemic Interpretations of the Quantum State and Contextuality*. Physical Review Letters **110**(12), 120401 (2013). arXiv:1208.5132 [quant-ph] , http://dx.doi.org/10.1103/PhysRevLett.110.120401
[32] O. J. E. Maroney, *How Statistical are Quantum States?* (2012). arXiv:1207.6906 [quant-ph]
[33] C. M. Caves, C. A. Fuchs, and R. Schack, *Conditions for Compatibility of Quantum-State Assignments*. Physical Review A **66**(6), 062111 (2002). arXiv:quant-ph/0206110 [quant-ph] , http://dx.doi.org/10.1103/PhysRevA.66.062111
[34] J. Barrett, E. G. Cavalcanti, R. Lal, and O. J. E. Maroney, *No ψ-Epistemic Model Can Fully Explain the Indistinguishability of Quantum States*. Physical Review Letters **112**(25), 250403 (2014). arXiv:1310.8302 [quant-ph] , http://dx.doi.org/10.1103/PhysRevLett.112.250403
[35] S. Kochen and E. P. Specker, *The Problem of Hidden Variables in Quantum Mechanics*. Journal of Mathematics and Mechanics **17**, 59–87 (1967)
[36] J.-M. A. Allen, *Reality, Causality, and Quantum Theory* (2017). Forthcoming doctoral thesis to be made available on the arXiv.
[37] R. Hermens and O. J. E. Maroney, *Constraints on Macroscopic Realism Without Assuming Non-invasive Measurability* (2017). arXiv:1706.01641 [quant-ph]

A A Useful Lemma

Here, a lemma is proved that will be useful in the following proofs.

Lemma. *Let $f(\lambda) : \Lambda \to [0,1]$ be any measurable function from an ontic state space onto the unit interval. Let μ be any probability measure over Λ. If*

$$\int_\Lambda \mathrm{d}\mu(\lambda)\, f(\lambda) = 1 \tag{24}$$

then it follows that

$$\mu\left(\ker(1-f)\right) = 1 \tag{25}$$

where the kernel of a measurable function $g : \Lambda \to [0,1]$ is defined

$$\ker g \stackrel{\text{def}}{=} \{\lambda' \in \Lambda \,:\, g(\lambda') = 0\} \tag{26}$$

and is necessarily a measurable set for measurable function g.

Proof. Let $\bar{f}(\lambda) \stackrel{\text{def}}{=} 1 - f(\lambda)$, which is a measurable function from Λ to $[0,1]$. As measurable functions, the kernels of both f and $\bar{f}$ are measurable sets in Λ.

Equation (24) implies that

$$1 = \int_{\ker \bar{f}} \mathrm{d}\mu(\lambda)\, f(\lambda) + \int_{\Lambda\setminus\ker \bar{f}} \mathrm{d}\mu(\lambda)\, f(\lambda) \tag{27}$$

$$= \mu(\ker \bar{f}) + \int_{\Lambda\setminus\ker \bar{f}} \mathrm{d}\mu(\lambda)\, f(\lambda) \tag{28}$$

since if $\lambda \in \ker \bar{f}$ then $f(\lambda) = 1$.

Consider the second term in Eq. (28); there are two possibilities. First, the term could be zero (if, for example, the only subsets of $\Lambda\setminus\ker \bar{f}$ where $f(\lambda) > 0$ are measure-zero subsets according to μ). If it's not zero then, since for all $\lambda \in \Lambda\setminus\ker \bar{f}$ $f(\lambda) < 1$ then $\int_{\Lambda\setminus\ker \bar{f}} \mathrm{d}\mu(\lambda)\, f(\lambda) < \mu(\Lambda\setminus\ker \bar{f})$. In the latter case this would imply

$$\mu(\ker \bar{f}) + \mu(\Lambda\setminus\ker \bar{f}) > 1 \tag{29}$$

$$\mu(\Lambda) > 1 \tag{30}$$

which is a contradiction as $\mu(\Lambda) = 1$ by definition of a probability measure. The only option is therefore for this term to vanish, in which case $1 = \mu(\ker \bar{f}) = \mu\left(\ker(1-f)\right)$ as desired. □

B Bounding Asymmetric Overlaps

The main text quotes a bound, Eq. (5), for the asymmetric overlap together with a sketch of the proof. This will now be proved fully.

The spirit of the proof is that if λ can be obtained by preparing some $\nu \in \Delta_{|\phi\rangle}$ then it should also return the outcome $|\phi\rangle$ with certainty in any measurement where that is an option (there are exceptions which make the proof more difficult than this). Thus, the probability of preparing μ and getting a λ which then returns the outcome $|\phi\rangle$ in a measurement is at least the probability of preparing μ and getting a λ which is accessible from some $\nu \in \Delta_{|\phi\rangle}$, which is a paraphrase of the desired result.

To prove Eq. (5), consider preparing $|\phi\rangle$ according to $\nu \in \Delta_{|\phi\rangle}$ and then performing some quantum measurement $M_\phi \ni |\phi\rangle$. For the ontological model to reproduce quantum probabilities, as in Eq. (3), it is therefore required that

$$\int_\Lambda \mathrm{d}\nu(\lambda) \mathbb{P}_{M_\phi}(|\phi\rangle \,|\, \lambda) = 1. \tag{31}$$

It is tempting to conclude from this that for every λ in the support of ν then $\mathbb{P}_{M_\phi}(|\phi\rangle \,|\, \lambda) = 1$. However, this is not generally possible because it is not generally possible to define a support for the measure ν. Therefore, a slightly different proof is required.

For notational convenience, define $g(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_{M_\phi}(|\phi\rangle \,|\, \lambda)$ as a measurable function from Λ to $[0,1]$. It follows from Eq. (31) and the lemma of Appendix A that $\nu(\ker(1-g)) = 1$. That is, $g(\lambda) = 1$ almost everywhere according to ν.

Now consider preparing $|\psi\rangle$ according to μ and then measuring with the same $M_\phi \ni |\phi\rangle$, by Eq. (1) and the Born rule

$$|\langle\phi|\psi\rangle|^2 = \int_\Lambda \mathrm{d}\mu(\lambda)\mathbb{P}_{M_\phi}(|\phi\rangle\,|\,\lambda). \tag{32}$$

Restricting the integral to $\ker(1-g)$ where, for every λ, $g(\lambda)=1$ one finds

$$|\langle\phi|\psi\rangle|^2 \geq \int_{\ker(1-g)} \mathrm{d}\mu(\lambda)\, g(\lambda) \tag{33}$$

$$= \mu\left(\ker(1-g)\right). \tag{34}$$

Thus, $|\langle\phi|\psi\rangle|^2$ bounds $\mu(\Omega)$ from above for some $\Omega \subseteq \Lambda$ for which $\nu(\Omega)=1$. It must therefore bound the smallest such $\mu(\Omega)$ from above:

$$|\langle\phi|\psi\rangle|^2 \geq \mu\left(\ker(1-g)\right) \geq \inf_{\Omega\subseteq\Lambda:\nu(\Omega)=1} \mu(\Omega) \stackrel{\text{def}}{=} \varpi(\nu\,|\,\mu) \tag{35}$$

which proves Eq. (5).

C Asymmetric Overlaps and Transformations

A similar manoeuvre to Appendix B can be used to prove Eq. (11). The gist of this proof is given in the text while the full proof is below.

It suffices to prove that for any measurable $\Omega' \subseteq \Lambda$ such that $\nu'(\Omega')=1, \forall \nu' \in \Delta_{U|\phi\rangle}$ then there exists some measurable $\Omega \subseteq \Lambda$ such that $\nu(\Omega)=1, \forall \nu \in \Delta_{|\phi\rangle}$ and $\mu'(\Omega') \geq \mu(\Omega)$. Since μ' is the measure obtained by following μ with γ then by Eq. (2)

$$\mu'(\Omega') \stackrel{\text{def}}{=} \int_\Lambda \mathrm{d}\mu(\lambda)\gamma(\Omega'|\lambda) \tag{36}$$

where Ω' is any such subset.

For any $\nu \in \Delta_{|\phi\rangle}$ there is some $\nu' \in \Delta_{U|\phi\rangle}$ such that ν' is obtained by performing ν followed by γ. Therefore, by Eq. (2)

$$1 = \nu'(\Omega') = \int_\Lambda \mathrm{d}\nu(\lambda)\gamma(\Omega'|\lambda) \tag{37}$$

Viewing $f(\lambda) \stackrel{\text{def}}{=} \gamma(\Omega'|\lambda)$ as a measurable function from Λ to $[0,1]$ and using the lemma of Appendix A, Eq. (37) implies $\nu(\ker(1-f))=1$. Note that this holds for every $\nu \in \Delta_{|\phi\rangle}$.

Therefore, let $\Omega = \ker(1-f) \subseteq \Lambda$ be a subset for which every $\nu(\Omega)=1, \forall \nu \in \Delta_{|\phi\rangle}$ and return to Eq. (36). One finds

$$\mu'(\Omega') \geq \int_{\ker(1-f)} \mathrm{d}\mu(\lambda)\, f(\lambda) \tag{38}$$

$$= \mu(\Omega) \tag{39}$$

which proves the desired statement and thereby Eq. (11).

D Anti-Distinguishability and Asymmetric Overlap

It is claimed (and roughly motivated) in the text that if $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ is an anti-distinguishable triple then Eq. (9) must hold with equality. This will now be proved fully.

Recall that $\{|\psi\rangle, |\phi\rangle, |0\rangle\}$ is an anti-distinguishable triple if and only if there is some quantum measurement M with three outcomes $E_{\neg\psi}, E_{\neg\phi}, E_{\neg 0}$ such that the outcome of getting $E_{\neg\psi}$ from a system prepared in state $|\psi\rangle$ is zero and similarly for the other state/outcome pairs. By Eq. (3) it therefore follows that for all $\mu \in \Delta_{|\psi\rangle}, \nu \in$

$\Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$

$$\int_\Lambda \mathrm{d}\mu(\lambda)\mathbb{P}_M(E_{\neg\psi}|\,\lambda) = 0 \tag{40}$$

$$\int_\Lambda \mathrm{d}\nu(\lambda)\mathbb{P}_M(E_{\neg\phi}|\,\lambda) = 0 \tag{41}$$

$$\int_\Lambda \mathrm{d}\chi(\lambda)\mathbb{P}_M(E_{\neg 0}|\,\lambda) = 0. \tag{42}$$

To prove that Eq. (9) holds with equality it suffices to show that given any measurable $\Omega \subseteq \Lambda$ for which $\nu(\Omega) = \chi(\Omega) = 1$ for all $\nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$ there exists some measurable $\Omega', \Omega'' \subseteq \Lambda$ for which $\nu(\Omega') = \chi(\Omega'') = 1$ for all $\nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$ for which

$$\mu(\Omega) \geq \mu(\Omega') + \mu(\Omega''). \tag{43}$$

This, together with Eq. (9) itself, would prove the desired result since the right-hand side bounds $\varpi(|\phi\rangle|\mu) + \varpi(|0\rangle|\mu)$ from above.

To prove that Eq. (43) holds define the following measurable functions from Λ to $[0, 1]$

$$g_\psi(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_M(E_{\neg\phi}|\,\lambda) + \mathbb{P}_M(E_{\neg 0}|\,\lambda) \tag{44}$$

$$g_\phi(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_M(E_{\neg\psi}|\,\lambda) + \mathbb{P}_M(E_{\neg 0}|\,\lambda) \tag{45}$$

$$g_0(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_M(E_{\neg\psi}|\,\lambda) + \mathbb{P}_M(E_{\neg\phi}|\,\lambda). \tag{46}$$

Using the fact that, for any $\lambda \in \Lambda$, the sum of probabilities of outcomes for any measurement must be unity it follows that $\mathbb{P}_M(E_{\neg\psi}|\lambda) = 1 - g_\psi(\lambda)$ and similarly for $|\phi\rangle$ and $|0\rangle$. Therefore Eqs. (40,41,42) are equivalent to

$$\int_\Lambda \mathrm{d}\mu(\lambda)\, g_\psi(\lambda) = 1 \tag{47}$$

$$\int_\Lambda \mathrm{d}\nu(\lambda)\, g_\phi(\lambda) = 1 \tag{48}$$

$$\int_\Lambda \mathrm{d}\chi(\lambda)\, g_0(\lambda) = 1. \tag{49}$$

By the lemma in Appendix A it immediately follows that $\nu(\ker(1 - g_\phi)) = \chi(\ker(1 - g_0)) = 1$ where, recall, ν and χ are arbitrary measures from $\Delta_{|\phi\rangle}$ and $\Delta_{|0\rangle}$ respectively.

With these definitions, consider $\mu(\Omega)$ for any measurable $\Omega \subseteq \Lambda$ for which $\nu(\Omega) = \chi(\Omega) = 1$ for all $\nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$

$$\mu(\Omega) = \int_\Omega \mathrm{d}\mu(\lambda) \tag{50}$$

$$= \int_\Omega \mathrm{d}\mu(\lambda)\Big(\mathbb{P}_M(E_{\neg\psi}|\lambda) + g_\psi(\lambda)\Big) \tag{51}$$

$$= \int_\Omega \mathrm{d}\mu(\lambda)\, g_\psi(\lambda) + \int_\Omega \mathrm{d}\mu(\lambda)\mathbb{P}_M(E_{\neg\psi}|\lambda) \tag{52}$$

By the definitions of g_ψ, g_ϕ, and g_0 this equals

$$\mu(\Omega) = \int_\Omega \mathrm{d}\mu(\lambda)\, g_\phi(\lambda) + \int_\Omega \mathrm{d}\mu(\lambda)\, g_0(\lambda) - \int_\Omega \mathrm{d}\mu(\lambda)\, \mathbb{P}_M(E_{\neg\psi}|\lambda) \tag{53}$$

$$= \int_\Omega \mathrm{d}\mu(\lambda)\, g_\phi(\lambda) + \int_\Omega \mathrm{d}\mu(\lambda)\, g_0(\lambda) \tag{54}$$

where the last term vanished because $0 \leq \int_\Omega \mathrm{d}\mu(\lambda)\mathbb{P}_M(E_{\neg\psi}|\lambda) \leq \int_\Lambda \mathrm{d}\mu(\lambda)\mathbb{P}_M(E_{\neg\psi}|\lambda) = 0$ by Eq. (40). Continuing

$$\begin{aligned}\mu(\Omega) &\geq \int_{\Omega\cap\ker(1-g_\phi)} \mathrm{d}\mu(\lambda)\, g_\phi(\lambda) \\ &\quad + \int_{\Omega\cap\ker(1-g_0)} \mathrm{d}\mu(\lambda)\, g_0(\lambda) \qquad (55)\\ &= \mu\left(\Omega\cap\ker(1-g_\phi)\right) + \mu\left(\Omega\cap\ker(1-g_0)\right) \qquad (56)\end{aligned}$$

by restricting the domains of integration and recalling that $\forall\lambda \in \ker(1-g_\phi)$, $g_\phi(\lambda) = 1$ (and similarly for g_0). Note that as both Ω and $\ker(1-g_\phi)$ are measure-one according to any $\nu \in \Delta_{|\phi\rangle}$, it follows that their intersection also satisfies $\nu\left(\Omega\cap\ker(1-g_\phi)\right) = 1$. Similarly, $\chi\left(\Omega\cap\ker(1-g_0)\right) = 1$. Thus what has been proved is that given any measurable Ω such that $\nu(\Omega) = \chi(\Omega) = 1$ for all $\nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$ there exist measurable sets $\Omega' \stackrel{\text{def}}{=} \Omega\cap\ker(1-g_\phi)$ and $\Omega'' \stackrel{\text{def}}{=} \Omega\cap\ker(1-g_0)$ satisfying $\nu(\Omega') = \chi(\Omega'') = 1$ for all $\nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$ for which

$$\mu(\Omega) \geq \mu(\Omega') + \mu(\Omega''). \qquad (57)$$

This is exactly what was to be proved.

E ESMR, EMMR, and Asymmetric Overlap

Here it is proved that for any state $|\psi\rangle \in \mathcal{P}(\mathcal{H})$ and any eigenstate $|0\rangle \in \mathcal{B}_Q$ for an ESMR/EMMR quantity Q then Eq. (12) must hold. This follows because in both ESMR and EMMR ontological models every ontic state that can be prepared can be obtained by preparing some operational eigenstate of Q.

This is easiest to state precisely if the asymmetric overlap is extended further to include arbitrary n-partite sets of states. That is, if $\mathcal{S}$ is a set of $n-1$ quantum states and $\mu \in \Delta_{|\psi\rangle}$ is a preparation measure for quantum state $|\psi\rangle$ then the n-partite asymmetric overlap $\varpi(\mathcal{S}|\mu)$ is the probability that the ontic state obtained by sampling μ could have been obtained by preparing some state in $\mathcal{S}$. Mathematically,

$$\varpi(\mathcal{S}\,|\,\mu) \stackrel{\text{def}}{=} \inf\left\{\mu(\Omega)\ :\ \Omega\subseteq\Lambda, \forall|\phi\rangle\in\mathcal{S}, \forall\nu\in\Delta_{|\phi\rangle}, \nu(\Omega)=1\right\}. \qquad (58)$$

As with the tripartite overlap, Boole's inequality gives the bound

$$\varpi(\mathcal{S}\,|\,\mu) \leq \sum_{|\phi\rangle\in\mathcal{S}} \varpi(|\phi\rangle\,|\,\mu). \qquad (59)$$

From Eq. (58), it follows immediately that for any ESMR or EMMR ontological model, every preparation $\mu \in \Delta_{|\phi\rangle}$ of every state $|\psi\rangle \in \mathcal{P}(\mathcal{H})$

$$\varpi(\mathcal{B}_Q\,|\,\mu) = 1, \qquad (60)$$

i.e. the probability of obtaining a $\lambda \in \Lambda$ accessible from some $|0\rangle \in \mathcal{B}_Q$ is unity. Let $\Omega \subseteq \Lambda$ be any measurable subset for which $\nu(\Omega) = 1, \forall\nu \in \Delta_{|0\rangle}, \forall|0\rangle \in \mathcal{B}_Q$. Let M_Q be any measurement for the basis $\mathcal{B}_Q$ and, for each $|0\rangle \in \mathcal{B}_Q$, let $f_{|0\rangle}(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_{M_Q}(|0\rangle\,|\,\lambda)$ be a measurable function from Λ to $[0,1]$. By the lemma of Appendix A it follows that for every $|0\rangle \in \mathcal{B}_Q$ and any $\nu \in \Delta_{|0\rangle}$

$$\begin{aligned}1 &= \int_\Lambda \mathrm{d}\nu(\lambda)\, f_{|0\rangle}(\lambda) \qquad (61)\\ \Rightarrow 1 &= \nu(\ker(1-f_{|0\rangle})). \qquad (62)\end{aligned}$$

Similarly, for any other $|1\rangle \in \mathcal{B}_Q$ and $\chi \in \Delta_{|1\rangle}$ ($|1\rangle \neq |0\rangle$) then

$$\begin{aligned}0 &= \int_\Lambda \mathrm{d}\chi(\lambda)\, f_{|0\rangle}(\lambda) \qquad (63)\\ \Rightarrow 1 &= \int_\Lambda \mathrm{d}\chi(\lambda)\,(1-f_{|0\rangle}(\lambda)) \qquad (64)\\ \Rightarrow 1 &= \chi(\ker f_{|0\rangle}). \qquad (65)\end{aligned}$$

Using these definitions, for any $|0\rangle \in \mathcal{B}_Q$

$$\mu(\Omega) = \mu\left(\Omega \cap \ker(1 - f_{|0\rangle})\right) + \mu(\Omega \backslash \ker(1 - f_{|0\rangle})). \tag{66}$$

Since , $\nu(\ker(1 - f_{|0\rangle})) = 1, \forall \nu \in \Delta_{|0\rangle}$ then $\mu(\Omega \cap \ker(1 - f_{|0\rangle}))$ upper bounds $\varpi(|0\rangle \,|\, \mu)$. Moreover, since $\ker f_{|0\rangle} \subseteq \Lambda \backslash \ker(1 - f_{|0\rangle})$ then for any $\chi \in \Delta_{|1\rangle}, |1\rangle \in \mathcal{B}_Q \backslash \{|0\rangle\}$ it follows that $\chi(\Omega \backslash \ker(1 - f_{|0\rangle})) = 1$. Thus this process can be continued taking $\Omega' = \Omega \backslash \ker(1 - f_{|0\rangle})$ as a set for which $\chi(\Omega') = 1, \forall \chi \in \Delta_{|1\rangle}, \forall |1\rangle \in \mathcal{B}_Q \backslash \{|0\rangle\}$ then for some $|1\rangle \in \mathcal{B}_Q \backslash \{|0\rangle\}$

$$\mu(\Omega') = \mu\left(\Omega' \cap \ker(1 - f_{|1\rangle})\right) + \mu(\Omega' \backslash \ker(1 - f_{|1\rangle})) \tag{67}$$

where $\mu(\Omega' \cap \ker(1 - f_{|1\rangle}))$ upper bounds $\varpi(|1\rangle \,|\, \mu)$.

By induction therefore, one finds,

$$\mu(\Omega) \geq \sum_{|0\rangle \in \mathcal{B}_Q} \varpi(|0\rangle \,|\, \mu). \tag{68}$$

Since this holds for any such Ω then

$$\varpi(\mathcal{B}_Q \,|\, \mu) \geq \sum_{|0\rangle \in \mathcal{B}_Q} \varpi(|0\rangle \,|\, \mu) \tag{69}$$

which, combined with Eqs. (59,60) gives

$$\varpi(\mathcal{B}_Q \,|\, \mu) = \sum_{|0\rangle \in \mathcal{B}_Q} \varpi(|0\rangle \,|\, \mu) = 1. \tag{70}$$

Finally, noting that $\varpi(|0\rangle \,|\, \mu) \leq |\langle 0|\psi\rangle|^2$ and $\sum_{|0\rangle \in \mathcal{B}_Q} |\langle 0|\psi\rangle|^2 = 1$ as $\mathcal{B}_Q$ is a basis and $|\psi\rangle$ is normalised, then Eq. (70) implies

$$\varpi(|0\rangle \,|\, \mu) = |\langle 0|\psi\rangle|^2, \quad \forall |0\rangle \in \mathcal{B}_Q \tag{71}$$

which is exactly Eq. (12).

F Tripartite Asymmetric Overlap and Quantum Measurements

In the proof of the main theorem, it is claimed that

$$\varpi(|\phi\rangle, |0\rangle \,|\, \mu) \leq \mathbb{P}_{\mathcal{B}'}(|0\rangle, |1'\rangle \,|\, |\psi\rangle), \tag{72}$$

where the right hand side is the quantum probability of the corresponding quantum measurement and the states and measurements are as defined in the text. This will now be proved.

The gist of the proof is that if $\lambda \in \Lambda$ is accessible by preparing two quantum states then it may only return measurement results that are compatible with both preparations. So if λ is accessible by preparing both $|\psi\rangle$ and $|0\rangle$ then it may only return $|0\rangle$ in a measurement in the $\mathcal{B}'$ basis and similarly if λ is accessible by preparing both $|\psi\rangle$ and $|\phi\rangle$ then it may only return either $|0\rangle$ or $|1'\rangle$ for a similar measurement. Thus, if one of these λ's is obtained in a preparation of $|\psi\rangle$ then the measurement result is necessarily either $|0\rangle$ or $|1'\rangle$. This will now be fully fleshed out.

Consider a measurement $M_{\mathcal{B}'}$ of the basis $\mathcal{B}'$ and let $f(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_{\mathcal{B}'}(|0\rangle \,|\, \lambda) + \mathbb{P}_{\mathcal{B}'}(|1\rangle \,|\, \lambda)$ and $g_3(\lambda) \stackrel{\text{def}}{=} \mathbb{P}_{\mathcal{B}'}(|3'\rangle \,|\, \lambda)$ be measurable functions from Λ to $[0, 1]$. Note that by definition of $|\phi\rangle$ and $|0\rangle$the following quantum probabilities are known for any $\nu \in \Delta_{|\phi\rangle}$ and $\chi \in \Delta_{|0\rangle}$

$$\int_\Lambda \mathrm{d}\nu(\lambda)\,(f(\lambda) + g_3(\lambda)) = 1 \tag{73}$$

$$\int_\Lambda \mathrm{d}\chi(\lambda) f(\lambda) = 1. \tag{74}$$

Therefore by the lemma of Appendix A it follows that $\nu(\ker(1 - f - g_3)) = \chi(\ker(1 - f)) = 1$. Since $\ker(1 - f) \subseteq \ker(1 - f - g_3)$ it follows that $\Omega \stackrel{\text{def}}{=} \ker(1 - f - g_3)$ is a measurable subset of Λ for which $\nu(\Omega) = \chi(\Omega) = 1$, $\forall \nu \in \Delta_{|\phi\rangle}, \chi \in \Delta_{|0\rangle}$.

The desired quantum probability is given by

$$\mathbb{P}_{\mathcal{B}'}(|0\rangle, |1'\rangle \,|\, |\psi\rangle) = \int_\Lambda \mathrm{d}\mu(\lambda)\, f(\lambda). \tag{75}$$

Since the quantum probability $\int_\Lambda \mathrm{d}\mu(\lambda) g_3(\lambda) = 0$ is also known (by definition of $|\psi\rangle$) it follows that

$$\begin{aligned}
\mathbb{P}_{\mathcal{B}'}(|0\rangle, |1'\rangle \,|\, |\psi\rangle) &= \int_\Lambda \mathrm{d}\mu(\lambda)\, (f(\lambda) + g_3(\lambda)) && (76)\\
&\geq \int_{\ker(1-f-g_3)} \mathrm{d}\mu(\lambda)\, (f(\lambda) + g_3(\lambda)) && (77)\\
&= \mu\left(\ker(1 - f - g_3)\right) = \mu(\Omega). && (78)
\end{aligned}$$

Since, by definition $\mu(\Omega)$ upper bounds $\varpi(|\phi\rangle, |0\rangle \,|\, \mu)$ this proves Eq. (72).